Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 1

"well-posedness"

Exercise 1.1

Consider the heat equation in one space dimension:

$$\begin{cases} u_t = \delta u_{xx}, \ 0 < x < 1, t > 0, \delta > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{1}$$

(a) show that: $||u(.,t)||_2^2 \leq ||u_0(.)||_2^2$. ("energy estimate")
(b) using (a), show that the solution of PDE model (1) is unique.
(c) also, show continuity with respect to initial conditions.
Note that the existence of the solution of PDE (1) can be established using Fourier series.

"PDE solutions"

Exercise 1.2

Solve the PDEs:

(a) $yu_y = u$.
(b) $cu_x - u_y = 0$. (introduce new variables)
(c)* $u_{xx} - u_{yy} = 0$. (introduce new variables)

Exercise 1.3

Consider the potential equation or Laplace equation $\Delta u = 0$. Identify the points $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$. Check that $u(x, y) = \Re(f(z))$ is the solution of the potential equation for, for example, $f(z) = 1$, $f(z) = z^2$ and $f(z) = \log(z - z_0)$, $z_0 \in \mathbb{C} \setminus \{0\}$ (polar coordinates!).

Exercise 1.4

Check that both $u(x, y) = \sin(cx)\mathrm{e}^{-c^2 y}$ and $u(x, y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} u_0(\xi)\mathrm{e}^{-\frac{(x-\xi)^2}{4y}} d\xi$ (with $u(x, 0) = u_0(x)$) solve the PDE $u_{xx} - u_y = 0$.

Exercise 1.5

Check that[1]:

(a) if $u(x,y)$ and $v(x,y)$ are solutions of the system:

$$\begin{cases} u_x + v_y = 0, \\ v_x + u_y = 0, \end{cases}$$

then they also solve the PDE in 1.1(c).
(b) if $u(x,y)$ and $v(x,y)$ are solutions of the Cauchy-Riemann differential equations:

$$\begin{cases} u_x + v_y = 0, \\ v_x - u_y = 0, \end{cases}$$

then they also solve the potential equation in exercise 1.3.
(c) if $u(x,y)$ and $v(x,y)$ are solutions of the PDE system:

$$\begin{cases} u_x + v_y = 0, \\ v_x + u = 0, \end{cases}$$

then they also solve the PDE in exercise 1.4.

"classification"

Exercise 1.6

Classify each of the PDEs below as either hyperbolic, parabolic, or elliptic, determine the characteristics (and transform the equations to canonical form*):

(a) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
(b) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$
(c) $yu_{xx} - 2u_{xy} + \mathrm{e}^x u_{yy} + x^2 u_x - u = 0$
(d) $u_{xx} + yu_{yy} = 0$
(e) $y^2 u_{xx} + x^2 u_{yy} = 0$
(f) $xu_{xx} + u_{yy} = x^2$.

Exercise 1.7

(a) Show that the nonlinear equation $u^2 u_{xx} + 2u_x u_y u_{xy} - u^2 u_{yy} = 0$ is hyperbolic for every solution $u(x,y)$.

(b) Show that the nonlinear equation $(1-u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1-u_y^2)u_{yy} = 0$ in certain kinds of compressible fluid flow models is elliptic, parabolic, or hyperbolic for those solutions $u(x,y)$ such that, respectively, $|\nabla u| < 1$, $|\nabla u| = 1$, or $|\nabla u| > 1$.

[1] you may interchange partial derivatives.

*Exercise 1.8

The following two PDEs are given:

(a) $2u_{xx} + 3u_{xy} + u_{yy} = 0$,
(b) $u_{xx} + 4u_x u_y u_{xy} + 4u_{yy} = 0$.

Determine the type of both PDEs (hyperbolic, parabolic, elliptic).
(*Also, find the general solution $u(x, y)$. For this, make use of a coordinate transformation $(x, y) \to (\xi, \eta)$ to write the PDEs in a canonical form.)

"Fourier"

Exercise 1.9

Use the Fourier transform method to solve the following two linear PDEs with initial condition $u(x, 0) = u_0(x)$:

(a) $u_t = u_{xx} - 12u$,
(b) $u_t = \kappa u_{xx} + \gamma u_x$.[2]

*Exercise 1.10

Show that for the linearized Korteweg-de Vries model[3]

$$u_t = u_{xxx}, \quad u(x, 0) = u_0(x)$$

we obtain (using Fourier transforms):

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u_0}(\xi) \mathrm{e}^{i\xi(x - \xi^2 t)} \, \mathrm{d}\xi,$$

with

$$|\hat{u}(\xi, t)| = |\hat{u_0}(\xi)| \quad \forall\ t \geq 0$$

just as for the advection equation. Here, we denote by $\hat{u}$ and $\hat{u_0}$ the Fourier-transforms of the functions u and u_0, respectively. Remark that the velocity at wavenumber ξ is given by ξ^2. These types of waves are called *dispersive* waves: smooth initial data turn into oscillatory solutions for later points of time. A big difference with the advection equation is that now the initial condition will not be preserved, in general. The Fourier-components disperse relative to each other.

