Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 3

Exercise 3.1

Show that

(a) the implicit Gear method

$$u^{n+1} = \frac{1}{3}(4u^n - u^{n-1}) + \frac{2\Delta t}{3} f(u^{n+1})$$

is *zero stable*,

(b) the explicit 3-step Adams method

$$u^{n+3} = u^{n+2} + \frac{\Delta t}{12}[5f(u^n) - 16f(u^{n+1}) + 23f(u^{n+2})]$$

is *zero stable*,

(c) and that the linear multistep method

$$u^{n+2} - 3u^{n+1} + 2u^n = -\Delta t f(u^n)$$

is <u>not</u> *zero stable*.

Exercise 3.2

Find the *stability polynomial* $\pi(\zeta; z)$ and its roots for the

(a) trapezoidal method:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}[f(u^n) + f(u^{n+1})].$$

(b) the midpoint (leapfrog) method:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = f(u^n).$$

Exercise 3.3

Sketch/plot[1] *the stability regions* in the complex plane of the following methods:

(a) *a* second-order explicit Runge-Kutta (two-stages) of the form:

$$\left\{ \begin{array}{l} k_1 = f(u^n) \\ k_2 = f(u^n + k_1 \Delta t), \\ u^{n+1} = u^n + \frac{\Delta t}{2}(k_1 + k_2). \end{array} \right\}$$

(b) *a* fourth-order explicit Runge-Kutta (four stages):

$$\left\{ \begin{array}{l} k_1 = f(u^n), \\ k_2 = f(u^n + k_1 \frac{\Delta t}{2}), \\ k_3 = f(u^n + k_2 \frac{\Delta t}{2}), \\ k_4 = f(u^n + k_3 \Delta t), \\ u^{n+1} = u^n + \frac{\Delta t}{6}[k_1 + 2k_2 + 2k_3 + k_4]. \end{array} \right\}$$

(c) *the* second-order Taylor method:

$$u^{n+1} = u^n + \Delta t f(u^n) + \frac{1}{2}(\Delta t)^2 f'(u^n) f(u^n).$$

Exercise 3.4

Show that the *boundary locus* of the method

$$u^{n+1} = u^n + \Delta t f(u^n + \Delta t f(u^n))$$

is defined by

$$[1 + x + x^2 - y^2]^2 + y^2[1 + 2x]^2 = 1.$$

Plot this curve, as described in Exercise 3.3.

Exercise 3.5

(a) Deduce the eigenvalues of the central finite-difference matrices $\mathcal{D}_{1c}$, $\mathcal{D}_{2c}$, $\mathcal{D}_{3c}$, $\mathcal{D}_{4c}$ and $\mathcal{D}_{6c}$.[2]

(b) Mark their positions in the complex plane.

(c) Comment on the stability properties of the methods *EF* (explicit Euler) and *EB* (implicit Euler), when applied to the semi-discrete ODE system:

$$\dot{\vec{u}} = \mathcal{D}_{mc} \vec{u}, \quad (m = 1, 2, 3, 4, 6).$$

[1] write $z = x + iy$ and make use of the Newton-Raphson method to find an approximation of the *boundary locus* for a finite set of (x, y)-values.

[2] You may assume that the eigenvectors of these matrices are identical.

Exercise 3.6

Work out the semi-discrete ODE system (applying the Method-of-Lines) for the following PDE models:

(a) $u_t = du_{xx} + (u^2)_x - \mu u_{xxt}$.

(b) $u_t = u_{xxx} + 6uu_x$.

(c) $u_{tt} = u_{xxx}$ with $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$.[3] Where in the complex plane do we find the eigenvalues of the semi-discrete ODE system? Comment on the applicability of explicit Euler, implicit Euler and other time-integration methods.

[3] first re-write the PDE as a system of two PDEs.