Lecture 10

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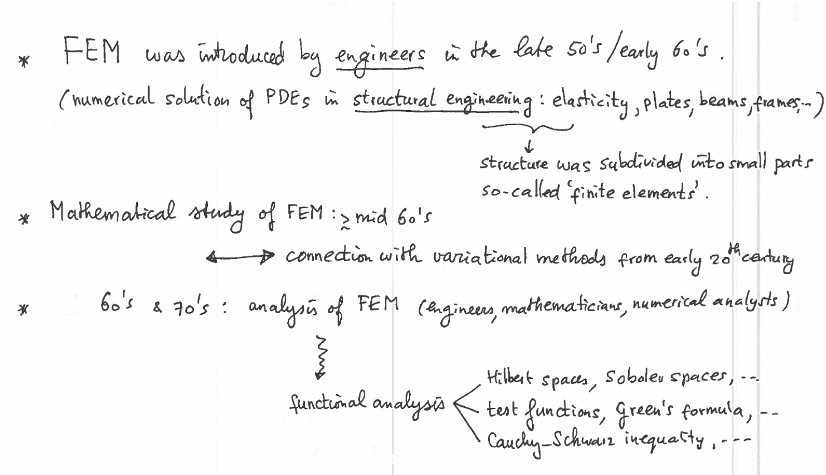
Numerical Methods for Time-Dependent PDEs, Spring 2024

Outline of Lecture 10

- exercises of Lecture 9
- historical background
- variational and minimization formulation
- a finite dimensional approximation & the stationary case
- stiffness matrix and load vector
- an abstract formulation & the time-dependent case
- Next week: Travelling Waves and Higher-order PDEs

Finite elements [1]



* FEM was introduced by engineers in the late 50's / early 60's.

(numerical solution of PDEs in structural engineering: elasticity, plates, beams, frames, --)

↓ structure was subdivided into small parts so-called 'finite elements'.

* Mathematical study of FEM: $\gtrsim$ mid 60's

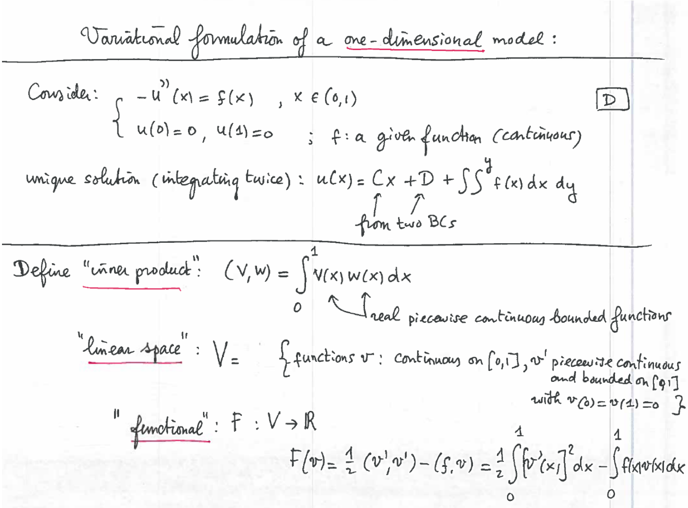
⟷ connection with variational methods from early 20th century

* 60's & 70's: analysis of FEM (engineers, mathematicians, numerical analysts)

↓ functional analysis

- Hilbert spaces, Sobolev spaces, --
- test functions, Green's formula, --
- Cauchy-Schwarz inequality, ---

Finite elements [2]



Variational formulation of a one-dimensional model :

Consider: $\begin{cases} -u''(x) = f(x) \;, & x \in (0,1) \\ u(0)=0 \,,\; u(1)=0 & ;\; f : \text{a given function (continuous)} \end{cases}$ $\boxed{D}$

unique solution (integrating twice) : $u(x) = \underset{\uparrow}{C}x + \underset{\uparrow}{D} + \int\int^{y} f(x)\,dx\,dy$

from two BCs

Define "inner product": $(v,w) = \int_0^1 v(x)\,w(x)\,dx$

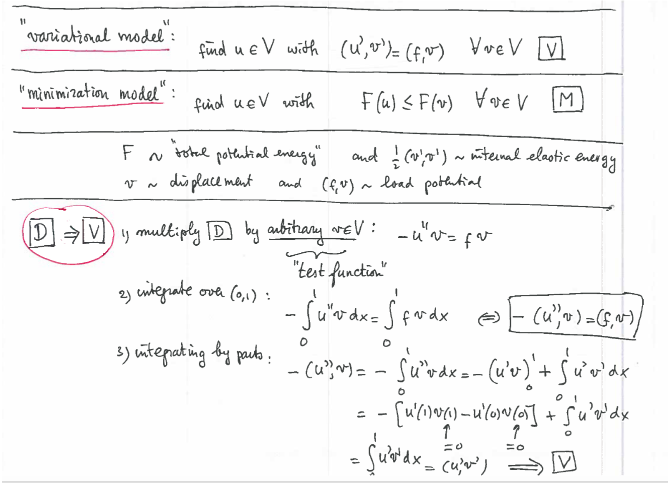
real piecewise continuous bounded functions

"linear space" : $V = \{$functions v : continuous on $[0,1]$, v' piecewise continuous and bounded on $[0,1]$ with $v(0)=v(1)=0$ $\}$

"functional" : $F : V \to \mathbb{R}$

$$F(v) = \frac{1}{2}(v',v') - (f,v) = \frac{1}{2}\int_0^1 [v'(x)]^2\,dx - \int_0^1 f(x)v(x)\,dx$$

Finite elements [3]



"variational model": find $u \in V$ with $(u', v') = (f, v) \quad \forall v \in V \quad \boxed{V}$

"minimization model": find $u \in V$ with $F(u) \leq F(v) \quad \forall v \in V \quad \boxed{M}$

F ~ "total potential energy" and $\frac{1}{2}(v', v')$ ~ internal elastic energy

v ~ displacement and (f, v) ~ load potential

$\boxed{D} \Rightarrow \boxed{V}$

1) multiply $\boxed{D}$ by $\underbrace{\text{arbitrary } v \in V}_{\text{"test function"}}$: $-u'' v = f v$

2) integrate over $(0,1)$: $-\int_0^1 u'' v \, dx = \int_0^1 f v \, dx \iff \boxed{-(u'', v) = (f, v)}$

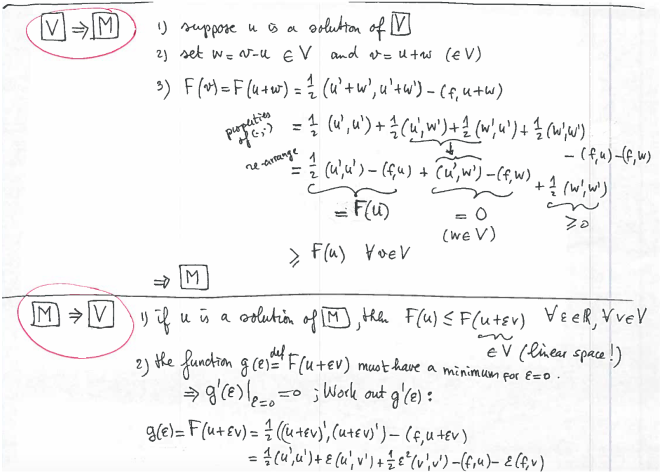
3) integrating by parts:

$$-(u'', v) = -\int_0^1 u'' v \, dx = -(u'v)\Big|_0^1 + \int_0^1 u' v' \, dx$$

$$= -[u'(1)\underset{=0}{\underset{\uparrow}{v(1)}} - u'(0)\underset{=0}{\underset{\uparrow}{v(0)}}] + \int_0^1 u' v' \, dx$$

$$= \int_0^1 u' v' \, dx = (u', v') \Longrightarrow \boxed{V}$$

Finite elements [4]



