Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 6

Consider the kinematic wave equation:

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty,\ t > 0 \tag{1}$$

with initial condition $u_0(x)$.

*Exercise 6.1

(a) Using the method of characteristics, show that the solution of (1) can be written as:

$$\begin{cases} u(x,t) = u_0(\xi), \\ \xi = x - tU_0(\xi), \\ U_0(\xi) = c(u_0(\xi)). \end{cases}$$

It can be shown (you do not need to prove this) that PDE (1) has a unique solution provided that

$$\{1 + tU_0'(\xi)\} \neq 0$$

and when u_0 and c are sufficiently smooth.

(b) Show that both u_x and u_t tend to infinity as $1 + tU_0'(\xi) \to 0$. Thus, on any characteristic for which $U_0(\xi) < 0$, a discontinuity occurs at time t given by

$$t = -\frac{1}{U_0'(\xi)},$$

which is positive because $U_0'(\xi) = c'(u_0)u_0'(\xi) < 0$.

(c) Check that the time $t = \tau$ when the solution first develops a discontinuity (singularity) for some value of ξ is given by:

$$\tau = -\frac{1}{\min_{-\infty<\xi<\infty}\{c'(u_0)u_0'(\xi)\}} > 0.$$

When $1 + tU_0'(\xi) = 0$, the solution develops a discontinuity known as a shock.

(d) Consider the special case $c(u) = u$ and

$$u_0(x) = \begin{cases} 1 - x^2, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

Solve this problem, sketch the characteristic curves in the (x,t)-plane and the solutions $u(x,t)$ as a function of x at $t = 0, 1, 2$.

Exercise 6.2

Show that for the nonlinear hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial [F(u)]}{\partial x} = 0 \tag{2}$$

the following property holds:

$$\int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty} u(x,0)\ dx \quad \forall t \geq 0,$$

if we assume that $\lim_{x\to\pm\infty} F(u(x,t)) = 0, \quad \forall t \geq 0$. When we apply a *finite-volume method* to equation (2), we write the approximation in flux-differencing form:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1}^n - F_i^n).$$

Show that the following discrete version of the conservation property holds:

$$\Delta x \sum_{i=I}^{J} u_i^{n+1} = \Delta x \sum_{i=I}^{J} u_i^n - \Delta t(F_{J+1}^n - F_I^n),$$

for all choices of indices I and $J > I$.

Exercise 6.3

Show that, taking $c(u) = u$ in PDE (1) (Burgers' equation), a slightly modified version of the upwind method

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

is also consistent with the two PDEs

$$u_t + (\tfrac{u^2}{2})_x = 0,$$

$$(u^2)_t + (\tfrac{2u^3}{3})_x = 0.$$

Next, consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{3}$$

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Exercise 6.4

Show, by using using d'Alembert's formula, that the solution $u(x,t) = u_0(x-ct)$ of the linear advection PDE

$$u_t + cu_x = 0, \quad u(x,0) = u_0(x), \quad -\infty < x < \infty, \ t > 0$$

is also a solution of the wave equation (3) with the special initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \\ u_t(x,0) &= u_0'(x). \end{aligned}$$

Describe the system of ODEs and its properties, after applying the first step in the Method-of-Lines to the wave equation (3).

Exercise 6.5

Assume that $u_i^0 = u(x_i,0)$ and $u_i^1(x_i,\Delta t)$ $\forall i$ are the exact solutions of PDE (3) at the grid points x_i. Prove that, for the special choice $\Delta t = \frac{\Delta x}{c}$, the FD solutions u_i^n of the method CTCS, satisfy the recursion

$$u_i^{n+1} = u_{i+1}^n + u_{i-1}^n - u_i^{n-1},$$

and that they are the exact solution values to the PDE at (x_i, t^n) (neglecting computer round off errors).

Exercise 6.6

Work out the Von Neumann stability analysis for the wave equation with the CTCS-scheme.

Exercise 6.7

Consider the following FD method, applied to PDE (3) with $c > 0$:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 [\frac{-u_{i+2}^n + 16u_{i+1}^n - 30u_i^n + 16u_i^{n-1} - u_{i-2}^n}{12(\Delta x)^2}]$$

Use the Von Neumann stability analysis to check that the scheme is stable, if and only if

$$c\frac{\Delta t}{\Delta x} \leq \frac{\sqrt{3}}{2}.$$

What is the accuracy of this FD scheme?

Exercise 6.8

Describe a central second-order FD scheme for the Euler-Bernoulli equation with $b > 0$:

$$u_{tt} = -b^2 u_{xxxx}.$$

This PDE models the vertical motion of a thin horizontal beam with small displacements from the rest position. Show that for stability we must have

$$b\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$