1. Introduction to FEM for elliptic problems

In this chapter we introduce FEM for some elliptic model problems and study the basic properties of the method. We first consider a simple one-dimensional problem and then some two-dimensional generalizations.

1.1 Variational formulation of a one-dimensional model problem

Let us consider the two-point boundary value problem

$$(D) \qquad \begin{aligned} -u''(x) &= f(x) \quad \text{for } 0<x<1, \\ u(0) &= u(1) = 0, \end{aligned}$$

where $v' = \frac{dv}{dx}$ and f is a given continuous function. By integrating the equation $-u''=f$ twice, it is easy to see that this problem has a unique solution u. We recall that the boundary value problem (D) can be viewed as modelling, in particular, the following situations in continuum mechanics:

A An elastic bar

Consider an elastic bar fixed at both ends subject to a tangential load of intensity $f(x)$ (see Fig 1.1). Let $\sigma(x)$ and $u(x)$ be the traction and tangential displacement at x, respectively, under the load f. Under the assumption of small displacements and a linearly elastic material, we have in the interval $(0,1)$

$$\begin{aligned} \sigma &= Eu' && \text{(Hooke's law)}, \\ -\sigma' &= f && \text{(equilibrium equation)}, \\ u(0) &= u(1) = 0 && \text{(boundary conditions)}, \end{aligned}$$

where E is the modulus of elasticity. If we take here $E=1$ and eliminate σ, we obtain (D).

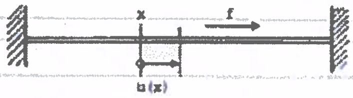

Fig 1.1

B An elastic cord

Consider an elastic cord with tension 1, fixed at both ends and subject to transversal load of intensity f (see Fig 1.2). Assuming again small displacements, we have that the transversal displacement u satisfies (D) (cf Problem 1.2).

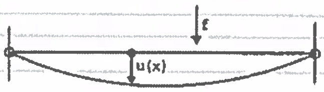

Fig 1.2

C Heat conduction

Let u be the temperature and q the heat flow in a heat conducting bar, subject to a distributed heat source of intensity f. Assuming the temperature to be zero at the end points, we have in the stationary case

$$\begin{aligned} -q &= ku' && \text{(Fourier's law)}, \\ q' &= f && \text{(conservation of energy)}, \\ u(0) &= u(1) = 0, \end{aligned}$$

where k is the heat conductivity, which again gives (D) if $k=1$.

We shall now show that the solution u of the boundary value problem or differential equation (D) also is the solution of a minimization problem (M) and a variational problem (V). To formulate the problems (M) and (V) we introduce the notation

$$(v,w)=\int_0^1 v(x)w(x)dx,$$

for real-valued piecewise continuous bounded functions. We also introduce the linear space

$$V=\{v:\ v \text{ is a continuous function on } [0,1],\ v' \text{ is piecewise continuous and bounded on } [0,1], \text{ and } v(0)=v(1)=0\},$$

and the linear functional $F: V\to R$ given by

$$F(v)=\frac{1}{2}(v',v')-(f,v).$$

The problems (M) and (V) are the following:

(M) Find $u\in V$ such that $F(u)\leq F(v)\quad \forall v\in V$,

(V) Find $u\in V$ such that $(u',v')=(f,v)\quad \forall v\in V$.

Let us notice that in the context of the problems A and B above, the quantity $F(v)$ represents the *total potential energy* associated with the displacement $v\in V$. The term $\frac{1}{2}(v',v')$ represents the internal elastic energy and (f,v) the load potential. Thus, the minimization problem (M) corresponds to the fundamental *Principle of minimum potential energy* in mechanics. Further the variational problem (V) corresponds to the *Principle of virtual work*.

Let us now first show that the solution u of (D) also is a solution of (V). To see this we multiply the equation $-u''=f$ by an arbitrary function $v\in V$, a so-called *test funtion* v, and integrate over the interval $(0,1)$ which gives

$$-(u'',v)=(f,v).$$

We now integrate the left-hand side by parts using the fact that $v(0)=v(1)=0$ to get

$$-(u'',v)=-u'(1)v(1)+u'(0)v(0)+(u',v')=(u',v'),$$

and we conclude that

$$(1.1)\qquad (u',v')=(f,v)\qquad \forall v\in V,$$

which shows that u is a solution of (V).

