

# Numerical Methods for Time-Dependent PDEs

Spring 2024

## Exercises for Lecture 9

### Exercise 9.1

Show that the Caputo fractional derivative is "consistent" with the traditional integer derivative: taking the limit for the fractional order derivative  $\alpha \in \mathbb{R} \rightarrow m \in \mathbb{N}$ , we obtain the well-known expression for the integer derivative  $m$ .

### Exercise 9.2

Define the space-fractional time-dependent PDE (in Caputo sense)

$$u_t = D_C^\alpha u, \quad 1 < \alpha < 2, \quad x \in \mathbb{R},$$

with initial solution  $u(x, 0) = e^{-x^2}$ . Discuss the solution behaviour for  $\lim_{\alpha \rightarrow 1}$ ,  $\lim_{\alpha \rightarrow 2}$  and intermediate values of  $\alpha$ . Also, describe the two cases " $\int_{-\infty}^x$ " and " $\int_x^\infty$ " in the definition. How do we get a symmetric fractional diffusion behaviour? In this case, what is the difference with ordinary diffusion?

### Exercise 9.3

(a) Check that the solution of the fractional ODE (not imposing any initial condition!):

$$\mathcal{D}_C^{\frac{3}{2}} u(t) = \frac{\Gamma(6)t^{\frac{7}{2}}}{\Gamma(\frac{9}{2})}$$

is given by:  $u(t) = t^5$ . (in fact, this is the only *analytic* solution of this ODE!)

(b) The solution of the space-fractional PDE:

$$\begin{cases} u_t = -(-\Delta u)^{\frac{\alpha}{2}}, & x \in [0, 1], \\ u|_{t=0} = u_0(x), \end{cases}$$

with homogeneous boundary conditions at  $x = 0$  and  $x = 1$  is given by the following expression:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-n^\alpha \pi^\alpha t},$$

where  $c_n = 2 \int_0^1 u_0(s) \sin(n\pi s) ds$ . Plot<sup>1</sup> the solution for  $\alpha = 2$  and  $\alpha = 3/2$ .

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<sup>1</sup>Check the Matlab file on the webpage of the course.

## Exercise 9.4

Derive the following (low-order) finite-difference approximation for  $D_C^\alpha u$ , where  $1 < \alpha < 2$ :

$$D_C^\alpha u|_{x_j} \approx \frac{1}{\Gamma(3-\alpha)h^\alpha} \sum_{j=1}^{i-1} \{j^{2-\alpha} - (j-1)^{2-\alpha}\} \{u_{i-j+1} - 2u_{i-j} + u_{i-j-1}\}$$

Describe the structure of the underlying finite-difference matrix.

## Exercise 9.5

- (a) Check that  $D_C^\alpha(\text{constant}) = 0$  and find  $D_{RL}^\alpha(\text{constant})$ .
- (b) Show that the fractional derivative is a linear operator.
- (c) Check that  $f(x) = \cos(2m\pi x) \Gamma(x)$  ( $m \in \mathbb{N}$ ) solves the functional equation:

$$\begin{cases} f(x+1) = xf(x), & x > 0, \\ f(1) = 1. \end{cases}$$

- (d) Plot the function  $f(x)$  in part (c) for  $m = 0, 1, 2$ .
- (e) Calculate the values  $\Gamma(\frac{1}{2})$ ,  $\Gamma(\frac{3}{2})$ ,  $\Gamma(\frac{5}{2})$ , ...
- (f) Sketch the Mittag-Leffler function  $E_\alpha(x)$  ( $x > 0$ ) for  $\alpha = 1, \frac{3}{4}, \frac{1}{2}$  and  $\frac{1}{4}$ .

## Exercise 9.6

Consider the space-fractional advection-diffusion (dispersion) PDE<sup>2</sup>:

$$\frac{\partial u(x, t)}{\partial t} = d(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} - v(x) \frac{\partial u(x, t)}{\partial x} + f(x, t), \quad x_L < x < x_R$$

- (a) Show that Euler-Forward combined with the Grünwald approximation defined by equation (3) in the mentioned extra file, applied to the advection-diffusion (dispersion) equation, is unstable.
- (b) Similar question for Euler-Backward combined with the Grünwald approximation defined by equation (3) in the extra file: it is unstable as well!
- (c) Show that the shifted Grünwald approximation defined by equation (10) in the extra file is consistent with the Riemann-Liouville fractional derivative of equation, defined in equation (2).

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<sup>2</sup>Check the article by Meerschaert and Tadjeran on the webpage = one of the extra files.

(d) Show that the shifted Grünwald approximation (10), applied to the advection-diffusion (dispersion) equation **is** unconditionally stable.

## Exercise 9.7

(a) Consider the ODE:

$$\begin{cases} u'(t) = \lambda u(t), & \lambda \in \mathbb{C}, \\ u(0) = u_0. \end{cases}$$

Work out the system of linear equations that is obtained when the following BV-method is applied:

- 1) Euler-Forward in the first time-step
- 2) explicit-midpoint for the intermediate time-steps
- 3) Euler-Backward for the final time-step

(b) The same question as in part (a) but now for the linear ODE system:

$$\begin{cases} \vec{u}'(t) = \mathcal{A} \vec{u}(t), & \mathcal{A} \in \mathbb{R}^{2 \times 2}, \\ \vec{u}(0) = \vec{u}_0. \end{cases}$$

(c) Apply the BV-method from (a) to the nonlinear ODE:

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0 \end{cases}$$

and describe the nonlinear system to be solved.

## Exercise 9.8

Show that the boundary locus of any consistent linear multistep method possesses the following properties:

⌘ it consists always the origin in the complex plane.

⌘ it is symmetric with respect to the real axis.

⌘ it is perpendicular to the real axis at the origin.

Moreover, show that the stability region of the (basic) midpoint BV-method, as discussed in the lecture, is the whole complex plane, excluding the imaginary axis.

### Exercise 9.9

Apply the doubling-splitting procedure (see lecture notes) for the model:

$$\begin{cases} u_t = -(-\Delta u)^{\frac{1}{2}}, \\ u|_{t=0} = u_0(x). \end{cases}$$

Describe the method-of-lines and the resulting ODE system. Comment on the eigenvalues of the matrix and the consequences/choices for the time-integration method.

### Exercise 9.10

The same questions as in exercise 9.9, but now for the *left*-space fractional heat equation of order  $5/4$ :

$$\begin{cases} u_t = \mathcal{D}_C^{\frac{5}{4}} u, \\ u|_{t=0} = u_0(x). \end{cases}$$