$\boxed{V} \Rightarrow \boxed{M}$

1) suppose u is a solution of $\boxed{V}$

2) set $w = v - u \in V$ and $v = u + w$ $(\in V)$

3) $F(v) = F(u+w) = \frac{1}{2}(u'+w', u'+w') - (f, u+w)$

properties of $(\cdot,\cdot)$ $= \frac{1}{2}(u',u') + \underbrace{\frac{1}{2}(u',w') + \frac{1}{2}(w',u')}_{\downarrow} + \frac{1}{2}(w',w') - (f,u) - (f,w)$

re-arrange $= \underbrace{\frac{1}{2}(u',u') - (f,u)}_{= F(u)} + \underbrace{\overbrace{(u',w')} - (f,w)}_{\substack{=0 \\ (w \in V)}} + \underbrace{\frac{1}{2}(w',w')}_{\geq 0}$

$\geqslant F(u) \quad \forall v \in V$

$\Rightarrow \boxed{M}$

$\boxed{M} \Rightarrow \boxed{V}$

1) if u is a solution of $\boxed{M}$, then $F(u) \leq F(\underbrace{u+\varepsilon v}_{\in V \text{ (linear space!)}}) \quad \forall \varepsilon \in \mathbb{R}, \forall v \in V$

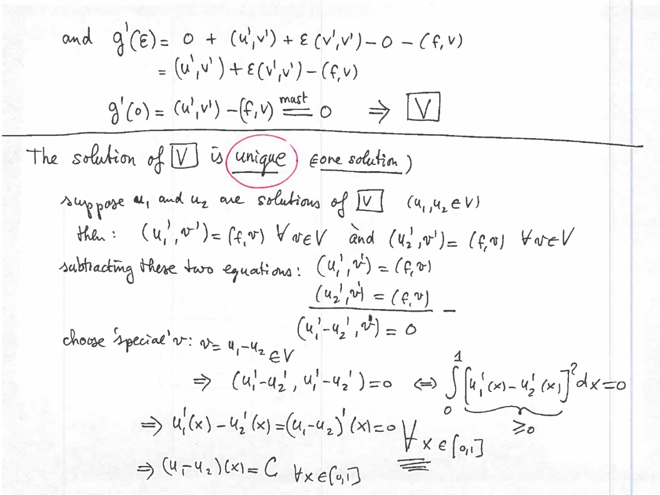
2) the function $g(\varepsilon) \overset{\text{def}}{=} F(u+\varepsilon v)$ must have a minimum for $\varepsilon = 0$.

$\Rightarrow g'(\varepsilon)\big|_{\varepsilon=0} = 0$; Work out $g'(\varepsilon)$:

$$g(\varepsilon) = F(u+\varepsilon v) = \frac{1}{2}\left((u+\varepsilon v)', (u+\varepsilon v)'\right) - (f, u+\varepsilon v)$$

$$= \frac{1}{2}(u',u') + \varepsilon(u',v') + \frac{1}{2}\varepsilon^2(v',v') - (f,u) - \varepsilon(f,v)$$

Finite elements [5]



and $g'(\varepsilon) = 0 + (u',v') + \varepsilon(v',v') - 0 - (f,v)$

$$= (u',v') + \varepsilon(v',v') - (f,v)$$

$$g'(0) = (u',v') - (f,v) \overset{\text{must}}{=} 0 \quad \Rightarrow \quad \boxed{V}$$

The solution of $\boxed{V}$ is unique (one solution)

suppose u_1 and u_2 are solutions of $\boxed{V}$ $(u_1, u_2 \in V)$

then: $(u_1', v') = (f,v)\ \forall v \in V$ and $(u_2', v') = (f,v)\ \forall v \in V$

subtracting these two equations:

$$\begin{array}{r} (u_1', v') = (f,v) \\ (u_2', v') = (f,v) \\ \hline (u_1' - u_2', v') = 0 \end{array} \; -$$

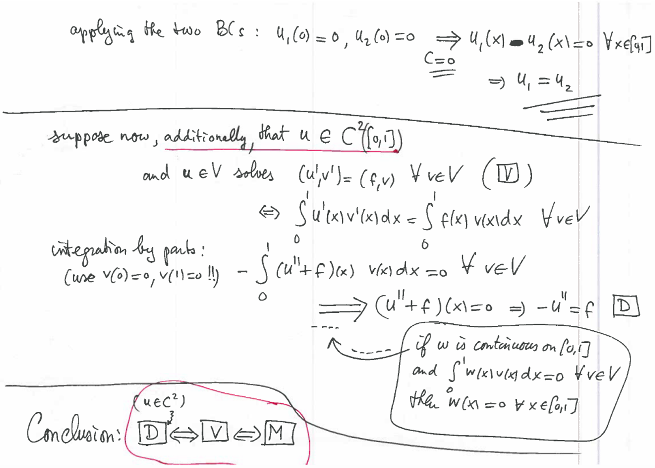
choose 'special' v: $v = u_1 - u_2 \in V$

$$\Rightarrow (u_1' - u_2', u_1' - u_2') = 0 \Leftrightarrow \int_0^1 \underbrace{\left[u_1'(x) - u_2'(x)\right]^2}_{\geq 0} dx = 0$$

$$\Rightarrow u_1'(x) - u_2'(x) = (u_1 - u_2)'(x) = 0 \quad \forall x \in [0,1]$$

$$\Rightarrow (u_1 - u_2)(x) = C \quad \forall x \in [0,1]$$

Finite elements [6]



applying the two BCs : $u_1(0)=0,\ u_2(0)=0 \underset{C=0}{\Rightarrow} u_1(x) - u_2(x)=0\ \forall x\in[0,1]$

$\Rightarrow u_1 = u_2$

suppose now, additionally, that $u \in C^2([0,1])$

and $u\in V$ solves $(u',v') = (f,v)\ \forall v\in V$ $(\boxed{V})$

$$\Leftrightarrow \int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)\,v(x)dx\quad \forall v\in V$$

integration by parts:
(use $v(0)=0, v(1)=0$!!)

$$-\int_0^1 (u''+f)(x)\ v(x)dx = 0\quad \forall\ v\in V$$

$$\Longrightarrow (u''+f)(x)=0 \Rightarrow -u''=f\quad \boxed{D}$$

if w is continuous on $[0,1]$
and $\int_0^1 w(x)v(x)dx=0\ \forall v\in V$
then $w(x)=0\ \forall x\in[0,1]$

Conclusion: $\boxed{D} \overset{(u\in C^2)}{\Leftrightarrow} \boxed{V} \Leftrightarrow \boxed{M}$

Finite elements [7]

A finite dimensional subspace V_h of $\underset{\sim}{V}$

"infinite dimensional"

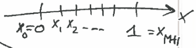
Many choices possible!

Basic example: piecewise linear functions

grid $0 = x_0 < x_1 < x_2 < \cdots < x_M < x_{M+1} = 1$

$h_j = x_j - x_{j-1}$, $j = 1, 2, \dots, M+1$

uniform grid $h_j = h$ (constant)

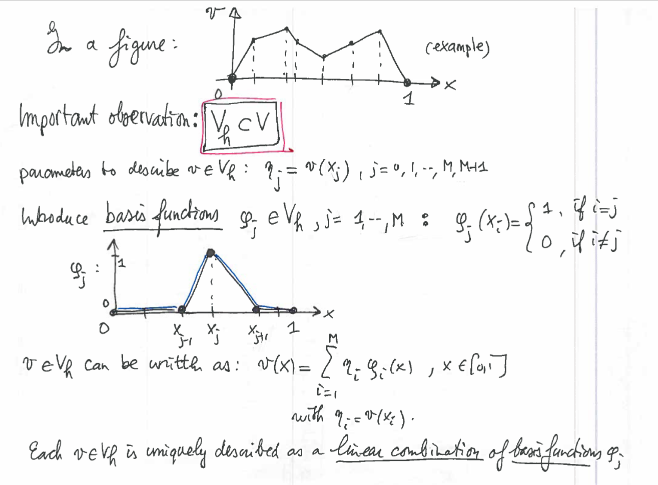


$V_h \overset{\text{def}}{=} \{$ functions v : v is $\underline{\text{linear}}$ on each subinterval (x_{j-1}, x_j),

v is continuous on $[0,1]$

$v(0) = 0$, $v(1) = 0$ $\}$

Finite elements [8]



In a figure:

(example)