Next, we show that the problems (V) and (M) have the same solutions. Suppose then first that u is a solution to (V), let $v\in V$ and set $w=v-u$ so that $v=u+w$ and $w\in V$. We have

$$F(v)=F(u+w)=\frac{1}{2}(u'+w',u'+w')-(f,u+w)$$
$$=\frac{1}{2}(u',u')-(f,u)+(u',w')-(f,w)+\frac{1}{2}(w',w')\geq F(u),$$

since by (1.1), $(u',w')-(f,w)=0$ and $(w',w')\geq 0$, which shows that u is a solution of (M). On the other hand, if u is a solution of (M) then we have for any $v\in V$ and real number ε

$$F(u)\leq F(u+\varepsilon v),$$

since $u+\varepsilon v\in V$. Thus, the differentiable function

$$g(\varepsilon)\equiv F(u+\varepsilon v)=\frac{1}{2}(u',u')+\varepsilon(u',v')+\frac{\varepsilon^2}{2}(v',v')-(f,u)-\varepsilon(f,v),$$

has a minimum at $\varepsilon=0$ and hence $g'(0)=0$. But

$$g'(0)=(u',v')-(f,v),$$

and we see that u is a solution of (V).

Let us also show that a solution to (V) is uniquely determined. Suppose then that u_1 and u_2 are solutions of (V), i.e. $u_1, u_2\in V$ and

$$(u_1',v')=(f,v)\qquad \forall v\in V,$$
$$(u_2',v')=(f,v)\qquad \forall v\in V.$$

Subtracting these equations and choosing $v=u_1-u_2\in V$, we get

$$\int_0^1 (u_1'-u_2')^2\,dx=0,$$

which shows that

$$u_1'(x)-u_2'(x)=(u_1-u_2)'(x)=0\qquad \forall x\in[0,1].$$

It follows that $(u_1-u_2)(x)$ is constant on $[0,1]$ which together with the boundary condition $u_1(0)=u_2(0)=0$ gives $u_1(x)=u_2(x)$, $\forall x\in[0,1]$, and the uniqueness follows.

To sum up, we have shown that if u is the solution to (D), then u is the solution to the equivalent problems (M) and (V) which we write symbolically as

$$(D)\Rightarrow(V)\Leftrightarrow(M)$$

Let us finally also indicate how to see that if u is the solution of (V) then u also satisfies (D). Thus, we assume that $u\in V$ satisfies

$$\int_0^1 u'v'dx-\int_0^1 fvdx=0\qquad \forall v\in V$$

If we now assume in addition that u'' exists and is continuous, then we can integrate the first term by parts to get, using the fact that $v(0)=v(1)=0$,

$$-\int_0^1 (u''+f)v\,dx=0 \qquad \forall v\in V.$$

But with the assumption that $(u''+f)$ is continuous this relation can only hold if (cf Problem 1.1)

$$(u''+f)(x)=0 \qquad 0<x<1,$$

and it follows that u is the solution of (D).

Thus we have seen that if u is the solution of (V) and in addition satisfies a regularity assumpton (u'' is continuous), then u is the solution of (D). It is now possible to show that if u is the solution of (V), then u in fact satisfies the desired regularity assumption and thus we have $(V)\Rightarrow(D)$ which shows that the three problems (D), (V) and (M) are equivalent (cf Section 1.5 below).

Problems

1.1 Show that if w is continuous on $[0, 1]$ and

$$\int_0^1 wv\,dx=0 \qquad \forall v\in V,$$

then $w(x)=0$ for $x\in[0, 1]$.

1.2 Show that under suitable assumptions the problem B above can be given the formulation (1.1).

1.2 FEM for the model problem with piecewise linear functions

We shall now construct a finite-dimensional subspace V_h of the space V defined above consisting of piecewise linear functions. To this end let $0=x_0<x_1\ldots<x_M<x_{M+1}=1$, be a partition of the interval (0,1) into subintervals $I_j=(x_{j-1}, x_j)$ of length $h_j=x_j-x_{j-1}$, $j=1,\ldots, M+1$ and set $h=\max h_j$. The quantity h is then a measure of how fine the partition is. We now let V_h be the set of functions v such that v is linear on each subinterval I_j, v is continuous on [0,1] and $v(0)=v(1)=0$ (cf Fig 1.3).

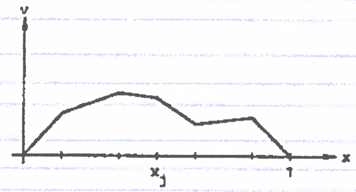

Fig 1.3 Example of a function $v\in V_h$

We observe that $V_h\subset V$. As parameters to describe a function $v\in V_h$ we may choose the values $\eta_j=v(x_j)$ at the node points x_j, $j=0,\ldots, M+1$. Let us introduce the *basis functions* $\varphi_j\in V_h$, $j=1,\ldots, M$, defined by

$$\varphi_j(x_i)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j,\ i, j=1,\ldots, M,\end{cases}$$

i.e., φ_j is the continuous piecewise linear function that takes the value 1 at node point x_j and the value 0 at other node points (see Fig 1.4).

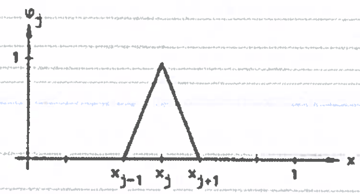

Fig 1.4 The basis function φ_j

A function $v\in V_h$ then has the representation

$$v(x)=\sum_{i=1}^{M}\eta_i\varphi_i(x), \qquad x\in[0,1],$$

where $\eta_i=v(x_i)$, i.e., each $v\in V_h$ can be written in a unique way as a *linear combination* of the basis functions φ_i. In particular it follows that V_h is a *linear space of dimension* M with *basis* $\{\varphi_i\}_{i=1}^{M}$.

