Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 8

Exercise 8.1[1]

(a) Apply Euler-Forward (EF) to the mathematical pendulum for small angles ($\sin(x) \approx x$). Show that the numerical energy of the system increases in time.

(b) Same question for Euler-Backward. Now, show that the numerical energy of the system decreases in time.

(c) Same question for the Implicit Midpoint method. What happens with the numerical energy?

(d) And also for the *symplectic* Euler method (the one from the lecture notes).

Exercise 8.2

Consider the linearized pendulum ODE system:

$$\begin{cases} x'(t) &= y(t), \\ y'(t) &= -x(t). \end{cases} \tag{1}$$

Check that the exact FD-scheme

$$\begin{cases} \frac{x^{n+1}-x^n}{\phi} &= \frac{y^{n+1}+y^n}{2}, \\ \frac{y^{n+1}-y^n}{\phi} &= -\frac{x^{n+1}+x^n}{2}, \end{cases} \tag{2}$$

with $\phi(\Delta t) = \frac{2(1-\cos(\Delta t))}{\Delta t \sin(\Delta t)}$ indeed conserves the energy of the system:

$$\mathcal{H}^{n+1} = \mathcal{H}^n, \quad \forall n \geq 0.$$

Exercise 8.3

Show that $E(p,q) = (q + p^2 - \frac{1}{2})\, \mathrm{e}^{2q}$ is a *constant of motion*[2] for the ODE system:

$$\left\{ \begin{array}{l} \dot{q} = p, \\ \dot{p} = -q - p^2. \end{array} \right\}$$

[1] See the first pages in the lecture notes of Lecture 8 for more details.
[2] $E(p(t), q(t)) = E(p(0), q(0)) = \text{ constant}, \quad \forall t \geq 0.$

Exercise 8.4

Consider the orbit of a small asteroid of mass m around a large star of mass M centered at the origin, where $M \gg m$. Let (p_1, p_2) and (q_1, q_2) be the momentum and position of the asteroid in the plane of motion. The Hamiltonian for this system is:

$$H(p,q) = \frac{p_1^2 + p_2^2}{2m} - \frac{GMm}{\sqrt{q_1^2 + q_2^2}},$$

where G is the gravitational constant. By rescaling length and time, the constants GM and m can be eliminated to obtain:

$$H(p,q) = \frac{p_1^2 + p_2^2}{2} - \frac{1}{\sqrt{q_1^2 + q_2^2}}.$$

(a) Derive the four governing differential equations[3] for p_1, p_2, q_1 and q_2.

(b) Show that H is constant for all $t \geq 0$.

Exercise 8.5

The area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by:

$$A_\Delta = \frac{1}{2} \mid \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \mid . \tag{3}$$

(a) Verify, using property (3), that the area $dA_\square$ of a rectangle with the three corners (x, y), $(x + dx, y)$ and $(x, y + dy)$ reads:

$$dA_\square = dx\, dy. \quad \text{(as expected)}$$

(b) Check that, for the dynamical system

$$\left\{ \begin{array}{ll} \dot{x} = y, & x(0) = x_0, \\ \dot{y} = -x, & y(0) = y_0, \end{array} \right\} \tag{4}$$

the area $dA_\diamond$ of a small parallelogram defined by the corners (x_1, y_1), (x_2, y_2) and (x_3, y_3) satisfies:

$$\frac{\mathrm{d}A_\diamond}{\mathrm{d}t} = 0 \quad \text{("the mapping is area preserving").}$$

[3] In the early 1600's, Johannes Kepler showed that the solutions are ellipses. More generally, the solutions could be any *conic* section, including parabolae and hyperbolae.

(c) Consider Euler-Forward (EF) with step size Δt, applied to system (4). For the three points (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) we find the mapping:

$$\left\{ \begin{array}{ll} X_r \to X_r + \Delta t Y_r, & r = 1, 2, 3, \\ Y_r \to Y_r - \Delta t X_r, & r = 1, 2, 3. \end{array} \right\}$$

Algebra can be simplified by choosing the points (X_1, Y_1), $(X_1 + h, Y_1)$ and $(X_1, Y_1 + k)$ and denoting the initial element of area as $A_0 = h\,k$.
Show that the vertices, changed according to EF, give the area:

$$A_1 = A_1(\Delta t) = (1 + (\Delta t)^2)\; h\; k.$$

This shows that EF will not preserve area and elements of area will increase exponentially (EF is not a "symplectic" method).

(d) Show that the *symplectic* ("hybrid Euler") method:

$$\left\{ \begin{array}{l} X_{n+1} = X_n + \Delta t Y_n \\ Y_{n+1} = Y_n - \Delta t X_{n+1}, \quad n = 0, 1, 2, ... \end{array} \right\}$$

does preserve area!

(e) Show that another *symplectic* method:

$$\left\{ \begin{array}{l} X_{n+1} = X_n + \Delta t Y_{n+1} \\ Y_{n+1} = Y_n - \Delta t X_n, \quad n = 0, 1, 2, ... \end{array} \right\}$$

preserves a *modified* Hamiltonian $\hat{H}(x, y)$. Which one?
(you may assume that $\hat{H}$ is a homogeneous second order polynomial[4] in x and y).

Exercise 8.6

(a) Calculate the variational derivatives of the functionals

$$\mathcal{H}_1(u, v) = \frac{1}{2} \int_{-\infty}^{\infty} [u_x^2 + v^2]\, \mathrm{d}x,$$

and

$$\mathcal{H}_2(p, q) = \int_{-\infty}^{\infty} [\frac{1}{2}p^2 + \frac{1}{2}q_x^2 + V(q)]\, \mathrm{d}x.$$

(b) Making use of part (a), show that both the *linear* wave equation $u_{tt} = u_{xx}$ and the *nonlinear* wave equation $u_{tt} - u_{xx} + V'(u) = 0$ are Hamiltonian PDEs.

