Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 2

Exercise 2.1

Use the *method of undetermined coefficients* to obtain a uniform grid FD approximation of

(a) u_x at $x = x_i$ of the form

$$u_{x,i} \approx [-\frac{1}{6}u_{i+2} + u_{i+1} - \frac{1}{2}u_i - \frac{1}{3}u_{i-1}]/\Delta x.$$

Show that the leading term in the error is $\mathcal{O}((\Delta x)^3)$.

(b) u_{xx} at $x = x_i$ of the form

$$u_{xx,i} \approx [u_{i-1} - u_i - u_{i+1} + u_{i+2}]/[2(\Delta x)^2].$$

Show that the leading term in the error is $\mathcal{O}(\Delta x)$.

Exercise 2.2

Construct, for the first-derivative operator u_x at the gridpoint $x = x_i$, an approximation of the form:

$$u_{x,i} \approx Au_{i+1} + Bu_i + Cu_{i-1}.$$

Which order of approximation do you find? How many solutions exist? Discuss a few well-known choices.

Exercise 2.3

Derive a *fourth*-order approximation of u_{xx} at $x = x_i$:

$$u_{xx,i} \approx Au_{i+2} + Bu_{i+1} + Cu_i + Du_{i-1} + Eu_{i-2}$$

and determine the constants A, B, C, D and E. What happens at the boundary grid points of the domain? Discuss the effects and how to include boundary conditions.

Exercise 2.4

(a) Show that the eigenvalues of the (tri-diagonal) $M \times M$-matrix $\mathcal{D}_{2c}$ (with *homogeneous Dirichlet* boundary conditions), given by

$$\mathcal{D}_{2c} = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & 0 & & & & & \\ 1 & -2 & 1 & 0 & & & \emptyset & \\ 0 & 1 & -2 & 1 & 0 & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & & \\ & & & 0 & 1 & -2 & 1 & 0 \\ & \emptyset & & & 0 & 1 & -2 & 1 \\ & & & & & 0 & 1 & -2 \end{pmatrix},$$

are:

$$\lambda_p = \frac{2}{(\Delta x)^2}(\cos(p\pi\Delta x) - 1), \quad \text{for } p = 1, 2, ..., M.$$

The eigenvector v^p corresponding to eigenvalue λ_p has components:

$$v_i^p = \sin(p\pi i\Delta x), \quad i = 1, 2, ..., M.$$

(b) Choose $M = 10$ and plot the eigenvalues in the complex plane. The same question for $M = 20$. What do you expect for $M \gg 1$?

*Exercise 2.5

(a) Show that the eigenvalues of the (circulant) $(M+1) \times (M+1)$-matrix $\mathcal{D}_{1c}$ (with *periodic* boundary conditions), given by

$$\mathcal{D}_{1c} = \frac{1}{2\Delta x} \begin{pmatrix} 0 & 1 & 0 & & & & & -1 \\ -1 & 0 & 1 & 0 & & & \emptyset & \\ 0 & -1 & 0 & 1 & 0 & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & & \\ & & & 0 & -1 & 0 & 1 & 0 \\ & \emptyset & & & 0 & -1 & 0 & 1 \\ 1 & & & & & 0 & -1 & 0 \end{pmatrix},$$

are:

$$\lambda_p = \frac{i}{\Delta x}\sin(2\pi p\Delta x), \text{ for } p = 1, ..., M + 1.$$

Note that the i-th component of the p-th eigenvector has the form:

$$v_i^p = \mathrm{e}^{\mathbf{i}2p\pi i\Delta x} \quad (\mathbf{i} = \sqrt{-1}).$$

(b) Choose $M = 10$ and plot the eigenvalues in the complex plane. The same question for $M = 20$. What do you expect for $M \gg 1$?

