Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 5

Consider the advection equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{1}$$

Exercise 5.1

Show that for the CTCS-method ('Leapfrog')

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

the local truncation error is of the form

$$\tau = -\frac{1}{6}(\Delta t)^2 u_{ttt}|_i^n - \frac{c}{6}(\Delta x)^2 u_{xxx}|_i^n + \text{H.O.T. in } \Delta t \text{ and } \Delta x.$$

Exercise 5.2

Compute the local truncation error for the Lax-Friedrichs method when applied to advection equation (1).

Exercise 5.3

Use the Von Neumann stability analysis to discuss the (in)stability of the Leapfrog method for PDE (1).

Exercise 5.4

Find a modified PDE for which the Lax-Wendroff method

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{(\Delta t)^2}{2(\Delta x)^2}c^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

applied to PDE (1) gives an $\mathcal{O}((\Delta t)^3)$ approximation.

Exercise 5.5

Compute the local truncation error for the Beam-Warming method when applied to advection equation (1):

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(3u_i^n - 4u_{i-1}^n + u_{i-2}^n) + \frac{1}{2}c^2(\frac{\Delta t}{\Delta x})^2(u_i^n - 2u_{i-1}^n + u_{i-2}^n).$$

Exercise 5.6

Show that the Beam-Warming method in exercise 5.5 is stable for $0 \leq c\frac{\Delta t}{\Delta x} \leq 2,$ if we assume that $c > 0$.

Exercise 5.7

In this exercise we apply the upwind method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

to advection equation (1). Show that the amplification factor in the Von Neumann stability analysis satisfies

$$|G| = |(1-\lambda) + \lambda \mathrm{e}^{-i\xi_m \Delta x}|,$$

with $\lambda = c\frac{\Delta t}{\Delta x}$. For which values of λ is this method stable?

Exercise 5.8

Apply the FTCS-scheme to the advection equation (1) and show that, for this case, the CFL-condition may give stable solutions (conditionally), but we know that the FTCS-solutions are *un*stable. This is an example of a numerical method for which the CFL-condition is *not sufficient* (although it is a *necessary* condition)! Illustrate this in a figure with computational stencils in the $x-t$-domain and with characteristics of PDE (1).

Exercise 5.9

Draw the domains of numerical and physical dependence for the FTBS and FTFS schemes applied to the linear advection equation (1).