The finite element method for the boundary value problem (D) can now be formulated as follows

(M_h) Find $u_h \in V_h$ such that $F(u_h) \leq F(v) \quad \forall v \in V_h$.

In the same way as above for the problems (M) and (V), we see that (M_h) is equivalent to the finite-dimensional variational problem (V_h): Find $u_h \in V_h$ such that

$$(1.2) \qquad (u_h', v') = (f, v) \qquad \forall v \in V_h.$$

Thus the finite element method for (D) can be formulated as (V_h) or equivalently (M_h). The problem (V_h) is usually referred to as *Galerkin's method* and (M_h) as *Ritz' method*. We observe that if $u_h \in V_h$ satisfies (1.2), then in particular

$$(1.3) \qquad (u_h', \varphi_j') = (f, \varphi_j) \qquad j = 1, \ldots, M,$$

and if these equations hold, then by taking linear combinations, we see that u_h satisfies (1.2). Since

$$u_h(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x), \qquad \xi_j = u_h(x_j),$$

we can write (1.3)

$$(1.4) \qquad \sum_{i=1}^{M} \xi_i (\varphi_i', \varphi_j') = (f, \varphi_j) \qquad j = 1, \ldots, M,$$

which is a linear system of equations with M equations in M unknowns $\xi_1, \ldots, \xi_M$. In matrix form the linear system (1.4) can be written as

$$(1.5) \qquad A\xi = b,$$

where $A = (a_{ij})$ is the $M \times M$ matrix with elements $a_{ij} = (\varphi_i', \varphi_j')$, and where $\xi = (\xi_1, \ldots, \xi_M)$ and $b = (b_1, \ldots, b_M)$ with $b_i = (f, \varphi_i)$ are M-vectors:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MM} \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_M \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

The matrix A is called the *stiffness matrix* and b the *load vector*, with terminology from early applications of FEM in structural mechanics.

The elements $a_{ij} = (\varphi_i', \varphi_j')$ in the stiffness matrix A can easily be computed. We first observe that $(\varphi_i', \varphi_j') = 0$ if $|i-j| > 1$ since in this case for all $x \in [0, 1]$ either $\varphi_i(x)$ or $\varphi_j(x)$ is equal to zero. Thus, the matrix A is tri-diagonal, i.e., only the elements in the main diagonal and the two adjoining diagonals may be different from zero. We have for $j = 1, \ldots, M$,

$$(\varphi_j', \varphi_j') = \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2}\,dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_{j+1}^2}\,dx = \frac{1}{h_j} + \frac{1}{h_{j+1}},$$

and for $j = 2, \ldots, M$,

$$(\varphi_j', \varphi_{j-1}') = (\varphi_{j-1}', \varphi_j') = -\int_{x_{j-1}}^{x_j} \frac{1}{h_j^2}\,dx = -\frac{1}{h_j}$$

Note also that the matrix A is *symmetric* and *positive definite* since $(\varphi_i', \varphi_j') = (\varphi_j', \varphi_i')$ and with $v(x) = \sum_{j=1}^{M} \eta_j \varphi_j(x)$, we have

$$\sum_{i,j=1}^{M} \eta_i (\varphi_i', \varphi_j') \eta_j = \left(\sum_{i=1}^{M} \eta_i \varphi_i', \sum_{j=1}^{M} \eta_j \varphi_j'\right) = (v', v') \geq 0,$$

with equality only if $v' \equiv 0$, that is since $v(0) = 0$ only if $v \equiv 0$, or $\eta_j = 0$ for $j = 1, \ldots, M$. We recall that a symmetric $M \times M$ matrix $S = (s_{ij})$ is said to be positive definite if

$$\eta \cdot S\eta = \sum_{i,j=1}^{M} \eta_i s_{ij} \eta_j > 0 \qquad \forall \eta \in R^M, \ \eta \neq 0,$$

where the dot denotes the scalar product in R^M. We also recall that a symmetric matrix S is positive definite if and only if the eigenvalues of S are strictly positive.

Since a positive definite matrix is non-singular it follows that the linear system (1.5) has a unique solution. We also note that A is *sparse*, i.e., only a few elements of A are different from zero (A is tridigagonal). This very important property depends, as we have seen, on the fact that a basis function φ_j of V_h is different from zero only on a few intervals and thus will interfere only with a few other basis functions. The fact that the basis functions may be chosen in this way is an important distinctive feature of the finite element method.

