Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 4

Consider the heat equation:

$$u_t = u_{xx}. \tag{1}$$

Exercise 4.1

Derive the Crank-Nicolson method for the heat equation (1) at the gridpoint (x_i, t^n):

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2(\Delta x)^2}.$$

Exercise 4.2

Derive the DuFort-Frankel scheme and its error term for the heat equation (1).

Exercise 4.3

Sketch the computational stencil for the Crank-Nicolson method, the Leapfrog method and the DuFort-Frankel scheme, when applied to (1).

Exercise 4.4

Show that, for the particular choice of the stepsizes $\Delta t = \frac{(\Delta x)^2}{\sqrt{12}}$, the DuFort-Frankel scheme is second-order in time and fourth-order accurate in space (hint: expand the terms more carefully and use a relation between u_{tt} and u_{xxxx}).

Exercise 4.5

Show that the local truncation error τ for the Crank-Nicolson method is second-order in space and time, i.e. $\tau = \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2)$ and compute the dominant term in τ.

Exercise 4.6

Suppose that an explicit (in time) finite difference method is used to approximate the heat equation. It can be written in the form:

$$\mathbf{u^{n+1}} = \mathcal{B}\mathbf{u^n} + \mathbf{b^n},$$

with an $(M-1) \times (M-1)$-matrix $\mathcal{B}$ and righthand-side vector $\mathbf{b}^n$ of length $M-1$. If we apply the finite difference equations to the exact solution u_* we can write:

$$\mathbf{u}_*^{n+1} = \mathcal{B}\mathbf{u}_*^n + \mathbf{b}^n + \Delta t \; \tau^n,$$

where τ^n denotes the vector of local truncation errors in each grid point x_i at time level t^n. Define the global error at time $t = t^n$ by $\mathbf{e}^n = \mathbf{u}_*^n - \mathbf{u}^n$.
Show that, if $\mathcal{B}^n$ (now as a power in n) is bounded for all Δt and indices n with $n\Delta t \leq T_{end}$ ('stability'), and if the method is consistent ($\tau^n \to 0, \;$ for $\Delta t \downarrow 0$), then the finite difference method is convergent: $\lim_{n\to\infty} \mathbf{e}^n = \mathbf{0}$.

Exercise 4.7

Show, using the Von Neumann-stability analysis, that the Crank-Nicolson method applied to the heat equation (1) is *un*conditionally stable.

Exercise 4.8

The same question as in exercise 4.7, but now for the DuFort-Frankel scheme applied to the heat equation (1).