Important observation: $\boxed{V_h \subset V}$

parameters to describe $v \in V_h$: $\eta_j = v(x_j)$, $j = 0, 1, \dots, M, M+1$

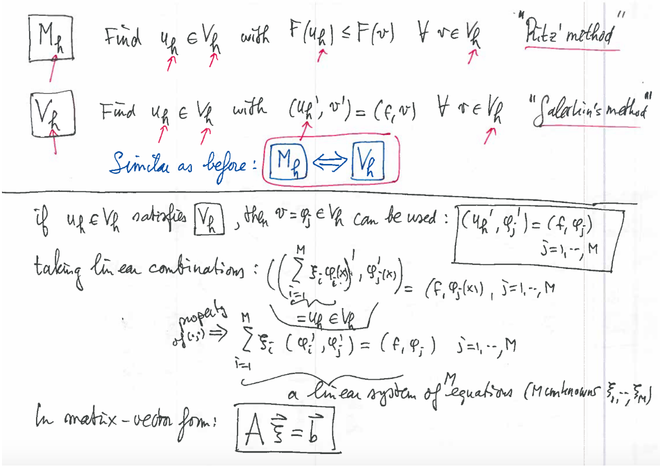
Introduce basis functions $\varphi_j \in V_h$, $j = 1 \dots, M$: $\varphi_j(x_i) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$

$v \in V_h$ can be written as: $v(x) = \sum_{i=1}^{M} \eta_i \varphi_i(x)$, $x \in [0,1]$

with $\eta_i = v(x_i)$.

Each $v \in V_h$ is uniquely described as a linear combination of basis functions φ_j

Finite elements [9]



$\boxed{M_h}$ Find $u_h \in V_h$ with $F(u_h) \le F(v)$ $\forall v \in V_h$ "Ritz' method"

$\boxed{V_h}$ Find $u_h \in V_h$ with $(u_h', v') = (f, v)$ $\forall v \in V_h$ "Galerkin's method"

Similar as before: $\boxed{M_h} \Leftrightarrow \boxed{V_h}$

---

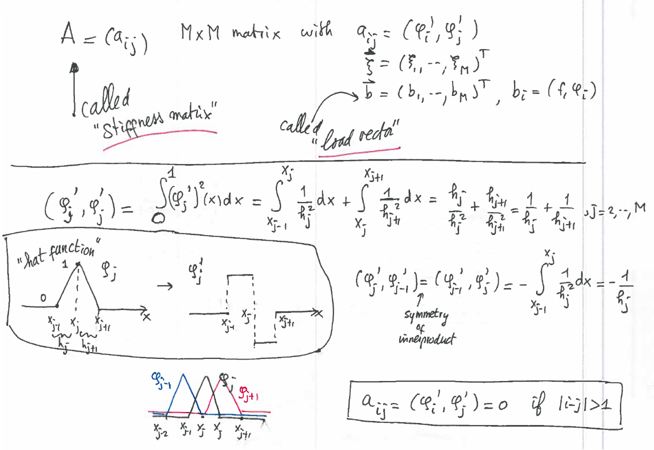
if $u_h \in V_h$ satisfies $\boxed{V_h}$, then $v = \varphi_j \in V_h$ can be used: $\boxed{(u_h', \varphi_j') = (f, \varphi_j) \quad j = 1, \dots, M}$

taking linear combinations: $\Big(\big(\underbrace{\sum_{i=1}^{M} \xi_i \varphi_i(x)}_{= u_h \in V_h}\big)', \varphi_j'(x)\Big) = (f, \varphi_j(x)),\ j = 1, \dots, M$

$$\overset{\text{property of } (\cdot,\cdot)}{\Longrightarrow} \underbrace{\sum_{i=1}^{M} \xi_i (\varphi_i', \varphi_j') = (f, \varphi_j) \quad j = 1, \dots, M}_{\text{a linear system of } M \text{ equations } (M \text{ unknowns } \xi_1, \dots, \xi_M)}$$

In matrix-vector form: $\boxed{A \vec{\xi} = \vec{b}}$

Finite elements [10]



$A = (a_{ij})$ M x M matrix with $a_{ij} = (\varphi_i', \varphi_j')$

$\vec{\xi} = (\xi_1, \dots, \xi_M)^T$

$\vec{b} = (b_1, \dots, b_M)^T, \quad b_i = (f, \varphi_i)$

called "stiffness matrix"

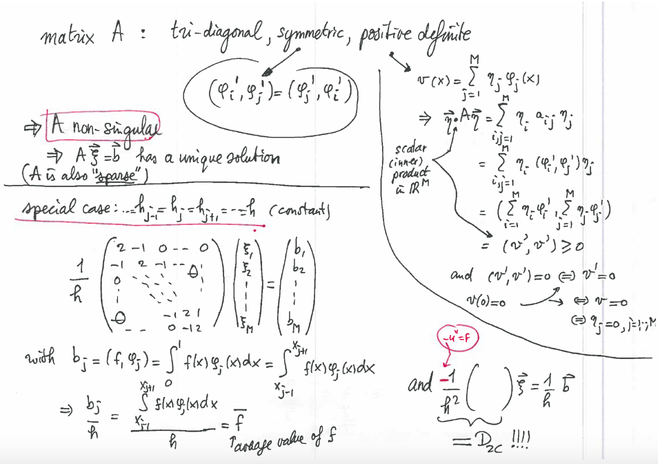
called "load vector"

$$(\varphi_j', \varphi_j') = \int_0^1 (\varphi_j')^2(x)\,dx = \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2}\,dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_{j+1}^2}\,dx = \frac{h_j}{h_j^2} + \frac{h_{j+1}}{h_{j+1}^2} = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad j = 2, \dots, M$$

$$(\varphi_j', \varphi_{j-1}') \underset{\substack{\uparrow \\ \text{symmetry of innerproduct}}}{=} (\varphi_{j-1}', \varphi_j') = -\int_{x_{j-1}}^{x_j} \frac{1}{h_j^2}\,dx = -\frac{1}{h_j}$$

$$\boxed{a_{ij} = (\varphi_i', \varphi_j') = 0 \quad \text{if } |i-j| > 1}$$

Finite elements [11]



matrix A : tri-diagonal, symmetric, positive definite

$(\varphi_i', \varphi_j') = (\varphi_j', \varphi_i')$

$\Rightarrow$ A non-singular

$\Rightarrow A\vec{\xi} = \vec{b}$ has a unique solution

(A is also "sparse")

$$v(x) = \sum_{j=1}^{M} \eta_j \varphi_j(x)$$

$$\Rightarrow \vec{\eta} \cdot A\vec{\eta} = \sum_{i,j=1}^{M} \eta_i a_{ij} \eta_j = \sum_{i,j=1}^{M} \eta_i (\varphi_i', \varphi_j') \eta_j = \left(\sum_{i=1}^{M} \eta_i \varphi_i', \sum_{j=1}^{M} \eta_j \varphi_j'\right) = (v', v') \geq 0$$

scalar (inner) product in $\mathbb{R}^M$

and $(v', v') = 0 \Leftrightarrow v' = 0$

$v(0) = 0$ $\rightarrow$ $\Leftrightarrow v = 0$

$\Leftrightarrow \eta_j = 0, j = 1 \cdots M$

special case: $\ldots = h_{j-1} = h_j = h_{j+1} = \ldots = h$ (constant)

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -1 & 2 & 1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix}$$

with $b_j = (f, \varphi_j) = \int_0^1 f(x)\varphi_j(x)dx = \int_{x_{j-1}}^{x_{j+1}} f(x)\varphi_j(x)dx$

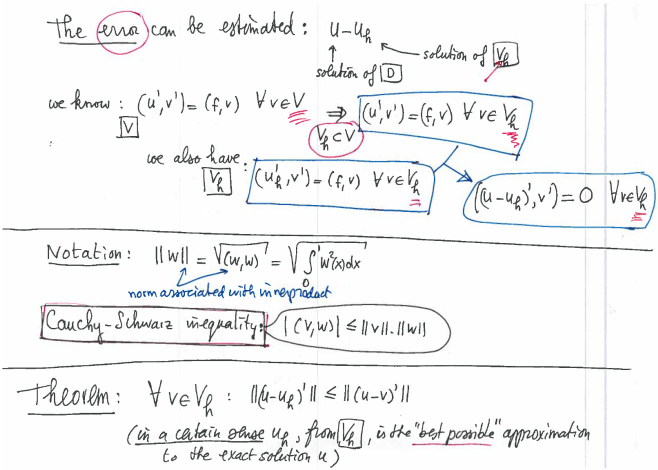
$$\Rightarrow \frac{b_j}{h} = \frac{\int_{x_{j-1}}^{x_{j+1}} f(x)\varphi_j(x)dx}{h} = \bar{f}$$