In the special case of a uniform partition with $h_j = h = \frac{1}{M+1}$ the system (1.5) takes the form

$$(1.6) \qquad \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \\ \\ \\ \\ \xi_M \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ \\ \\ \\ b_M \end{bmatrix}$$

After division by h this may be interpreted as a variant of a standard difference method for (D) where the elements of the right hand side b_j/h are mean values of f over the intervals (x_{j-1}, x_{j+1}) (cf Problem 1.4).

To sum up, we have seen that the finite element method (V_h) for (D) leads to a linear system of equations with a sparse, symmetric and positive definite stiffness matrix.

Problems

1.3 Construct a finite-dimensional subspace V_h of V consisting of functions which are quadratic on each subinterval I_j of a partition of $I=(0, 1)$. How can one choose the parameters to describe such functions? Find the corresponding basis functions. Then formulate a finite element method for (D) using the space V_h and write down the corresponding linear system of equations in case of a uniform partition.

1.4 Formulate a difference method for (D) and compare with (1.6).

1.5 Consider the boundary value problem

$$\frac{d^4u}{dx^4}=f, \qquad 0<x<1, \tag{1.7}$$
$$u(0)=u'(0)=u(1)=u'(1)=0.$$

Here u represents e g the deflection of a clamped beam subject to a transversal force with intensity f (see Fig 1.5).

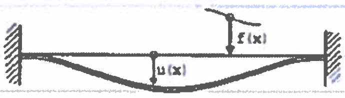

Fig 1.5

(a) In mechanics this beam problem would naturally be formulated as follows:

(1.8a) $M=u''$, $0<x<1$,
(1.8b) $M''=f$, $0<x<1$,
(1.8c) $u(0)=u'(0)=u(1)=u'(1)=0$.

What does here the quanity M represent and what is the mechanical interpretation of (1.8a–c)?

(b) Show that the problem (1.7) can be given the following variational formulation: Find $u\in W$ such that

$$(u'', v'')=(f, v) \quad \forall v\in W,$$

where $W=\{v: v$ and v' are continuous on $[0,1]$, v'' is piecewise continuous and $v(0)=v'(0)=v(1)=v'(1)=0\}$.

(c) For $I=[a, b]$ an interval, define

$$P_3(I)=\{v: v \text{ is a polynomial of degree} \leq 3 \text{ on } I, \text{ i e, } v \text{ has the form } v(x)=a_3x^3+a_2x^2+a_1x+a_0,\ x\in I \text{ where } a_i\in R\}.$$

Show that $v\in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a)=1$, $v'(a)=0$, $v(b)=v'(b)=0$, etc).

(d) Starting from (c) construct a finite-dimensional subspace W_h of W consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions.

(e) Formulate a finite element method for (1.7) based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition. Determine the solution in e g the case of two intervals and f constant. Compare with the exact solution.

1.3 An error estimate for FEM for the model problem

We shall now study the error $u-u_h$ where u is the solution of (D) and u_h is the solution of the finite element problem (V_h), i e, $u_h\in V_h$ and u_h satisfies (1.2). The proof is based on the following equation for the error:

$$((u-u_h)', v')=0 \qquad \forall v\in V_h. \tag{1.9}$$

This follows by recalling that $(u', v')=(f,v)$, $\forall v\in V$, so that in particular since $V_h\subset V$

$$(u', v')=(f, v) \qquad \forall v\in V_h. \tag{1.10}$$

Subtracting (1.2) from (1.10) we obtain (1.9).

We shall use the notation

$$\|w\| = (w, w)^{1/2} = \left(\int_0^1 w^2 dx\right)^{1/2},$$

$\|\cdot\|$ is the *norm* associated with the *scalar product* $(\cdot,\cdot)$. We also recall *Cauchy's inequality*

(1.11) $\quad |(v, w)| \leq \|v\| \, \|w\|.$

We shall prove the following estimate for $u - u_h$ which shows that in a certain sense u_h is the best possible approximation to the exact solution u.

Theorem 1.1. For any $v \in V_h$ we have

$$\|(u-u_h)'\| \leq \|(u-v)'\|.$$

Proof Let $v \in V_h$ be arbitrary and set $w = u_h - v$. Then $w \in V_h$ and using (1.9) with v replaced by w, we get, using Cauchy's inequality also,

$$\begin{aligned}\|(u-u_h)'\|^2 &= ((u-u_h)', (u-u_h)') + ((u-u_h)', w') \\ &= ((u-u_h)', (u-u_h+w)') = ((u-u_h)', (u-v)') \\ &\leq \|(u-u_h)'\| \, \|(u-v)'\|.\end{aligned}$$

Dividing by $\|(u-u_h)'\|$ we obtain the statement of the theorem (if $\|(u-u_h)'\| = 0$, then the theorem clearly holds). □

From Theorem 1.1 we can obtain a quantitative estimate for the error $\|(u-u_h)'\|$ by estimating $\|(u-\tilde{u}_h)'\|$ where $\tilde{u}_h \in V_h$ is a suitably chosen function. We shall choose $\tilde{u}_h \in V_h$ to be the *interpolant* of u, i.e. $\tilde{u}_h$ interpolates u at the nodes x_j, i.e.,

$$\tilde{u}_h(x_j) = u(x_j) \qquad j = 0, \ldots, M+1.$$

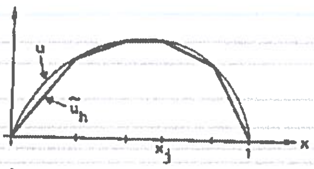

Fig 1.6 The interpolant $\tilde{u}_h$.

