Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 10

Exercise 10.1

Consider the following stationary convection-diffusion PDE:

$$-\Delta u + \beta \cdot \nabla u = g(x, y), \ (x, y) \in \Omega := [0, 1] \times [0, 1] \tag{1}$$

with $g \in \mathcal{L}_2(\Omega)$, constant velocity vector β and homogeneous Dirichlet boundary conditions $u|_{\partial\Omega} = 0$.

(a) Formulate the variational form connected to PDE (1):

$$b(u, v) = (g, v) := L(g), \quad \forall \ v \in V. \tag{2}$$

Which space V would be appropriate? Describe the norm $||h||_V$ for an arbitrary function $v \in V$.

Let V_h be the space of piecewise linear functions: $V_h \subset V$.

(b) What kind of subdivision of the domain would you choose? Sketch a typical basisfunction $\phi_i(x, y) \in V_h$.

Next, choose $\beta = (0, 0)^T$.

(c) Show that the bilinear form $b(.,.)$ in (2) is V-elliptic and continuous.

(d) Prove that the solution $u(x, y)$ of (2) satisfies the stability estimate:

$$||u||_V \leq \frac{\Lambda}{\alpha}$$

for certain positive constants Λ and α. Where do Λ and α come from?

(e) Show that, for the finite element solution $u_h(x, y) \in V_h$ of the discrete system,

$$b(u_h, v) = L(v), \quad \forall \ v \in V_h \tag{3}$$

the following two estimates hold:

$$||u_h||_V \leq \tfrac{\Lambda}{\alpha} \quad (\textit{stability}),$$

$$||u - u_h||_V \leq \tfrac{\gamma}{\alpha}||u - v||_V, \quad \forall \ v \in V_h, \gamma > 0.$$

(the second inequality indicates that u_h is the *'best approximation'* of u in the space V_h)

(f) Check that both problem (2) and problem (3) have a unique solution.

Consider now again the case $\beta \neq (0,0)^T$ and the finite element approximation $u_h = \sum_j \xi_j \phi_j \in V_h$.

(g) Work out the general form of the linear system $\mathcal{C}\vec{\xi} = \vec{g}$. In particular, describe the structure of the matrix $\mathcal{C}$ in terms of the innerproducts. Is the matrix $\mathcal{C}$ symmetric?

(h) For the one-dimensional case, calculate the innerproducts from part (g).

Exercise 10.2

In this exercise we consider the bi-harmonic model

$$\begin{aligned} &\Delta\Delta u = f(x,y), \quad (x,y) \in \Omega, \\ &u = \tfrac{\partial u}{\partial n} = 0, \quad (x,y) \in \partial\Omega, \end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded region, n the outward normal vector and $f \in \mathcal{L}^2(\Omega)$. This PDE models a thin elastic plate with external forces.

Introduce the function space $\mathcal{H}_0^{2,2}(\Omega) := \{v \in \mathcal{H}^{2,2}(\Omega) : v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$, where $\mathcal{H}^{2,2}(\Omega) = \{v \in \mathcal{L}^2(\Omega) : D^\alpha v \in \mathcal{L}^2, \forall\ |\alpha| \leq 2\}$.

(a) Show that the variational problem for this model can be written as

$$\begin{aligned} &\text{find the solution } u \text{ that satisfies:} \\ &a(u,v) := \textstyle\int_\Omega \Delta u \Delta v \; d\Omega = \int_\Omega f \; v \; d\Omega := L(v), \quad \forall v \in \mathcal{H}_0^{0,0}(\Omega). \end{aligned} \tag{5}$$

(b) Which space of piecewise polynomials[1] would be useful for the finite element space V_h. Discuss why piecewise linear polynomials are not suitable for this purpose.

[1] answer this question for the 1D case.

Exercise 10.3

Consider the following one-dimensional time-dependent convection-diffusion PDE model:

$$\frac{\partial u}{\partial t} = \epsilon \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial x} + f(t),\ x \in \Omega := [0, \pi] \tag{6}$$

with $f \in \mathcal{L}_2(\Omega)$, initial condition $u(x,0) = u^0(x)$, constant velocity β, constant $\epsilon > 0$ and homogeneous Dirichlet boundary conditions $u|_{\partial\Omega} = 0$.

(a) First, show that, for $\beta = f = 0$ and $\epsilon = 1$, the following two relations hold:

$$\begin{aligned} &u(x,t) = \textstyle\sum_{j=1}^{\infty} u_j^0 \mathrm{e}^{-j^2 t} \sin(jx) \quad \text{with} \quad u_j^0 = \sqrt{\tfrac{2}{\pi}} \int_0^{\pi} u^0(x) \sin(jx)\ dx, \\ &\qquad \text{and} \\ &||u(t)||_{\mathcal{L}^2(\Omega)} \leq ||u(0)||_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{7}$$

(b) Re-write PDE (9) in the variational form:

$$(\dot{u}(t), v) + a(u, v) = (f(t), v), \quad \forall\ v \in V, t > 0. \tag{8}$$

Give the bilinear form $a(.,.)$.

