Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 11

Exercise 11.1 (TWs)

Consider the Fisher equation from population dynamics ($u > 0$!):

$$u_t = u_{xx} + \gamma u(1-u), \quad -\infty < x < \infty$$

with boundary conditions $u(-\infty, t) = 1$ and $u(+\infty, t) = 0$. Determine the travelling wave ODE, analyze the phase plane and sketch the corresponding TW solution of the PDE.

Exercise 11.2 (TWs)

The same question as in exercise 11.1, but now for the non-equilibrium geohydrology model from porous media:

$$u_t = u_{xx} + [u^2]_x + \tau u_{xxt}, \quad -\infty < x < \infty, \quad \tau > 0$$

with boundary conditions $u(-\infty, t) = u_-$ and $u(+\infty, t) = u_+$ (with $u_+ > u_-$).

Exercise 11.3 (TWs)

As in exercise 11.1, but now for the Fisher PDE with density-dependent diffusion:

$$u_t = [uu_x]_x + u(1-u), \quad -\infty < x < \infty$$

with boundary conditions $u(-\infty, t) = 1$ and $u(+\infty, t) = 0$.
Assume monotonically decreasing solutions.

Exercise 11.4 (higher-order PDEs)

a) Work out a central second-order FD approximation for u_{xxxxx} on a uniform grid with step size Δx. Alternatively, you could work with the D_5-matrix from lecture 2. Plot the eigenvalues in the complex plane.
b) Same question as in a), but now for a fourth-order approximation.
c) As in part a), but now on a non-uniform grid with grid points $\{x_{i-3}, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$.

Exercise 11.5 (higher-order PDEs)

Derive a second-order central finite difference approximation for the spatial derivatives in the 'stable-unstable' PDE:

$$u_t = -p u_{xxxx} - q u_{xx}, \qquad p,\ q > 0$$

and discuss its stability properties (in a method-of-lines context). Which time-integrators do you recommend?

Exercise 11.6 (higher-order PDEs)

Consider the following PDE model which describes vibrations in a thin beam:

$$\psi_{tt} + \psi_{xxxx} = 0,$$

supplemented with appropriate boundary and initial conditions. Develop an explicit (in time) finite difference approximation using differences at $x = j\Delta x$ and $t = n\Delta t$. Show that a sufficient condition for stability is $\frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$. Sketch the computational molecule.

*Exercise 11.7 (higher-order PDEs)

Apply a Fourier stability analysis ("Von Neumann"-analysis) to the scheme

$$u_j^{n+1} = u_j^{n-1} - \mu D_0[\eta + \frac{\epsilon}{(\Delta x)^2} D_+ D_-] u_j^n, \quad \text{with } \mu = \frac{\Delta t}{\Delta x},$$

where η and ϵ are given parameters. This is a leapfrog discretization of the linearized KdV equation

$$u_t + \eta u_x + \epsilon u_{xxx} = 0.$$

Note that we have used the following notation:

$$D_0 u_j = u_{j+1} - u_{j-1},\ D_+ u_j = u_{j+1} - u_j,\ D_- u_j = u_j - u_{j-1}.$$

*Exercise 11.8 (higher-order PDEs)

Check that the Zabusky-Kruskal scheme

$$\begin{aligned} \tfrac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) + \tfrac{1}{6\Delta x}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n) \\ + \tfrac{\epsilon}{2(\Delta x)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n) = 0 \end{aligned}$$

is a consistent scheme for the KdV model

$$u_t + u u_x + \epsilon u_{xxx} = 0,$$

and that the truncation error is of the form $\mathcal{O}((\Delta t)^3) + \mathcal{O}(\Delta t(\Delta x)^2)$.