Exercise 2.6

(a) Check that the following two identities hold:

$$\mathcal{D}_{2c} = \mathcal{D}_{1+}\mathcal{D}_{1-} \quad \& \quad \mathcal{D}_{1c} = \frac{1}{2}(\mathcal{D}_{1+} + \mathcal{D}_{1-}).$$

(b) Work out the matrix: $\mathcal{D}_{4c} = (\mathcal{D}_{2c})^2$.
(the matrix $\mathcal{D}_{4c}$ approximates $u_{4x} := u_{xxxx}$)

(c) Derive a finite difference matrix $\mathcal{D}_{6c}$ for $u_{6x} := u_{xxxxxx}$.

Exercise 2.7

Consider the one-dimensional stationary convection-diffusion model:

$$\begin{cases} \epsilon\ u''(x) - u'(x) = 0,\ x \in [0,1],\ 0 < \epsilon < 1, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{1}$$

We approximate (1) by

$$\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - u_{x,i} = 0, \quad i = 1, ..., M-1, \tag{2}$$

where $u_0 = 0, u_M = 1$ and $u_{x,i}$ is an approximation for $u'(x_i)$.

(a) Find the analytical solution of model (1). Sketch a few solutions for decreasing values of ϵ.

(b) Use central differences for $u_{x,i}$ on a three-point stencil. Check that the exact solution of the (linear) numerical system (2) satisfies:

$$u_i = \frac{(\frac{1+P_e}{1-P_e})^i - 1}{(\frac{1+P_e}{1-P_e})^M - 1}, \quad i = 1, ..., M-1,$$

where the mesh-Péclet number is defined by: $P_e = \frac{\Delta x}{2\epsilon}$. Sketch a few numerical solutions for $P_e > 1$ and $P_e < 1$ (choose M not too large).

(c) Next, use first-order upwind (backward finite differences) for $u_{x,i}$. Check that the exact solution of the (linear) numerical system (2) now becomes:

$$u_i = \frac{(1 + 2P_e)^i - 1}{(1 + 2P_e)^M - 1}, \quad i = 1, ..., M-1.$$

Choose the same values for M and ϵ as in part (b) and compare both types of numerical solutions. Discuss the influence of P_e.

*Exercise 2.8

Consider the one-dimensional *non*linear Gelfand-Bratu model:

$$\begin{cases} u''(x) + \lambda \mathrm{e}^{u(x)} = 0, \ x \in [0,1], \\ u(0) = 0, \quad u(1) = 0, \ \lambda \in \mathbb{R}. \end{cases} \tag{3}$$

We approximate (3) by

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \lambda \mathrm{e}^{u_i} = 0, \quad i = 1, ..., M-1, \tag{4}$$

with $u_0 = 0$ and $u_M = 0$.

(a) Check that, for $\lambda \geq 0$, the exact solution of (3) satisfies:

$$\begin{cases} u(x) = -2\ln[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}], \\ \theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4}). \end{cases}$$

Draw the *two*(!) solutions for $\lambda = 1$.

(b) Apply the Matlab-routine *fsolve.m*, with an appropriate initial guess and $M = 10$, to the nonlinear system (4). Reproduce the two solutions for $\lambda = 1$ numerically.

*Exercise 2.9

Derive a *non*-uniform grid approximation at $x = x_i$ for both $u_{x,i}$ and $u_{xx,i}$ on a three-point stencil $\{x_{i-1}, x_i, x_{i+1}\}$. Express the truncation error in terms of derivatives of u at x_i and the non-uniform grid variables $\{\Delta x_{i-1}, \Delta x_i, \Delta x_{i+1}\}$.

Exercise 2.10

In two dimensions the Laplacian Δu can be approximated at the gridpoint $(x,y) = (x_{i,j}, y_{i,j})$ on a *nine-point* stencil (take $\Delta x = \Delta y := h$):

$$\begin{aligned} \Delta u_{i,j} \approx \tfrac{1}{6h^2}[4(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) + \\ u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{i,j}]. \end{aligned}$$

Check that this approximation is of second order (similar to the five point central approximation), but becomes *fourth* order, when the Laplace equation $\Delta u = 0$ is solved, or when the Poisson equation $\Delta u = f(x,y)$ is solved with a harmonic function f, i.e., $\Delta f = 0$.