Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 7

Exercise 7.1

Consider the logistic ODE model:

$$\dot{u} = u - u^2$$

with initial condition $u(0) = u^0$. First, check that the exact solution satisfies:

$$u(t) = \frac{u^0}{u^0 + (1 - u^0)\mathrm{e}^{-t}}.$$

Show that we obtain, from this expression, the following *exact* finite-difference scheme:

$$\frac{u^{n+1} - u^n}{[1 - \mathrm{e}^{-\Delta t}]} = u^{n+1}(1 - u^n).$$

Exercise 7.2

Consider the nonlinear PDE:

$$u_t + u_x = u(1 - u)$$

with initial condition $u(x, 0) = f(x)$. Check that the exact solution satisfies:

$$u(x, t) = \frac{f(x - t)}{\mathrm{e}^{-t} + (1 - \mathrm{e}^{-t})f(x - t)}.$$

Derive the *exact* (explicit!) finite difference scheme:

$$u^n_{i+1} = \frac{u^n_{i-1}}{1 + (\mathrm{e}^{\Delta t} - 1)u^n_{i-1}}.$$

Exercise 7.3

Consider the nonlinear ODE model:

$$\dot{u} = u^2 - u^3$$

with initial condition $u(t^0) = u^0$. Derive the *nonstandard* finite-difference scheme:

$$u^{n+1} = \frac{(1 + 2\phi(\Delta t)u^n)u^n}{1 + \phi(\Delta t)(u^n + (u^n)^2)}.$$

Which function $\phi(\Delta t)$ would be a good choice?

Exercise 7.4

Consider Fisher's PDE

$$u_t = u_{xx} + u(1-u).$$

The solution $u(x,t)$ satisfies "the boundedness condition":

$$0 \le u(x,0) \le 1 \Rightarrow 0 \le u(x,t) \le 1, \ \forall t > 0.$$

Show that the *non-standard* finite-difference scheme with the *nonlocal* approximation[1]

$$2\bar{u}_i^n - u_i^{n+1} - \bar{u}_i^n u_i^{n+1}$$

for the reaction term yields:

$$0 \le u_i^0 \le 1 \Rightarrow 0 \le u_i^n \le 1, \quad \forall n \ge 1, \ \forall \text{ relevant } i.$$

Use the standard FT and CS approximations for u_t and u_{xx}, respectively. It is convenient to first work out an explicit expression $u_i^{n+1} = ...$ (do this for $\frac{\Delta t}{(\Delta x)^2} = \frac{1}{2}$).

Exercise 7.5

(a) Check that the *Leapfrog* method

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \sqrt{u^n}, \quad u^0 = 1, \ u^1 = \frac{1}{4}(\Delta t)^2 + \Delta t + 1$$

is an *exact* finite difference (FD) scheme for: $\dot{u}(t) = \sqrt{u(t)}$ with $u(0) = 1$.

(b) Give two important ingredients of a *nonstandard* FD scheme, when compared to a standard FD scheme.

Exercise 7.6

Verify that the scheme:

$$\left\{ \begin{array}{l} \frac{u^{n+1} - u^n}{e^{\pi \Delta t} - 1} = u^n, \quad n = 0, 1, 2, ...; \Delta t > 0, \\ \\ u^0 = 1, \\ \\ u^n \approx u(t^n) = u(n\Delta t), \end{array} \right\}$$

is an *exact* finite difference (FD) scheme for the ODE:

$$\left\{ \begin{array}{l} \dot{u}(t) = \pi \ u(t), \\ u(0) = 1. \end{array} \right\}$$

[1] $\bar{u}_i^n := \frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}$.

Exercise 7.7

Show that the local truncation error ρ_n for the *first*-order *splitting* method

$$w(t^{n+1}) = \mathrm{e}^{\tau A_2}\mathrm{e}^{\tau A_1} w^n$$

with $\tau := \Delta t$ and $A = A_1 + A_2$ for the linear ODE

$$w'(t) = Aw(t), w(0) = w^0$$

satisfies:

$$\rho_n = \frac{\tau}{2}[A_1, A_2]w(t^n) + \mathcal{O}(\tau^2),$$

where $[*, *]$ denotes the *commutator* of A_1 and A_2.

Exercise 7.8

Show that the symmetric *splitting* method ("Strang-splitting")

$$w^{n+1} = \mathrm{e}^{\frac{1}{2}\tau A_1}\mathrm{e}^{\tau A_2}\mathrm{e}^{\frac{1}{2}\tau A_1} w^n$$

has consistency order *two*. Work out the term in front of τ^2, where $\tau := \Delta t$.

Exercise 7.9

Work out:

$$[A_2, [A_2, A_1] + [A_1, [A_1, A_2]]]$$

and

$$[A_2, [A_1, [A_1, A_2]]].$$

Check that, if the matrices A_1 and A_2 commute, that then all higher-order terms in the Baker-Campbell-Hausdorff formula[2] vanish. And, that we obtain, in that case: $\tilde{A} = A = A_1 + A_2$, where $\tau = \Delta t$ and

$$\mathrm{e}^{\tau A_2}\mathrm{e}^{\tau A_1} = \mathrm{e}^{\tau \tilde{A}}.$$

*Exercise 7.10

Consider the nonlinear PDE

$$u_t = \mathcal{L}u + \mathcal{N}(u, t).$$

Derive the ETD-Euler method [3]. Which function ϕ plays a role in this method?

[2] see lecture notes.
[3] ETD stands for Exponential-Time-Differencing.

*Exercise 7.11

Give a few other choices for the function ϕ in the ETD-method. You may use the recurrence relation for these functions from the lecture notes.