↑ average value of f

$-u'' = f$

$$\text{and } \underbrace{-\frac{1}{h^2}\left(\qquad\right)}_{= D_{2c}\ !!!!} \vec{\xi} = \frac{1}{h}\vec{b}$$

Finite elements [12]



The error can be estimated: $\underset{\substack{\uparrow \\ \text{solution of } \boxed{D}}}{u} - u_h \leftarrow$ solution of $\boxed{V_h}$

we know $\boxed{V}$: $(u',v')=(f,v) \quad \forall v\in V \;\underset{V_h\subset V}{\Rightarrow}\; \boxed{(u',v')=(f,v) \quad \forall v\in V_h}$

we also have $\boxed{V_h}$: $\boxed{(u_h',v')=(f,v) \quad \forall v\in V_h}$

$\Rightarrow \boxed{((u-u_h)',v')=0 \quad \forall v\in V_h}$

Notation: $\|w\| = \sqrt{(w,w)} = \sqrt{\int_0^1 w^2(x)dx}$

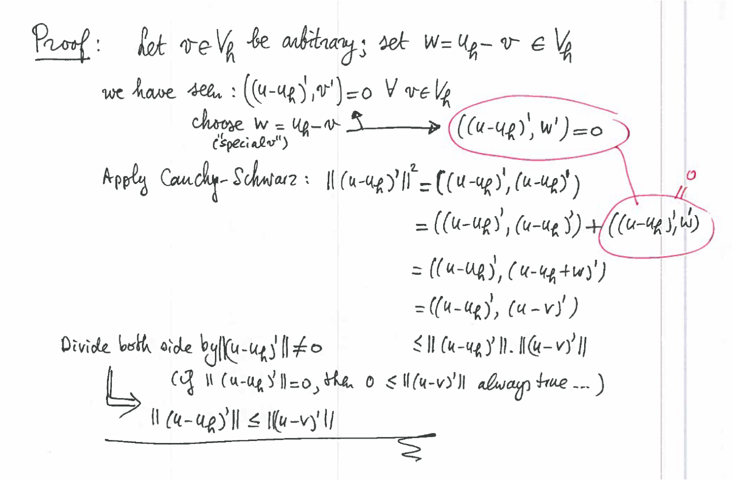
norm associated with innerproduct

Cauchy-Schwarz inequality: $|(v,w)| \leq \|v\|.\|w\|$

Theorem: $\forall v\in V_h$: $\|(u-u_h)'\| \leq \|(u-v)'\|$

(in a certain sense u_h, from $\boxed{V_h}$, is the "best possible" approximation to the exact solution u)

Finite elements [13]



Proof: Let $v \in V_h$ be arbitrary; set $w = u_h - v \in V_h$

we have seen: $((u-u_h)', v') = 0 \ \forall v \in V_h$

choose $w = u_h - v$ ("special v") $\longrightarrow \boxed{((u-u_h)', w') = 0}$

Apply Cauchy-Schwarz: $\|(u-u_h)'\|^2 = ((u-u_h)', (u-u_h)')$

$$= ((u-u_h)', (u-u_h)') + \overset{\overset{0}{\parallel}}{\boxed{((u-u_h)', w')}}$$

$$= ((u-u_h)', (u-u_h+w)')$$

$$= ((u-u_h)', (u-v)')$$

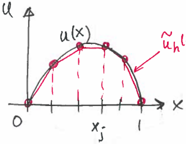
Divide both side by $\|(u-u_h)'\| \neq 0$ $\quad \le \|(u-u_h)'\| \cdot \|(u-v)'\|$

(if $\|(u-u_h)'\| = 0$, then $0 \le \|(u-v)'\|$ always true ...)

$\hookrightarrow \|(u-u_h)'\| \le \|(u-v)'\|$

Finite elements [14]

How do we obtain a "real" error estimate ?



$\tilde{u}_h(x)$: a suitably chosen function

estimate : $\|(u-\tilde{u}_h)\|$

$\tilde{u}_h \in V_h$: the interpolant of u, i.e.

$\tilde{u}_h$ interpolates u at the grid points x_j : $\tilde{u}_h(x_j)=u(x_j)$

$j=0,1,\cdots,M,M+1$

From numerical analysis (Bachelor course) :

$0\le x\le 1$: $|u'(x)-\tilde{u}_h'(x)| \le h.\max\limits_{0\le y\le 1}|u''(y)|$ from "intermediate value theorem"

or $|u(x)-\tilde{u}_h(x)| \le \frac{h^2}{8}.\max\limits_{0\le y\le 1}|u''(y)|$

combine with $\|(u-u_h)'\| \le \|(u-v)'\|$

$\|(u-u_h)'\| \le h.\max\limits_{0\le y\le 1}|u''(y)|$ $\Longrightarrow$ $|u(x)-u_h(x)| \le h.\max\limits_{0\le y\le 1}|u''(y)|$ for $0\le x\le 1$

$(u-u_h)(0)=0$

$(u-u_h)(x)=\int_0^x (u-u_h)'(y)d$

Cauchy-Schwarz inequality

Abstract formulation [1]

The two-dimensional case

< general theory

Poisson equation in 2 D: $\begin{cases} -\Delta u = \overset{\text{given}}{f}(x,y)\,, & (x,y) \in \Omega \\ u|_{\partial\Omega} = 0 & \end{cases}$ for example

∗ multiply with "arbitrary" function v : $-\Delta u . v = f . v$

and integrate over Ω : $-\int_\Omega \Delta u . v \, d\Omega = \int_\Omega f . v \, d\Omega$

∗ use Green's formula : $+\int_\Omega \nabla u \bullet \nabla v \, d\Omega - \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = \int_\Omega f v \, d\Omega$ (holds for all $\underset{\text{testfunctions}}{v} \in V$)

∗ for example : $V = \left\{ v \text{ continuous in } \Omega,\ v|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} \text{ piecewise continuous} \right\}$

Abstract formulation [2]

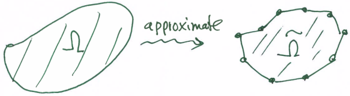
$\Rightarrow \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS = 0 \quad \Rightarrow \quad \boxed{\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \quad \forall v \in V}$

$*$ define: $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega$ (a bilinear form)

and $\langle f, v \rangle = \int_\Omega f v \, d\Omega$

$\Rightarrow$ find $u \in V$ such that $a(u,v) = \langle f, v \rangle \quad \forall v \in V$

$*$



and subdivide $\tilde{\Omega}$ into triangles K

for example

Define $h = \max_{K \in T_h} \{\text{longest side of } K\}$

$*$ approximate the function u : $u \approx u_h$

Find $u_h \in V_h$ such that $a(u_h, v) = \langle f, v \rangle \quad \forall v \in V_h$

$= \{ v \text{ continuous in } \Omega, \; v|_K \text{ is linear}, \; v|_{\partial\Omega} = 0 \} \subset V$

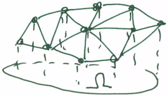
for example

restricted to

Abstract formulation [3]

* the properties of $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ in V also hold for the approximating V_h !!