It is easy to see (cf any basic course in numerical analysis or Problem 4.1 below) that if $\tilde{u}_h \in V_h$ is chosen in this way, then for $0 \leq x \leq 1$,

(1.12) $\quad |u'(x) - \tilde{u}_h'(x)| \leq h \max_{0 \leq y \leq 1} |u''(y)|,$

(1.13) $\quad |u(x) - \tilde{u}_h(x)| \leq \frac{h^2}{8} \max_{0 \leq y \leq 1} |u''(y)|.$

Using (1.12) and Theorem 1.1 we now obtain the following estimate for the derivative of the error $u - u_h$:

(1.14) $\quad \|(u-u_h)'\| \leq h \max_{0 \leq y \leq 1} |u''(y)|.$

Since $(u-u_h)(0) = 0$ we obtain from (1.14) by integration the following estimate for the error itself (cf Problem 1.6)

(1.15) $\quad |u(x) - u_h(x)| \leq h \max_{0 \leq y \leq 1} |u''(y)| \quad$ for $0 \leq x \leq 1$.

We observe that this latter estimate is less sharp than the estimate (1.13) for the interpolation error where we have a factor h^2. With a more precise analysis it is possible to show that in fact also the finite element method gives a factor h^2 for the error $u - u_h$ (cf also Problem 1.19 below).

Let us note that the quantity u', representing a deformation or a force in Examples A and B above, is usually of more (or at least no less) practical interest than the quantity u itself, representing in these cases a displacement. Thus the estimate (1.14) is of independent interest and not just a step on the way to an estimate of $u - u_h$.

Let us also notice that to prove (1.14) we do not need to concretely construct $\tilde{u}_h$ (which would require knowledge of the exact solution u); we only have to be able to give an estimate of the interpolation error, for instance of the form (1.12), (1.13).

To sum up, by Theorem 1.1 we have the qualitative information that $\|(u-u_h)'\|$ is "as small as possible" and by using also the interpolation estimate (1.12) we obtain the quantitative error estimate (1.14), which in particular shows that the error tends to zero as the maximum length of the subintervals I_j tends to zero if u'' is bounded on [0,1].

Problem

1.6 Prove (1.15) using (1.14) and the boundary conditions $u(0) = u_h(0) = 0$. Hint: Use the relation

$$(u-u_h)(x) = \int_0^x (u-u_h)'(y)\,dy$$

together with Cauchy's inequality.

1.4 FEM for the Poisson equation

We will now consider the following boundary value problem for the Poisson equation:

(1.16a) $\quad -\Delta u = f \quad \text{in } \Omega,$

(1.16b) $\quad u = 0 \quad \text{on } \Gamma,$

where Ω is a bounded open domain in the plane $R^2 = \{x = (x_1, x_2) : x_i \in R\}$ with boundary Γ, f is a given function and as usual,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

A number of problems in physics and mechanics are modelled by (1.16); u may represent for instance a temperature, an electro-magnetic potential or the displacement of an elastic membrane fixed at the boundary under a transversal load of intensity f (see Fig 1.7 and compare also with problem B of Section 1.1).

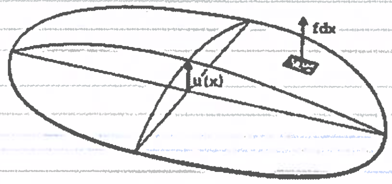

Fig 1.7

Let us now before continuing recall a certain *Green's formula* which will be of fundamental importance in what follows. Let us start from the *divergence theorem* (in two dimensions):

$$\int_\Omega \operatorname{div} A \, dx = \int_\Gamma A \cdot n \, ds,$$

where $A = (A_1, A_2)$ is a vector-valued function defined on Ω,

$$\operatorname{div} A = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2},$$

and $n = (n_1, n_2)$ is the outward unit normal to Γ. Here dx denotes the element of area in R_2 and ds the element of arc length along Γ. If we apply the divergence theorem to $A = (vw, 0)$ and $A = (0, vw)$, we find that

(1.17) $$\int_\Omega \frac{\partial v}{\partial x_i} w \, dx + \int_\Omega v \frac{\partial w}{\partial x_i} dx = \int_\Gamma v w n_i \, ds, \quad i = 1, 2.$$

Denoting by ∇v the *gradient* of v, i.e. $\nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\right)$, we get from (1.17) the following Green's formula:

$$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Omega \left[\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}\right] dx$$

$$= \int_\Gamma \left[v \frac{\partial w}{\partial x_1} n_1 + v \frac{\partial w}{\partial x_2} n_2\right] ds - \int_\Omega v \left[\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right] dx$$

$$= \int_\Gamma v \frac{\partial w}{\partial n} ds - \int_\Omega v \Delta w \, dx,$$

i.e.