[2] This equation models heat transfer in a long heated bar that is exchanging heat with the surrounding medium.
[3] This PDE models water waves on shallow water surfaces.

Exercise 1.11

Consider the PDE

$$u_t = -u_{xxxx}, \quad u(x,0) = u_0(x).$$

Show that

$$\hat{u}(\xi,t) = \mathrm{e}^{-\xi^4 t}\hat{u_0}(\xi)$$

and comment on the damping effect on the initial condition compared to the damping in the heat equation. What happens for the PDEs $u_t = -u_{xx}$ or $u_t = +u_{xxxx}$?

Exercise 1.12

Use the Fourier series method (with separation of variables) to solve the following two linear PDEs with $x \in [0,1]$, initial condition $u(x,0) = u_0(x)$ and zero-Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$:

(a) $u_t = u_{xx} + 4u_x$ (convection-diffusion PDE)
(b) $u_{tt} - c^2 u_{xx} + d^2 u = 0$ (the Klein-Gordon problem)[4].

*Exercise 1.13

In this exercise we are looking for a solution of the PDE model:

$$u_{tt} = -u_{xx},$$

with $u(x,0) = \sin(2\pi x)$, $u_t(x,0) = 0$ and $u(0,t) = u(1,t)$, $u_x(0,t) = u_x(1,t)$ (periodic boundary conditions).

(a) we write the solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\sin(2\pi kx) + \sum_{k=0}^{\infty} b_k(t)\cos(2\pi kx).$$

Show that $a_k = b_k = 0$ for all $k \neq 1$ and that:

$$\begin{cases} \ddot{a}_1(t) - 4\pi^2 a_1(t) = 0, \\ \ddot{b}_1(t) - 4\pi^2 b_1(t) = 0. \end{cases}$$

(b) re-write these two second-order ODEs as a system of four first-order ODEs. Determine the eigenvalues and eigenvectors of this system and conclude that $u(x,t) = \cosh(2\pi t)\sin(2\pi x)$.

[4] extra initial condition: $u_t(x,0) = 0$.

(c) A similar question for the PDE $u_{tt} = u_x$ with $u(x, 0) = \sin(2\pi x), u_t(x, 0) = 0$ and $u(0, t) = u(1, t)$. Check that in this case: $a_1(t) = \cosh(\sqrt{\pi}t)\cos(\sqrt{\pi}t)$ and $b_1(t) = \sinh(\sqrt{\pi}t)\sin(\sqrt{\pi}t)$.

Exercise 1.14

Use the Fourier series method (with separation of variables) to solve the linear heat equation

$$u_t = \kappa u_{xx}, \quad \kappa \in \mathbb{R}$$

with $x \in [0, 1]$, initial condition $u(x, 0) = \sin(\pi x)$ and zero-Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$. Comment on the cases $\kappa < 0$ (backward heat equation) and $\kappa > 0$ (forward heat equation).

"characteristics"

Exercise 1.15

Consider the hyperbolic PDE:

$$u_t + au_x + bu = f(x, t),$$

with initial condition $u(x, 0) = u_0(x)$, constants a, b and given function f. Apply a special coordinate transformation $(x, t) \to (\xi, \tau)$ to solve this PDE.

Exercise 1.16

Solve the PDE:

$$u_t + \frac{1}{1 + \frac{1}{2}\cos(x)} u_x = 0,$$

with initial condition $u(x, 0) = u_0(x)$. Show that the solution is given by $u(x, t) = u_0(\xi)$, where ξ is the unique solution of the equation $\xi + \frac{1}{2}\sin(\xi) = x + \frac{1}{2}\sin(x) - t$.

*Exercise 1.17

Consider the equation:

$$u_t + xu_x = 0,$$

with initial condition

$$u(x,0) = u_0(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that the solution is given by $u(x,t) = u_0(\xi)$, where ξ is the unique solution of the equation $\xi + \frac{1}{2}\sin(\xi) = x + \frac{1}{2}\sin(x) - t$.
Show that, for $t > 0$, the solution is given by

$$u(x,t) = \begin{cases} 1, & 0 \le x \le \mathrm{e}^t, \\ 0, & \text{elsewhere.} \end{cases}$$