* example of an "element" (function) from V_h:



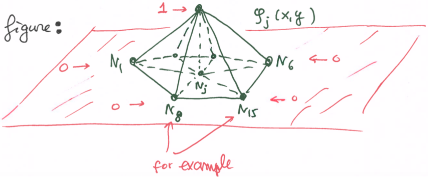
* Example of a basisfunction from V_h: $\varphi_j(x,y) \in V_h$

such that $\varphi_j(N_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

with N_i the gridpoint with index i (in our case: edge of a triangle)

Call $M \overset{\text{def}}{=} \# \{N_i\}$ (the total number of gridpoints)

* Basisfunction in a figure:



Abstract formulation [4]

- a function $v \in V_h$ can be written as $v(x,y) = \sum_{i=1}^{M} \underbrace{v(N_i)}_{\text{"amplitudes"}} \cdot \varphi_i(x,y)$
- special choice for functions $v \in V_h$: $\{\varphi_j\}_{j=1}^{M}$ ← the φ_j's themselves!
- we try to find $u_h \in V_h$, therefore: $u_h(x,y) = \sum_{i=1}^{M} u_h(N_i) \cdot \varphi_i(x,y)$
- substitute into $a(u_h, v) = \langle f, v \rangle \ \forall v \in V_h$

$$\Rightarrow \quad a\Big(\sum_{i=1}^{M} u_h(N_i)\,\varphi_i, \varphi_j\Big) = \langle f, \varphi_j \rangle \quad , j = 1, \dots, M$$

use bilinearity of $a(\cdot,\cdot)$

$$\Leftrightarrow \quad \sum_{i=1}^{M} u_h(N_i)\; a(\varphi_i, \varphi_j) = \langle f, \varphi_j \rangle \quad , j = 1, \dots, M$$

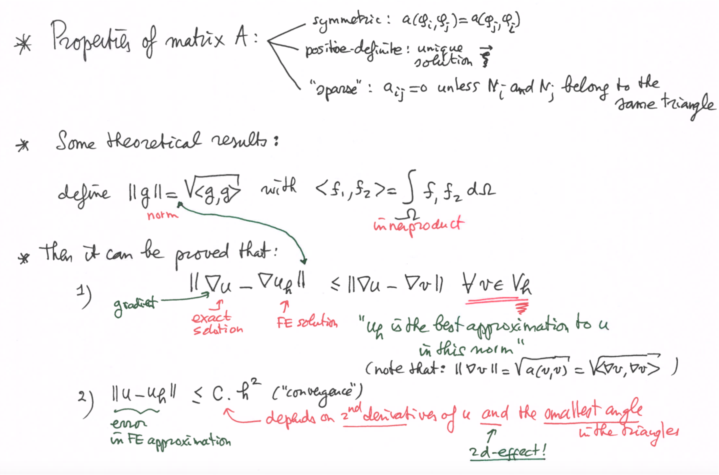
- Define $A = (a_{ij})_{ij} = (a(\varphi_i, \varphi_j))_{ij}$ the stiffness matrix

  and $\vec{b} = (b_i)_i = (\langle f, \varphi_i \rangle)_i$, $\vec{\xi} = (\xi_i)_i = (u_h(N_i))_i$

$\Rightarrow$ we must solve the $M \times M$ linear system of equations for $\vec{\xi}$: $\boxed{A\vec{\xi} = \vec{b}}$

$$\vec{\xi} \Rightarrow u_h(N_i) \Rightarrow u_h = \sum_{i=1}^{N} \cdots \quad .$$

Abstract formulation [5]



* Properties of matrix A:

- symmetric: $a(\varphi_i,\varphi_j)=a(\varphi_j,\varphi_i)$
- positive-definite: unique solution $\vec{\xi}$
- "sparse": $a_{ij}=0$ unless N_i and N_j belong to the same triangle

* Some theoretical results:

define $\|g\|=\sqrt{\langle g,g\rangle}$ with $\langle f_1,f_2\rangle=\int_\Omega f_1 f_2\, d\Omega$

norm

innerproduct

* Then it can be proved that:

1) $\|\nabla u-\nabla u_h\| \le \|\nabla u-\nabla v\| \quad \underline{\forall v\in V_h}$

gradient

exact solution

FE solution

"u_h is the best approximation to u in this norm"

(note that: $\|\nabla v\|=\sqrt{a(v,v)}=\sqrt{\langle\nabla v,\nabla v\rangle}$)

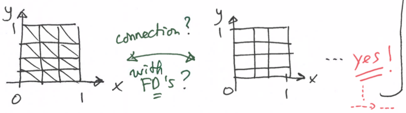
2) $\underbrace{\|u-u_h\|}_{\text{error in FE approximation}} \le C\cdot h^2$ ("convergence")

← depends on 2nd derivatives of u and the smallest angle in the triangles

2d-effect!

Abstract formulation [6]

Example : $\Omega = [0,1] \times [0,1]$



Theory : It is useful to work with a slightly "larger" space V
(for the analysis of FE's)
($\Rightarrow$ we can use functional analysis)

Examples ① 1d

a) $\Omega = I = (a,b)$, space $L_2(I) = \{ v \mid \int_I v^2 dx < \infty \}$

inner product: $\langle v,w \rangle_{L_2(I)} = \int_I vw\,dx$

norm: $\|v\|_{L_2(I)} = \sqrt{\langle v,v\rangle_{L_2(I)}} = \sqrt{\int_I v^2 dx}$

b) $H^1(I) = \{ v \mid v \in L_2(I) \text{ and } v' \in L_2(I) \}$

inner product: $\langle v,w \rangle_{H^1(I)} = \int_I (vw + v'w')\,dx$

norm: $\|v\|_{H^1(I)} = \sqrt{\langle v,v\rangle_{H^1(I)}} = \sqrt{\int_I (v^2 + (v')^2)\,dx}$

Abstract formulation [7]

c) $H_0^1(I) = \{v \in H^1(I) \text{ with } v(a)=v(b)=0\} \subset H^1(I) \subset L_2(I)$

with the same innerproduct and norm as for $H^1(I)$

$L_2(I), H^1(I), H_0^1(I), H^2(I), \ldots, H_0^4(I)$, etcetera are "Sobolev-spaces" (of functions)

---

② the 2d-case

$\Omega \subset \mathbb{R}^2$

$L_2(\Omega) = \{v \mid \int_\Omega v^2 \, \underset{\uparrow}{d\Omega} < \infty\}$ ← double integral

$H^1(\Omega) = \{v \mid v \in L_2(\Omega) \text{ and } \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L_2(\Omega)\} \subset L_2(\Omega)$

note: $\langle v,w\rangle_{H^1(\Omega)} = \int_\Omega (vw + \underset{\uparrow}{\nabla} v \underset{\uparrow}{\bullet} \underset{\uparrow}{\nabla} w) \, d\Omega$

$H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\} \subset H^1(\Omega) \subset L_2(\Omega)$

"Hilbert space theory"
V

→ linear space with inner product (and, therefore, also a norm)

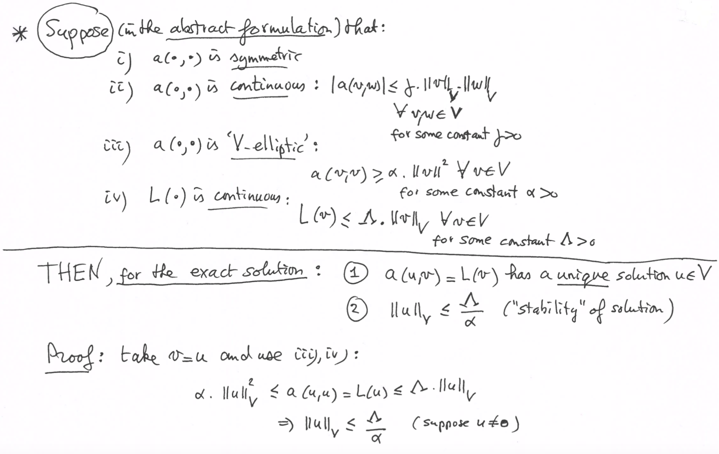
→ "complete" (every Cauchy sequence converges to a limit in the space itself)

$a(\cdot,\cdot)$ bilinear form : $V \times V \to \mathbb{R}$

$L(\cdot)$ linear form : $V \to \mathbb{R}$

(Banach space: only a norm, not an innerproduct)

Abstract formulation [8]



* Suppose (in the abstract formulation) that:

i) $a(\cdot,\cdot)$ is symmetric

ii) $a(\cdot,\cdot)$ is continuous: $|a(v,w)| \le \gamma \cdot \|v\|_V \cdot \|w\|_V$ $\forall\, v,w \in V$ for some constant $\gamma > 0$

iii) $a(\cdot,\cdot)$ is 'V-elliptic': $a(v,v) \ge \alpha \cdot \|v\|^2$ $\forall\, v \in V$ for some constant $\alpha > 0$

iv) $L(\cdot)$ is continuous: $L(v) \le \Lambda \cdot \|v\|_V$ $\forall\, v \in V$ for some constant $\Lambda > 0$