(1.18) $$\int_\Omega \nabla v \cdot \nabla w \, dx = \int_\Gamma v \frac{\partial w}{\partial n} ds - \int_\Omega v \Delta w \, dx,$$

where

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2$$

is the *normal derivative*, i.e. the derivative in the outward normal direction to the boundary Γ.

We shall now give a variational formulation of problem (1.16). We shall first show that if u satisfies (1.16), then u is the solution of the following variational problem: Find $u \in V$ such that

(1.19) $\quad a(u, v) = (f, v) \quad \forall v \in V,$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \left[\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right] dx,$$

$$(f, v) = \int_\Omega f v \, dx,$$

$$V = \{v : v \text{ is continuous on } \Omega, \frac{\partial v}{\partial x_1} \text{ and } \frac{\partial v}{\partial x_2} \text{ are piecewise continuous on } \Omega \text{ and } v = 0 \text{ on } \Gamma\}.$$

In exactly the same way as in Section 1.1, we see that $u \in V$ satisfies (1.19) if and only if u is the solution of the following minimization problem: Find $u \in V$ such that $F(u) \le F(v)$, $\forall v \in V$, where $F(v)$ is the total potential energy

$$F(v) = \frac{1}{2} a(v, v) - (f, v).$$

To see that (1.19) follows from (1.16) we multiply (1.16a) with an arbitrary test function $v \in V$ and integrate over Ω. According to Green's formula (1.18) we then have

$$(f, v) = -\int_\Omega \Delta u \, v \, dx = -\int_\Omega \frac{\partial u}{\partial n} v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = a(u, v),$$

where the boundary integral vanishes since $v=0$ on Γ. On the other hand, if $u \in V$ satisfies (1.19) and u is sufficiently regular, then we see as in Section 1.1 that u also satisfies (1.16) (cf Problem 1.10).

Let us now construct a finite dimensional subspace V_h of V. For simplicity we shall assume that Γ is a polygonal curve, in which case we say that Ω is a polygonal domain (if Γ is curved we may first approximate Γ with a polygonal curve, see Chapter 12). Let us now make a *triangulation* of Ω, by subdividing Ω into a set $T_h = K_1, \ldots, K_m$ of non-overlapping triangles K_i,

$$\Omega = \bigcup_{K \in T_h} K = K_1 \cup K_2 \ldots \cup K_m,$$

such that no vertex of one triangle lies on the edge of another triangle (see Fig 1.8)

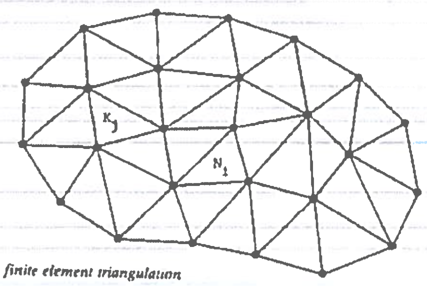

Fig 1.8 A finite element triangulation

We introduce the mesh parameter

$$h = \max_{K \in T_h} \operatorname{diam}(K), \quad \operatorname{diam}(K) = \text{diameter of } K = \text{longest side of } K.$$

We now define V_h as follows:

$$V_h = \{v : v \text{ is continuous on } \Omega, \; v|_K \text{ is linear for } K \in T_h, \; v=0 \text{ on } \Gamma\}.$$

Here $v|_K$ denotes the restriction of v to K, i.e. the function defined on K agreeing with v on K. The space V_h consists of all continuous functions that are linear on each triangle K and vanish on Γ. We notice that $V_h \subset V$. As parameters to describe a function $v \in V_h$ we choose the values $v(N_i)$ of v at the *nodes* N_i, $i=1, \ldots, M$, of T_h (see Fig 1.8) but exclude the nodes on the boundary since $v=0$ on Γ. The corresponding basis functions $\varphi_j \in V_h$, $j=1, \ldots, M$, are then defined by (see Fig 1.9)

$$\varphi_j(N_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad i, j = 1, \ldots, M$$

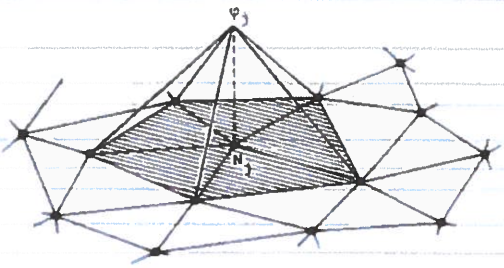

Fig 1.9 The basis function φ_j.

