Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 12

Exercise 12.1

Given the equidistribution principle

$$[x_\xi \omega]_\xi = 0, \quad x(0) = 0, \ x(1) = 1.$$

Show that for the arclength-type monitor function $\omega = \sqrt{1 + \alpha u_x^2}$ we obtain a uniform grid distribution, if $\alpha \to 0$.
What happens with the grid distribution for the monitor function $\omega = u_x$, if the solution has flat parts $u_x \approx 0$?

Exercise 12.2

Show that the equidistribution principle in exercise 8.1 is obtained by minimizing the 'grid energy' functional $E = \frac{1}{2}\int_0^1 \omega x_\xi^2 \, d\xi$.

*Exercise 12.3

The deformation method in one space dimension can be described in terms of grid velocities:

$$\frac{\partial}{\partial t} x(\xi, t) = \frac{v(x,t)}{\omega(x,t)},$$

with $v(\xi, t) = -\int_0^\xi \frac{\partial \omega(\eta,t)}{\partial t} \, d\eta$ and ω is a normalized monitor function such that $\int_0^1 \omega(x,t) = 1, \quad \forall t \in (0,T)$. Show that, if $x_\xi \ \omega = 1$ at $t = 0$, then the grid distribution for the deformation method satisfies $x_\xi \ \omega = 1, \quad \forall t \geq 0$ (an integrated version of the equidistribution principle).

Exercise 12.4

Consider the heat equation $u_t = u_{xx}$. Apply a coordinate transformation of the form $x(\xi, \theta), \ t = \theta$ and work out the transformed PDE.
Suppose we want to use an equidistribution principle as in exercises 8.1 and 8.2. Which monitor function ω must be used to cancel the first term in the local truncation error of u_{xx} on a non-uniform grid? This is called *supra-convergence*.

Exercise 12.5

Consider the hyperbolic PDE $u_t + c(x)u_x = f(u)$. Apply a general coordinate transformation of the form

$$x = x(\xi, \theta), \quad t = \theta$$

and derive the transformed advection PDE:

$$u_\theta + \beta x_\theta = \gamma u_\xi + g.$$

Specify the functions β, γ and g. Also, give the Jacobian matrix (and its determinant) of this transformation. Which system of two PDEs would define the transformation for the '*method of characteristics*'?

Exercise 12.6

(a) Work out a general *non-uniform grid* approximation for u_x at the grid point x_i, only making use of the values u_{i-1}, u_i and u_{i+1} at x_{i-1}, x_i and x_{i+1}, respectively. Show at least three different approximations $u_x|_i \approx Au_{i-1}+Bu_i+Cu_{i+1}$.

(b) Derive the central difference approximation (as a special case from part ii): $u_x|_i \approx \frac{u_{i+1}-u_{i-1}}{x_{i+1}-x_{i-1}}$, work out its local truncation error τ in terms of the transformation derivatives x_ξ, $x_{\xi\xi}$, ... and solution derivatives u_{xx}, u_{xxx}, ...:

$$\tau = \epsilon H^2 + \mathcal{O}(H^3),$$

where $H := \Delta\xi$, the constant stepsize in the transformed variable ξ (find the factor ϵ).

(c) Considering only u_x as in part (b): which monitor function ω of the form $[\frac{\partial^p u}{\partial x^p}]^q$ in the equidistribution relation

$$[\omega x_\xi]_\xi = 0$$

yields '*supra-convergence*' on the non-uniform grid x_i? In other words, set $\epsilon = 0$ in part (b) and find the appropriate ω, for which $\tau = \mathcal{O}(H^2) \Rightarrow \tau = \mathcal{O}(H^3)$.

Exercise 12.7

Consider the time-dependent grid transformation

$$\begin{aligned} x &= x(\xi, \eta, \theta), \\ y &= y(\xi, \eta, \theta), \\ t &= \theta. \end{aligned} \tag{1}$$

Show that the determinant of this transformation is given by:

$$J := \det(\mathcal{J}) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}.$$

*Exercise 12.8

(a) Work out the two-dimensional Laplacian Δ in 'curvilinear' coordinates

$$(\xi, \eta) = (\xi(x, y), \eta(x, y)).$$

(b) Check with the formula in part (a) that, if we choose $(\xi, \eta) = (\rho, \phi)$ with

$$x = \rho \sin(\phi),$$
$$y = \rho \cos(\phi),$$

we obtain the well-known formula in polar coordinates:

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}.$$

*Exercise 12.9

The deformation method in d space dimensions can be described in terms of grid velocities:

$$\frac{\partial \vec{x}}{\partial t}(\vec{\xi}, t) = \vec{v}(\vec{x}, t) f(\vec{x}, t), \ \ t > 0, \quad \vec{x}(\vec{\xi}, 0) = \vec{x}_0(\vec{\xi}), \tag{2}$$

where the velocity field $\vec{v}$ satisfies

$$\nabla_{\vec{\zeta}} \cdot \vec{v}(\vec{\zeta}, t) = -\frac{\partial}{\partial t}[\frac{1}{f(\vec{\zeta}, t)}] \tag{3}$$

and

$$\int_0^1 \int_0^1 ... \int_0^1 \frac{1}{f(\vec{x}, t)} \ dx_1 dx_2 ... dx_d = 1, \quad \forall t \in (0, T).$$

Suppose that at $t = 0$ we have: $J(\vec{x}_0(\vec{\xi}), 0) = f(\vec{x}_0(\vec{\xi}), 0)$, where J is the determinant of the Jacobian matrix of the grid transformation in d space dimensions and $f = \frac{1}{\omega}$ (remember that ω is the monitor function which determines the adaptivity of the method.
Prove that for the time-dependent adaptive grid obtained from formulas (2) and (3) the following relation holds:

$$J(\vec{x}, t) = f(\vec{x}, t) \quad \forall t \geq 0.$$

Hints: 1) prove that $\mathcal{H} = \frac{J}{f}$ is independent of t, 2) consult also the proof of the 1d-version, 3) it is useful to make use of the following theorem (Abel-Jacobi-Liouville theorem):
Let A be a $d \times d$-matrix with continuous elements on an interval I: $a \leq t \leq b$ and suppose Φ is a matrix satisfying the matrix differential equation: $\Phi'(t) = A(t)\Phi(t)$, $t \in I$. Then $det(\Phi)$ satisfies on I the first order differential equation: $(det(\Phi))' = trace(A) det(\Phi)$.

*Exercise 12.10

Derive the moving finite element ODEs for piecewise linear approximations via a least squares minimization procedure. Comment on the regularity of the extended mass-matrix and of the right-handside vector.