THEN, for the exact solution: ① $a(u,v) = L(v)$ has a unique solution $u \in V$

② $\|u\|_V \le \frac{\Lambda}{\alpha}$ ("stability" of solution)

Proof: take $v = u$ and use iii), iv):

$$\alpha \cdot \|u\|_V^2 \le a(u,u) = L(u) \le \Lambda \cdot \|u\|_V$$

$$\Rightarrow \|u\|_V \le \frac{\Lambda}{\alpha} \quad (\text{suppose } u \ne 0)$$

Abstract formulation [9]

suppose u_1 and $u_2 \in V$ are solutions, then $u_1 - u_2 \in V$ (because V is a linear space)

write:

bilinear form

$$\begin{array}{ll} a(u_1,v) = L(v) & \forall v \in V \\ a(u_2,v) = L(v) & \forall v \in V \quad - \\ \hline a(u_1-u_2,v) = 0 & \forall v \in V \end{array}$$

now use the stability inequality with $\Lambda = 0$ $\Rightarrow \|u_1 - u_2\| \leqslant 0$

but always: $\|u_1 - u_2\| \geqslant 0$

$\Rightarrow \|u_1 - u_2\| = 0$

$\Leftrightarrow u_1 - u_2 = 0$

$\Leftrightarrow u_1 = u_2$ (a unique solution in V)

[Note that the symmetry i) of $a(\cdot,\cdot)$ is not needed here]

Examine now: $\boxed{\text{find } u_h \in V \text{ such that } a(u_h,v) = \langle f,v\rangle \ \forall v \in V_h}$ the numerical approximations via FE's

Choose $v = u_h \in V_h \Rightarrow \alpha \|u_h\|_V^2 \leq a(u_h,u_h) = L(u_h) \leq \Lambda \cdot \|u_h\|_V$

$$\Rightarrow \|u_h\|_V \leq \frac{\Lambda}{\alpha}$$

The discrete (numerical, FE) solution $u_h^{\in V_h}$ satisfies exactly the same stability estimate as the exact solution $u \in V$

Abstract formulation [10]

Another result: we have $a(u,v) = L(v) \quad \forall v \in V \quad \Rightarrow$ also $\forall v \in V_h \subset V$

and $a(u_h,v) = L(v) \quad \forall v \in V_h \subset V$

$$\Rightarrow \quad a(u-u_h, w) = 0 \quad \forall w \in V_h$$

and (because $V_h \subset V$): $a(u,w) = L(w) \quad \forall w \in V_h$

$\boxed{\text{Take}}$ $w \overset{\text{def}}{=} u_h - v \in V_h$ (for arbitrary $v \in V_h$)

$$\Rightarrow \quad \alpha \cdot \|u-u_h\|_V^2 \le a(u-u_h, u-u_h) = a(u-u_h, u-u_h) + \underbrace{a(u-u_h, w)}_{=0} \;!$$

$$\overset{\text{bilinear}}{=} a(u-u_h, u-u_h+w)$$

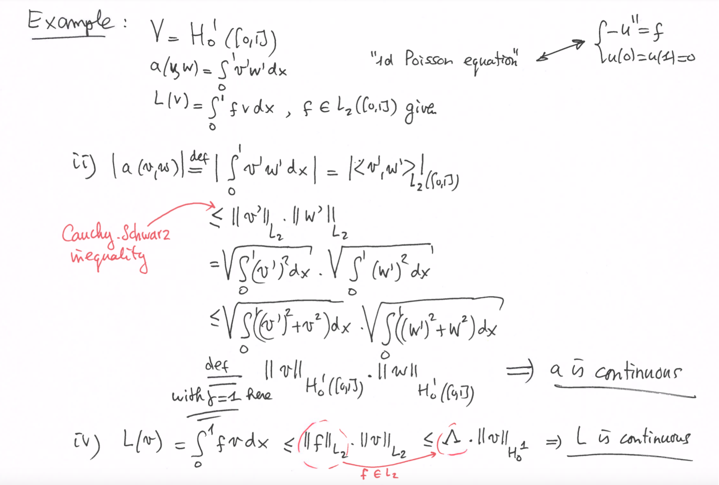
$$\overset{\text{def of } w}{=} a(u-u_h, u-v)$$

$$\le \gamma \cdot \|u-u_h\|_V \cdot \|u-v\|_V$$

$$\xrightarrow[u \neq u_h]{} \quad \|u-u_h\|_V \le \frac{\gamma}{\alpha} \|u-v\|_V \quad \forall v \in V_h$$

"best approximation in $\|\cdot\|_V$"

Abstract formulation [11]



Example: $V = H_0^1([0,1])$

$a(v,w) = \int_0^1 v'w'\,dx$

$L(v) = \int_0^1 f v\,dx$, $f \in L_2([0,1])$ given

"1d Poisson equation" $\leftrightarrow$ $\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$

iii) $|a(v,w)| \overset{def}{=} \left|\int_0^1 v'w'\,dx\right| = |\langle v', w'\rangle_{L_2([0,1])}|$

$\le \|v'\|_{L_2} \cdot \|w'\|_{L_2}$

Cauchy-Schwarz inequality

$= \sqrt{\int_0^1 (v')^2 dx} \cdot \sqrt{\int_0^1 (w')^2 dx}$

$\le \sqrt{\int_0^1 ((v')^2 + v^2) dx} \cdot \sqrt{\int_0^1 ((w')^2 + w^2) dx}$

$\underset{\text{with } \gamma = 1 \text{ here}}{\overset{def}{=}} \|v\|_{H_0^1([0,1])} \cdot \|w\|_{H_0^1([0,1])}$ $\Rightarrow$ a is continuous

iv) $L(v) = \int_0^1 f v\,dx \le \|f\|_{L_2} \cdot \|v\|_{L_2} \le \Lambda \cdot \|v\|_{H_0^1}$ $\Rightarrow$ L is continuous

$f \in L_2$

Abstract formulation [12]

iii) $|v(x)| = |\underbrace{v(0)}_{=0} + \int_0^x v'(y)\,dy| = |\int_0^x v'(y)\,dy| \le |\int_0^1 v'(y)\,dy| \le \underbrace{\sqrt{\int_0^1 1^2\,dy}}_{=1}\cdot\sqrt{\int_0^1 (v')^2\,dy}$

$v \in H_0^1$ $\quad$ $x \le 1$ $\quad$ Cauchy-Schwarz

take square and $\int$'s on both sides:

$$\int_0^1 |v(x)|^2\,dx \le \int_0^1\int_0^1 (v'(y))^2\,dy\,dx = \int_0^1 (v')^2\,dy$$

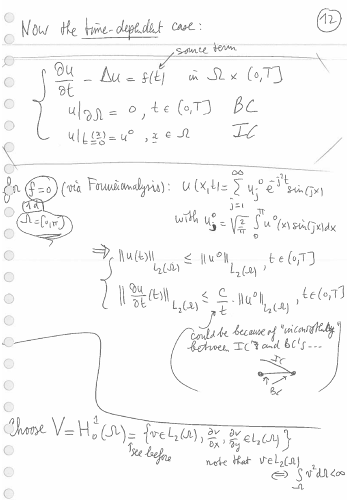
$$\Rightarrow \|v\|_{H_0^1}^2 \overset{def}{=} \int_0^1 v^2\,dx + \int_0^1 (v')^2\,dx \le 2\cdot\int_0^1 (v')^2\,dx$$

$$= 2\cdot a(v,v)$$

So, here $\alpha = \frac{1}{2}$ $\Rightarrow$ a is V-elliptic

$(V = H_0^1)$

Time-dependent case [1]