We see that the *support* of φ_j (the set of points x for which $\varphi_j(x) \neq 0$) consists of the triangles with the common node N_j (the shaded area in Fig 1.9). A function $v \in V_h$ now has the representation

$$v(x) = \sum_{j=1}^{M} \eta_j \varphi_j(x), \quad \eta_j = v(N_j), \text{ for } x \in \Omega \cup \Gamma.$$

We can now formulate the following finite element method for (1.16) starting from the variational formulation (1.19): Find $u_h \in V_h$ such that

(1.20) $\quad a(u_h, v) = (f, v) \quad \forall v \in V_h$

Exactly as in Section 1.2 we see that (1.20) is equivalent to the linear system of equations

(1.21) $\quad A\xi = b,$

where $A=(a_{ij})$, the stiffness matrix, is an $M\times M$ matrix with elements $a_{ij}=a(\varphi_i, \varphi_j)$ and $\xi=(\xi_i)$, $b=(b_i)$ are M-vectors with elements $\xi_i=u_h(N_i)$, $b_i=(f, \varphi_i)$.

Clearly A is symmetric and as in Section 1.2 we see that A is positive definite and thus in particular non-singular so that (1.21) admits a unique solution ξ. Moreover, A is again sparse; we have that $a_{ij}=0$ unless N_i and N_j are nodes of the same triangle.

In the same way as in Section 1.2 we realize that $u_h\in V_h$ is the best approximation of the exact solution u in the sense that

(1.22) $$\|\nabla u-\nabla u_h\|\leq\|\nabla u-\nabla v\| \qquad \forall v\in V_h,$$

where

$$\|\nabla v\|=a(v,v)^{1/2}=(\int_\Omega|\nabla v|^2dx)^{1/2}.$$

In particular we have

(1.23) $$\|\nabla u-\nabla u_h\|\leq\|\nabla u-\nabla \tilde{u}_h\|,$$

where $\tilde{u}_h$ is the interpolant of u, i.e. $\tilde{u}_h\in V_h$ and

$$\tilde{u}_h(N_i)=u(N_i) \qquad i=1, \ldots, M.$$

In Chapter 4 we prove that if the triangles $K\in T_h$ are not allowed to become too thin, then

(1.24) $$\|\nabla u-\nabla \tilde{u}_h\|\leq Ch.$$

Here and below we denote by C a positive constant, possibly different at different occurences, that does not depend on the mesh parameter h. In the case (1.24) the constant C depends on the size of the second partial derivatives of u and the smallest angle of the triangles $K\in T_h$. One can also prove (see Section 4.7) that

$$\|u-u_h\|=(\int_\Omega(u-u_h)^2dx)^{1/2}\leq Ch^2$$

with a similar dependence of C. In particular these estimates show that if the exact solution u is sufficiently regular, then the error and the gradient of the error $u-u_h$ tend to zero in the norm $\|\cdot\|$ as h tends to zero.

Example 1.1. Let Ω be a square with side length 1 and let T_h be the uniform triangulation of Ω according to Fig 1.10 with the indicated enumeration of the nodes of T_h.

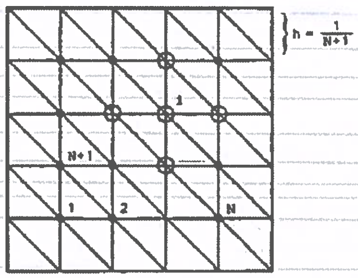

Fig 1.10

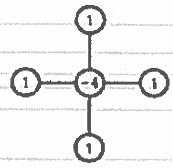

Fig 1.11

In this case the linear system (1.21) reads as follows:

(1.25) $$\text{row } N+1\quad \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & \cdot & \cdot & 0\\ -1 & 4 & -1 & 0 & -1 & 0 & \cdot & \cdot\\ 0 & -1 & 4 & -1 & 0 & -1 & \cdot & \cdot\\ -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0\\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1\\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & -1 & 0\\ \cdot & \cdot & \cdot & -1 & 0 & -1 & 4 & -1\\ 0 & \cdot & \cdot & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} \xi_1\\ \\ \cdot\\ \cdot\\ \cdot\\ \\ \\ \xi_M \end{bmatrix} = \begin{bmatrix} b_1\\ \\ \cdot\\ \cdot\\ \cdot\\ \\ \\ b_M \end{bmatrix}.$$

Note that here the left-hand side of equation i is a linear combination of the values of u_h at the 5 nodes indicated in Fig 1.10 with coefficients given in Fig 1.11. Dividing by h^2 we recognize this as the linear system obtained by applying the so-called 5-point difference method for (1.16) with the components of the right-hand side being weighted averages of f around the nodes N_i (cf Problem 1.7 below). □

The elements $a_{ij}=a(\varphi_i, \varphi_j)$ in the stiffness matrix A are usually in practice computed by summing the contributions from the different triangles

(1.26) $$a(\varphi_i, \varphi_j)=\sum_{K\in T_h} a_K(\varphi_i, \varphi_j),$$

where

$$a_K(\varphi_i, \varphi_j)=\int_K\nabla\varphi_i\cdot\nabla\varphi_j dx.$$

We notice that $a_K(\varphi_i, \varphi_j)=0$ unless both nodes N_i and N_j are vertices of K.

