Numerical Methods for Time-Dependent PDEs

Spring 2024

Exercises for Lecture 9

Exercise 9.1

Show that the Caputo fractional derivative is ”consistent” with the traditional integer derivative: taking the limit for the fractional order derivative $\alpha \in \mathbb{R} \to m \in \mathbb{N}$, we obtain the well-known expression for the integer derivative m.

Exercise 9.2

Define the space-fractional time-dependent PDE (in Caputo sense)

$$u_t = D_C^{\alpha} u, \quad 1 < \alpha < 2, \quad x \in \mathbb{R},$$

with initial solution $u(x, 0) = \sin^{50}(\pi x)$. Discuss the solution behaviour for $\lim_{\alpha \to 1}$, $\lim_{\alpha \to 2}$ and intermediate values of α. Also, describe the two cases ”$\int_{-\infty}^{x}$” and ”$\int_{x}^{\infty}$” in the definition. How do we get a symmetric fractional diffusion behaviour? In this case, what is the difference with ordinary diffusion?

Exercise 9.3

(a) Check that the solution of the fractional ODE (not imposing any initial condition!):

$$\mathcal{D}_C^{\frac{3}{2}} u(t) = \frac{\Gamma(6) t^{\frac{7}{2}}}{\Gamma(\frac{9}{2})}$$

is given by: $u(t) = t^5$. (in fact, this is the only *analytic* solution of this ODE!)

(b) Check that the solution of the space-fractional PDE:

$$\begin{cases} u_t = -(-\Delta u)^{\frac{\alpha}{2}}, & x \in [0, 1], \\ u_{|t=0} = u_0(x), \end{cases}$$

with homogeneous boundary conditions at $x = 0$ and $x = 1$ is given by the following expression:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \mathrm{e}^{-n^{\alpha} \pi^{\alpha} t},$$

where $c_n = 2 \int_0^1 u_0(s) \sin(n\pi s)\, ds$. Plot[1] the solution for $\alpha = 2$ and $\alpha = 3/2$.

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[1] Check the Matlab file on the webpage of the course.

Exercise 9.4

Derive the following (low-order) finite-difference approximation for $D_C^\alpha u$, where $1 < \alpha < 2$:

$$D_C^\alpha u|x_j \approx \frac{1}{\Gamma(3-\alpha)h^\alpha}\sum_{j=1}^{i-1}\{j^{2-\alpha}-(j-1)^{2-\alpha}\}\{u_{i-j+1}-2u_{i-j}+u_{i-j-1}\}$$

Describe the structure of the underlying finite-difference matrix.

Exercise 9.5

(a) Check that $D_C^\alpha(constant) = 0$ and find $D_{RL}^\alpha(constant)$.

(b) Show that the fractional derivative is a linear operator.

(c) Check that $f(x) = \cos(2m\pi x)\ \Gamma(x)$ $(m \in \mathbb{N})$ solves the functional equation:

$$\begin{cases} f(x+1) = xf(x), & x > 0, \\ f(1) = 1. \end{cases}$$

(d) Plot the function $f(x)$ in part (c) for $m = 0, 1, 2$.

(e) Calculate the values $\Gamma(\frac{1}{2})$, $\Gamma(\frac{3}{2}))$, $\Gamma(\frac{5}{2})$, ...

(f) Sketch the Mittag-Leffler function $E_\alpha(x)$ $(x > 0)$ for $\alpha = 1, \frac{3}{4}, \frac{1}{2}$ and $\frac{1}{4}$.

Exercise 9.6

Consider the space-fractional advection-diffusion (dispersion) PDE[2]:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - v(x)\frac{\partial u(x,t)}{\partial x} + f(x,t), \quad x_L < x < x_R$$

(a) Show that Euler-Forward combined with the Grünwald approximation defined by equation (3) in the mentioned extra file, applied to the advection-diffusion (dispersion) equation, is unstable.

(b) Similar question for Euler-Backward combined with the Grünwald approximation defined by equation (3) in the extra file: it is unstable as well!

(c) Show that the shifted Grünwald approximation defined by equation (10) in the extra file is consistent with the Riemann-Liouville fractional derivative of equation, defined in equation (2).

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[2] Check the article by Meerschaert and Tadjeran on the webpage = one of the extra files.

(d) Show that the shifted Grünwald approximation (10), applied to the advection-diffusion (dispersion) equation **is** unconditionally *stable.*

Exercise 9.7

(a) Consider the ODE:

$$\begin{cases} u'(t) = \lambda\; u(t), \quad \lambda \in \mathbb{C}, \\ u(0) = u_0. \end{cases}$$

Work out the system of linear equations that is obtained when the following BV-method is applied:

1) Euler-Forward in the first time-step
2) explicit-midpoint for the intermediate time-steps
3) Euler-Backward for the final time-step

(b) The same question as in part (a) but now for the linear ODE system:

$$\begin{cases} \vec{u}'(t) = \mathcal{A}\; \vec{u}(t), \quad \mathcal{A} \in \mathbb{R}^{2\times 2}, \\ \vec{u}(0) = \vec{u}_0. \end{cases}$$

(c) Apply the BV-method from (a) to the nonlinear ODE:

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0 \end{cases}$$

and describe the nonlinear system to be solved.

Exercise 9.8

Show that the boundary locus of any consistent linear multistep method possesses the following properties:

⋕ it consists always the origin in the complex plane.

⋕ it is symmetric with respect to the real axis.

⋕ it is perpendicular to the real axis at the origin.

Moreover, show that the stability region of the (basic) midpoint BV-method, as discussed in the lecture, is the whole complex plane, excluding the imaginary axis.

Exercise 9.9

Apply the doubling-splitting procedure (see lecture notes) for the model:

$$\begin{cases} u_t = -(-\Delta u)^{\frac{1}{2}}, \\ u_{|t=0} = u_0(x). \end{cases}$$

Describe the method-of-lines and the resulting ODE system. Comment on the eigenvalues of the matrix and the consequences/choices for the time-integration method.

Exercise 9.10

The same questions as in exercise 9.9, but now for the *left*-space fractional heat equation of order 5/4:

$$\begin{cases} u_t = \mathcal{D}_C^{\frac{5}{4}} u, \\ u_{|t=0} = u_0(x). \end{cases}$$