⑫

Now the *time-dependent* case:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t) & \text{in } \Omega \times (0,T] \\ u|_{\partial\Omega} = 0, \; t \in (0,T] & BC \\ u|_{t=0}(\underline{x}) = u^0, \; \underline{x} \in \Omega & IC \end{cases}$$

source term

for $\boxed{f=0}$ $\boxed{1d}$ $\boxed{\Omega=[0,\pi]}$ (via Fourieranalysis): $u(x,t) = \sum_{j=1}^{\infty} u_j^0 e^{-j^2 t} \sin(jx)$

with $u_j^0 = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u^0(x) \sin(jx) dx$

$$\Rightarrow \begin{cases} \|u(t)\|_{L_2(\Omega)} \le \|u^0\|_{L_2(\Omega)}, \; t \in (0,T] \\ \|\frac{\partial u}{\partial t}(t)\|_{L_2(\Omega)} \le \frac{c}{t} \cdot \|u^0\|_{L_2(\Omega)}, \; t \in (0,T] \end{cases}$$

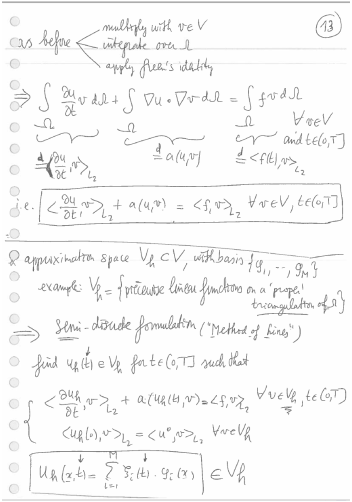
could be because of "inconsistency" between IC's and BC's ...

Choose $V = H_0^1(\Omega) = \{ v \in L_2(\Omega), \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L_2(\Omega) \}$

see before

note that $v \in L_2(\Omega)$
$\Leftrightarrow \int_\Omega v^2 d\Omega < \infty$

Time-dependent case [2]



⑬

as before: multiply with $v \in V$; integrate over Ω; apply Green's identity

$$\Rightarrow \underbrace{\int_\Omega \frac{\partial u}{\partial t} v \, d\Omega}_{\overset{d}{=}\langle \frac{\partial u}{\partial t}, v\rangle_{L_2}} + \underbrace{\int_\Omega \nabla u \cdot \nabla v \, d\Omega}_{\overset{d}{=} a(u,v)} = \underbrace{\int_\Omega f v \, d\Omega}_{\overset{d}{=} \langle f(t), v\rangle_{L_2}} \quad \forall v \in V \text{ and } t \in (0,T]$$

i.e. $$\boxed{\langle \frac{\partial u}{\partial t}, v\rangle_{L_2} + a(u,v) = \langle f, v\rangle_{L_2} \quad \forall v \in V, t \in (0,T]}$$

approximation space $V_h \subset V$, with basis $\{\varphi_1, \ldots, \varphi_M\}$

example: $V_h = \{$piecewise linear functions on a 'proper' triangulation of $\Omega\}$

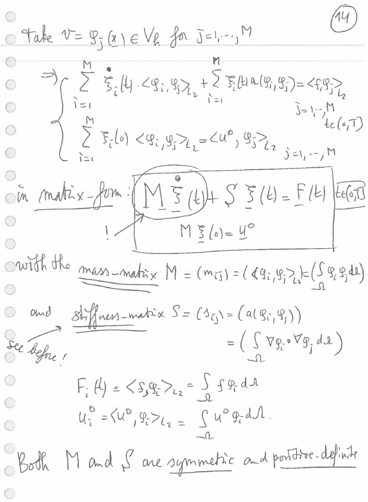
$\Rightarrow$ Semi-discrete formulation ("Method of Lines")

find $u_h(\overset{\downarrow}{t}) \in V_h$ for $t \in (0,T]$ such that

$$\begin{cases} \langle \frac{\partial u_h}{\partial t}, v\rangle_{L_2} + a(u_h(t), v) = \langle f, v\rangle_{L_2} & \forall v \in \underline{V_h}, t \in (0,T] \\ \langle u_h(0), v\rangle_{L_2} = \langle u^0, v\rangle_{L_2} & \forall v \in V_h \end{cases}$$

$$\boxed{u_h(x, \overset{\downarrow}{t}) = \sum_{i=1}^{M} \overset{\downarrow}{\xi_i}(t) \cdot \varphi_i(x)} \in V_h$$

Time-dependent case [3]



(14)

Take $v = \varphi_j(\underline{x}) \in V_h$ for $j = 1, \dots, M$

$$\Rightarrow \begin{cases} \sum_{i=1}^{M} \dot{\xi}_i(t) \cdot \langle \varphi_i, \varphi_j \rangle_{L_2} + \sum_{i=1}^{M} \xi_i(t)\, a(\varphi_i, \varphi_j) = \langle f, \varphi_j \rangle_{L_2} & j = 1, \dots, M \quad t \in (0,T) \\ \sum_{i=1}^{M} \xi_i(0) \langle \varphi_i, \varphi_j \rangle_{L_2} = \langle u^0, \varphi_j \rangle_{L_2} & j = 1, \dots, M \end{cases}$$

in matrix-form:

$$\boxed{\begin{matrix} \underline{\underline{M}}\, \dot{\underline{\xi}}(t) + \underline{\underline{S}}\, \underline{\xi}(t) = \underline{F}(t) \\ M\, \underline{\xi}(0) = \underline{u}^0 \end{matrix}} \quad \boxed{t \in (0,T]}$$

!

with the mass-matrix $M = (m_{ij}) = (\langle \varphi_i, \varphi_j \rangle_{L_2}) = (\int_\Omega \varphi_i \varphi_j \, d\Omega)$

and stiffness-matrix $S = (s_{ij}) = (a(\varphi_i, \varphi_j))$

see before!

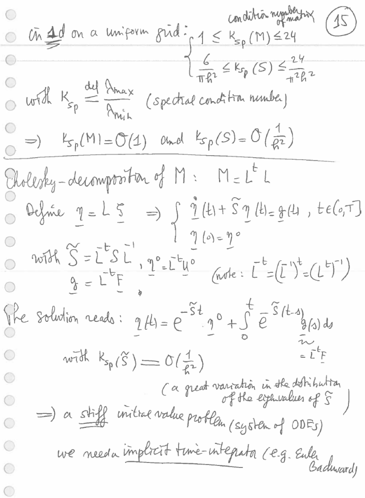
$$= \left(\int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega \right)$$

$$F_i(t) = \langle f, \varphi_i \rangle_{L_2} = \int_\Omega f \varphi_i \, d\Omega$$

$$u_i^0 = \langle u^0, \varphi_i \rangle_{L_2} = \int_\Omega u^0 \varphi_i \, d\Omega .$$

Both M and S are symmetric and positive-definite

Time-dependent case [4]



in $\underline{\underline{1}}$d on a uniform grid: $\begin{cases} 1 \le \kappa_{sp}(M) \le 24 \\ \frac{6}{\pi h^2} \le \kappa_{sp}(S) \le \frac{24}{\pi^2 h^2} \end{cases}$ condition number of matrix

with $\kappa_{sp} \overset{\text{def}}{=} \frac{\lambda_{max}}{\lambda_{min}}$ (spectral condition number)

$\Rightarrow$ $\kappa_{sp}(M) = O(1)$ and $\kappa_{sp}(S) = O\left(\frac{1}{h^2}\right)$

Cholesky-decomposition of M: $M = L^t L$

Define $\underline{\eta} = L\,\underline{\xi}$ $\Rightarrow$ $\begin{cases} \dot{\underline{\eta}}(t) + \tilde{S}\,\underline{\eta}(t) = \underline{g}(t), & t \in (0,T] \\ \underline{\eta}(0) = \underline{\eta}^0 \end{cases}$

with $\tilde{S} = L^{-t} S L^{-1}$, $\underline{\eta}^0 = L^{-t}\underline{u}^0$

$\underline{g} = L^{-t}\underline{F}$.