(c) Consider the finite-dimensional space of piecewise linear functions $V_h \subset V$ and choose the testfunctions v from the same space V_h. Give the semi-discrete ODE system in terms of the finite element solution u_h.

(d) For $\beta = f = 0$ and $\epsilon = 1$, show that

$$||u_h(t)||_{\mathcal{L}^2(\Omega)} \leq ||u(0)||_{\mathcal{L}^2(\Omega)}.$$

Exercise 10.4

Consider the two-dimensional version of PDE (9):

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \epsilon \Delta u + f(t),\ x \in \Omega \subset \mathbb{R}^2 \tag{9}$$

with $u|_{\partial\Omega} = 0$ and $u(x,y,0) = u^0(x,y)$, $(x,y) \in \Omega$.

(a) Work out the variational formulation with test functions $v \in V_h$. (V_h a finite dimensional subspace of $V = \mathcal{H}_0^1(\Omega)$).

(b) Apply Euler-Backward to the semi-discrete ODE system and show that, for the case $f = 0$:

$$||u_h^n|| \leq ||u_h^0|| \leq ||u^0||, \quad n = 1, ..., N.$$

(c) A similar question as in part (b), but now for the Crank-Nicolson method applied to the ODE system.

Exercise 10.5

Formulate the Galerkin method for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1), t > 0.$$

Show that we obtain an ODE system of the form:

$$\mathcal{A}\ddot{\vec{u}} + \mathcal{B}\vec{u} + \vec{f} = \vec{0}.$$

Describe the finite element matrices $\mathcal{A}$, $\mathcal{B}$ and the vector $\vec{f}$.

*Exercise 10.6

Consider again PDE (9). Define the operator:

$$L(u) := \epsilon\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial x} + f(t).$$

(a) Choose the piecewise linear function space V_h for the finite element approximation u_h. Calculate $\frac{\partial u_h}{\partial t}$.

(b) Minimize (using a least-squares approach and the $\mathcal{L}^2$ norm): $\frac{\partial u_h}{\partial t} - L(u_h)$, i.e, write:

$$\min_{\dot{u}_1,\ldots,\dot{u}_N} ||\frac{\partial u_h}{\partial t} - L(u_h)||_{\mathcal{L}^2(\Omega)}$$

and find the ODE system in terms of $\dot{\vec{u}} := (\dot{u}_1, ..., \dot{u}_N)^T$.

(c) Discus the correspondence with the ODE system from part (b) and the one from exercise 7.3 (c).

*Exercise 10.7

The *Sobolev* space $\mathcal{H}^{k,k}(\Omega)$ is defined by:

$$\mathcal{H}^{k,k}(\Omega) = \{v \in \mathcal{L}^2(\Omega) : D^\alpha v \in \mathcal{L}^2, \forall\ |\alpha| \le k\}.$$

(a) Check that, for the function $g(x) = x^{-\beta}, x \in [0,1]$, the following two relations hold:

$$g \in \mathcal{L}^2([0,1]) \Leftrightarrow \beta < \tfrac{1}{2},$$

$$g \in \mathcal{H}^{1,2}([0,1]) \Leftrightarrow \beta < -\tfrac{1}{2}.$$

Consider the unit ball in $\mathbb{R}^3$:

$$\mathcal{B} := \{\vec{x} \in \mathbb{R}^3 : ||\vec{x}||_2 \leq 1, \}$$

and the function $v(\vec{x}) = ||\vec{x}||_2^\lambda$ ($\lambda \in \mathbb{R}$).

(b) Show that[2]:

$$v \in \mathcal{L}^2(\mathcal{B}) \Leftrightarrow \lambda > -\tfrac{3}{2},$$

and

$$v \in \mathcal{H}^{k,2}(\mathcal{B}) \Leftrightarrow \lambda > k - \tfrac{3}{2}.$$

*Exercise 10.8

What do we mean with the *'Poincaré-Friedrichs inequality'* ?

(a) Consider an bounded open region in $\mathbb{R}^d$ (with a sufficiently smooth boundary $\partial\Omega$). Let $u \in \mathcal{H}_0^{1,2}(\Omega)$. Then there is a positive constant $c_*(\Omega)$ such that for all $u \in \mathcal{H}_0^{1,2}(\Omega)$ the following inequality holds:

$$\int_\Omega |u(\vec{x})|^2 \; d\vec{x} \leq c_* \sum_{i=1}^{d} \int_\Omega |\frac{\partial u}{\partial x_i}|^2 \; d\vec{x}. \tag{10}$$

This inequality is being used very frequently in the analysis of the finite element method, among others, to verify the so-called V-ellipticity property.

(b) Take $\Omega = [0, 1] \subset \mathbb{R}$, i.e. $d = 1$. Show that inequality (10) holds[3] for $c_* = \frac{1}{2}$.

(c) The same question as in part (b), but now for $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, i.e. $d = 2$, and $c_* = \frac{1}{4}$.

[2] use spherical coordinates in $\mathbb{R}^3$.
[3] use the inequality of Cauchy-Schwarz.