[4] A homogeneous polynomial, sometimes called quantic in older texts, is a polynomial whose nonzero terms all have the same degree.

Exercise 8.7

(a) Calculate the variational derivative of the functional

$$\mathcal{H}_3 = \int_{-\infty}^{\infty} [\frac{1}{3}u^3 - 3u_x^2 + \frac{1}{15}u_{xx}^2 + \frac{1}{2}u\, u_x^2]\, \mathrm{d}x.$$

(b) Making use of part (a), show that the extended fifth-order KdV-equation

$$u_t + \frac{2}{15}u_{xxxxx} + u\, u_{xxx} + 3u\, u_x + 2u_{xx}u_x = 0$$

is a Hamiltonian PDE.

(c) Verify that $\mathcal{I}_1 = \int_{-\infty}^{\infty} u\, \mathrm{d}x$ and $\mathcal{I}_2 = \int_{-\infty}^{\infty} u^2\, \mathrm{d}x$ are also preserved.

Exercise 8.8

Consider the beam equation:

$$\left\{ \begin{array}{l} u_{tt} + u_{xxxx} + f(x,u) = 0, \\ \\ u(0,t) = u(\pi,t) = 0, \\ u_{xx}(0,t) = u_{xx}(\pi,t) = 0. \end{array} \right\}$$

Show that this is a Hamiltonian PDE with:

$$\mathcal{H}(u,p) = \int_0^{\pi} [\frac{p^2}{2} + \frac{u_{xx}^2}{2} + F(x,u)]\, \mathrm{d}x,$$

and

$$J = \begin{pmatrix} \mathcal{O} & \mathcal{I} \\ -\mathcal{I} & \mathcal{O} \end{pmatrix}$$

with $\frac{\partial F(x,u)}{\partial u} = f(x,u)$.

Exercise 8.9

Consider the Korteweg-de Vries (KdV) equation:

$$u_t - 6\, u\, u_x + u_{xxx} = 0$$

with $u(x,t) \to 0$ as $|x| \to +\infty$.
Show that this is a Hamiltonian PDE with:

$$\mathcal{H}(u) = \int_{-\infty}^{\infty} [\frac{u_x^2}{2} + u^3]\, \mathrm{d}x,$$

and

$$J = \frac{\partial}{\partial x}.$$

Exercise 8.10

Consider the nonlinear Schroedinger (NLS) equation:

$$\mathrm{i}\,\psi_t = -\frac{\partial^2 \psi}{\partial x^2} - 2\,|\psi|^2 \psi, \quad x \in [0, L] \tag{5}$$

with $\psi(x,t) = q(x,t) + \mathrm{i}\,p(x,t)$ and periodic boundary conditions. The NLS is an example of an *integrable* PDE. It possesses a Hamiltonian structure and an *infinite* number of invariant integrals.

(a) Demonstrate that PDE (5) is a Hamiltonian PDE with

$$\mathcal{H} = \int_0^L [\frac{1}{2}|\psi_x|^2 - \frac{1}{2}|\psi|^4]\,\mathrm{d}x.$$

What is $\mathcal{J}$ in the Hamiltonian PDE?

(b) Show that the following two quadratic integrals are conserved quantities as well:

$$\mathcal{N} = \int_0^L |\psi|^2\,\mathrm{d}x, \quad \text{and} \quad \mathcal{M} = \int_0^L \bar{\psi}\psi_x\,\mathrm{d}x.$$

(c) Construct a Hamiltonian *spatial discretization* of the NLS equation. Check which of the integrals $\mathcal{H}$, $\mathcal{M}$, $\mathcal{N}$ possess conserved discrete counterparts H, M, N in your discretization.

(d) Suppose you would discretize *in time* using the *implicit midpoint* method. Which of the discrete integrals H, M, N do you expect to be preserved to machine precision?

Exercise 8.11

Consider the Kawahara PDE (a special fifth-order KdV equation):

$$2u_t + \alpha u_{xxx} + \beta u_{xxxxx} = \frac{\partial}{\partial x} h(u, u_x, u_{xx}), \quad x \in [0, L]. \tag{6}$$

Define $f = \frac{1}{2}\frac{\delta}{\delta u}\int_0^L h\,\mathrm{d}x$, $\mathcal{J} = -\frac{1}{2}\frac{\partial}{\partial x}$ and

$$\mathcal{H} = \int_0^L [\frac{1}{2}\beta u_{xx}^2 - \frac{1}{2}\alpha u_x^2 + h(u, u_x, u_{xx})]\,\mathrm{d}x.$$

(a) Show that PDE (6) is equivalent to $u_t = \mathcal{J}\frac{\delta \mathcal{H}}{\delta u}$, i.e., (6) is a *symplectic* (=Hamiltonian) PDE.

*(b) Show that (6) can be written as a *multi-symplectic* PDE:

$$M\,\vec{z}_t + K\vec{z}_x = \nabla_{\vec{z}} S(\vec{z})$$

with $K \in \mathbb{R}^{6\times 6}$, $M \in \mathbb{R}^{6\times 6}$ and $\vec{z} \in \mathbb{R}^6$.

*Exercise 8.12

Re-write the KdV equation from exercise (13.8) as a *multi-symplectic* PDE:

$$M\ \vec{z}_t + K\vec{z}_x = \nabla_{\vec{z}} S(\vec{z})$$

with $K \in \mathbb{R}^{4\times4}$, $M \in \mathbb{R}^{4\times4}$ and $\vec{z} \in \mathbb{R}^4$.