(note: $L^{-t} = (L^{-1})^t = (L^t)^{-1}$)

The solution reads: $\underline{\eta}(t) = e^{-\tilde{S}t}\,\underline{\eta}^0 + \int_0^t e^{-\tilde{S}(t-s)}\,\underbrace{\underline{g}(s)}_{=L^{-t}\underline{F}}\,ds$

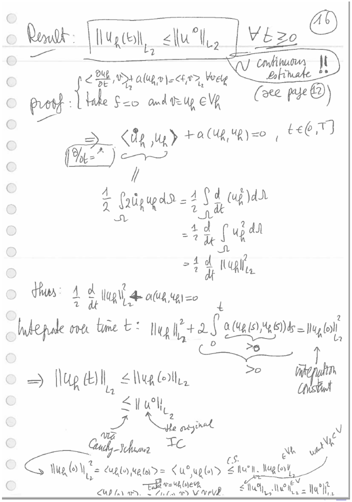
with $\kappa_{sp}(\tilde{S}) = O\left(\frac{1}{h^2}\right)$

(a great variation in the distribution of the eigenvalues of $\tilde{S}$)

$\Rightarrow$ a $\underline{\underline{\text{stiff}}}$ initial value problem (system of ODEs)

we need a <u>implicit time-integrator</u> (e.g. Euler Backward)

Time-dependent case [5]



(16)

Result: $\boxed{\|u_h(t)\|_{L_2} \leq \|u^0\|_{L_2}}$ $\forall t \geq 0$

$\sim$ continuous estimate !!

(see page (12))

proof: $\begin{cases} \langle \frac{\partial u_h}{\partial t}, v\rangle_{L_2} + a(u_h, v) = \langle f, v\rangle_{L_2} \; \forall v \in V_h \\ \text{take } f=0 \text{ and } v = u_h \in V_h \end{cases}$

$\Rightarrow \underbrace{\langle \dot{u}_h, u_h \rangle} + a(u_h, u_h) = 0, \quad t \in (0, T]$

$\boxed{\partial/\partial t = \dot{}}$

$$\| \\ \frac{1}{2}\int_\Omega 2\dot{u}_h u_h \, d\Omega = \frac{1}{2}\int_\Omega \frac{d}{dt}(u_h^2)\, d\Omega$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 \, d\Omega$$

$$= \frac{1}{2}\frac{d}{dt}\|u_h\|_{L_2}^2$$

thus: $\frac{1}{2}\frac{d}{dt}\|u_h\|_{L_2}^2 + a(u_h, u_h) = 0$

Integrate over time t: $\underbrace{\|u_h\|_{L_2}^2 + 2\int_0^t \underbrace{a(u_h(s), u_h(s))}_{>0} ds}_{>0} = \|u_h(0)\|_{L_2}^2$

↑ integration constant

$\Rightarrow \|u_h(t)\|_{L_2} \leq \|u_h(0)\|_{L_2}$

$\leq \|u^0\|_{L_2}$

via Cauchy-Schwarz

the original IC

$\|u_h(0)\|_{L_2}^2 = \langle u_h(0), u_h(0)\rangle = \langle u^0, u_h(0)\rangle \overset{C.S.}{\leq} \|u^0\|_{L_2} \cdot \|u_h(0)\|_{L_2}$

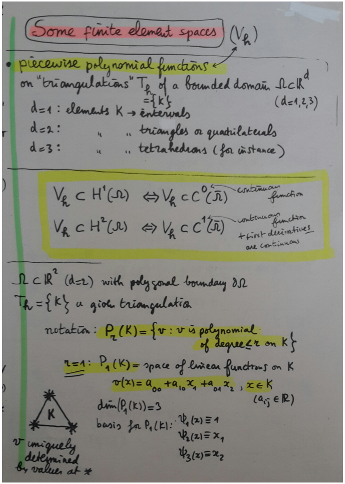
$\in V_h$ want $V_h \subset V$

Take $v = u_h(0) \in V_h$

$\langle u_h(0), v\rangle = \langle u(0), v\rangle \; \forall v \in V_h$

$\leq \|u^0\|_{L_2} \cdot \|u^0\|_{L_2} = \|u^0\|_{L_2}^2$ $\in V$

Examples [1]



Some finite element spaces (V_h)

- piecewise polynomial functions

on "triangulations" $T_h = \{K\}$ of a bounded domain $\Omega \subset \mathbb{R}^d$ $(d=1,2,3)$

$d=1$: elements $K \to$ intervals

$d=2$: " " triangles or quadrilaterals

$d=3$: " " tetrahedrons (for instance)

$V_h \subset H^1(\Omega) \Leftrightarrow V_h \subset C^0(\bar{\Omega})$ ← continuous function

$V_h \subset H^2(\Omega) \Leftrightarrow V_h \subset C^1(\bar{\Omega})$ ← continuous function + first derivatives are continuous

$\Omega \subset \mathbb{R}^2$ $(d=2)$ with polygonal boundary $\partial\Omega$

$T_h = \{K\}$ a given triangulation

notation: $P_r(K) = \{v : v \text{ is polynomial of degree} \leq r \text{ on } K\}$

$r=1$: $P_1(K)$ = space of linear functions on K

$v(x) = a_{00} + a_{10}x_1 + a_{01}x_2$, $x \in K$ $(a_{ij} \in \mathbb{R})$

$\dim(P_1(K)) = 3$

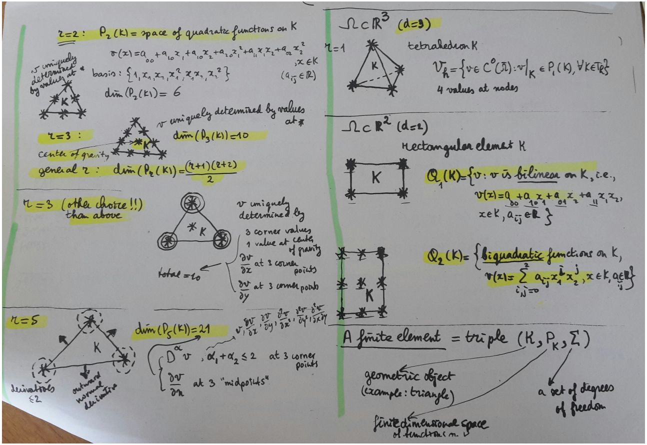
basis for $P_1(K)$: $\psi_1(x) \equiv 1$

$\psi_2(x) \equiv x_1$

$\psi_3(x) \equiv x_2$

v uniquely determined by values at $*$

Examples [2]



$r=2$: $P_2(K)$ = space of quadratic functions on K

$v(x)=a_{00}+a_{10}x_1+a_{10}x_2+a_{20}x_1^2+a_{11}x_1x_2+a_{02}x_2^2$, $x\in K$ $(a_{ij}\in\mathbb{R})$

basis: $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$

$\dim(P_2(K))=6$

v uniquely determined by values at *

$r=3$: v uniquely determined by values at *

$\dim(P_3(K))=10$

center of gravity

general r: $\dim(P_r(K))=\frac{(r+1)(r+2)}{2}$

$r=3$ (other choice!!) than above

v uniquely determined by

- 3 corner values
- 1 value at center of gravity
- $\frac{\partial v}{\partial x}$ at 3 corner points
- $\frac{\partial v}{\partial y}$ at 3 corner points

total = 10

$r=5$

$\dim(P_5(K))=21$

$v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial x\partial y}$

$D^\alpha v$, $\alpha_1+\alpha_2\leq 2$ at 3 corner points

$\frac{\partial v}{\partial n}$ at 3 "midpoints"

derivatives ≤ 2

outward normal derivative

$\Omega\subset\mathbb{R}^3$ $(d=3)$

$r=1$ tetrahedron K

$V_h=\{v\in C^0(\bar\Omega): v|_K\in P_1(K), \forall K\in T_h\}$

4 values at nodes

$\Omega\subset\mathbb{R}^2$ $(d=2)$

rectangular element K

$Q_1(K)=\{v: v$ is bilinear on K, i.e.,
$v(x)=a_{00}+a_{10}x_1+a_{01}x_2+a_{11}x_1x_2$, $x\in K$, $a_{ij}\in\mathbb{R}\}$

$Q_2(K)=\{$biquadratic functions on K,
$v(x)=\sum_{i,j=0}^{2}a_{ij}x_1^ix_2^j$, $x\in K$, $a_{ij}\in\mathbb{R}\}$

A finite element = triple (K, P_K, Σ)

geometric object (example: triangle)

finite dimensional space of functions on K

a set of degrees of freedom