Let N_i, N_j and N_k be the vertices of the triangle K. We call the 3×3-matrix

$$(1.27)\qquad \begin{bmatrix} a_K(\varphi_i, \varphi_i) & a_K(\varphi_i, \varphi_j) & a_K(\varphi_i, \varphi_k) \\ & a_K(\varphi_j, \varphi_j) & a_K(\varphi_j, \varphi_k) \\ \text{sym} & & a_K(\varphi_k, \varphi_k) \end{bmatrix}$$

the *element stiffness matrix* for K. The global stiffness matrix A may thus be computed by first computing the element stiffness matrices for each $K\in T_h$ and then summing the contributions from each triangle according to (1.26). In a corresponding way we compute the right-hand side b. This process of computing A and b by summation is called the *assembly* of A and b.

To compute the elements in the stiffness matrix (1.27) we clearly work with the restrictions of the basis functions φ_i, φ_j and φ_k to the triangle K. Denoting these restrictions by ψ_i, ψ_j and ψ_k, we have that each ψ is a linear function on K that takes the value one at one vertex and vanishes at the other two vertices of K. We call ψ_i, ψ_j and ψ_k the *basis functions on the triangle* K, cf Fig 1.12. If w is a linear function on K, then w has the representation

$$w(x)=w(N_i)\psi_i(x)+w(N_j)\psi_j(x)+w(N_k)\psi_k(x), \qquad x\in K.$$

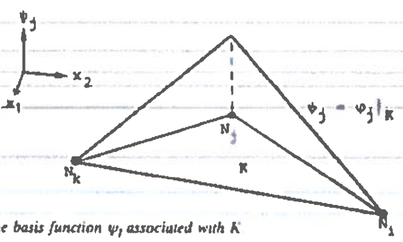

Fig 1.12 The basis function ψ_j associated with K

Problems

1.7 Formulate a difference method for (1.16) in the case when Ω is a square using the difference approximation

$$\frac{\partial^2 u}{\partial x_1^2}(x_1, x_2) \sim \frac{u(x_1+h, x_2)-2u(x_1, x_2)+u(x_1-h, x_2)}{h^2},$$

and a corresponding approximation for $\frac{\partial^2 u}{\partial x_2^2}$. Compare with Example 1.1.

1.8 Find the linear basis functions for the triangle K with vertices at (0, 0), (h, 0) and (0, h). Show that the corresponding element stiffness matrix (1.27) is given by

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Using this result show that the linear system (1.25) of Example 1.1 has the stated form.

1.9 Find the element stiffness matrix (1.27) for a general triangle K in terms of the coordinates $a^i=(a_1^i, a_2^i)$, $i=1, 2, 3$, of the vertices of K.

1.10 Show that if $u\in V$ satisfies (1.19) and u is twice continuously differentiable, then u satisfies (1.16).

1.11 Find the element stiffness matrix for the problem

$$-u''=f \quad \text{for } 0<x<1, \quad u(0)=u(1)=0,$$

if we use piecewise quadratic functions according to Problem 1.3. Then determine the corresponding global stiffness matrix in the case of a uniform subdivision. Can you interpret the resulting equations as difference approximations of the equation $-u''=f$?

1.5 The Hilbert spaces $L_2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$

When giving variational formulations of boundary value problems for partial differential equations, it is from the mathematical point of view natural and very useful to work with function spaces V that are slightly larger (i.e. contain somewhat more functions) than the spaces of continuous functions with piecewise continuous derivatives used in the preceeding sections. It is also useful to endow the spaces V with various *scalar products* with the scalar product related to the boundary value problem. More precisely, V will be a *Hilbert space* (see below).

Before introducing these Hilbert spaces let us recall a few simple concepts from linear algebra. If V is a linear space, then we say that L is a *linear form* on V if $L: V \rightarrow R$, i.e., $L(v) \in R$ for $v \in V$, and L is *linear*, i.e., for all $v, w \in V$ and $\beta, \theta \in R$

$$L(\beta v+\theta w)=\beta L(v)+\theta L(w).$$

Furthermore, we say that $a(\cdot,\cdot)$ is a *bilinear form* on $V \times V$ if $a: V \times V \rightarrow R$, i.e., $a(v, w) \in R$ for $v, w \in V$, and a is linear in each argument, i.e., for all $u, v, w \in V$ and $\beta, \theta \in R$ we have

$$a(u, \beta v+\theta w)=\beta a(u, v)+\theta a(u, w),$$

$$a(\beta u+\theta v, w)=\beta a(u, w)+\theta a(v, w).$$

The bilinear form $a(\cdot,\cdot)$ on $V \times V$ is said to be *symmetric* if

$$a(v,w)=a(w,v) \quad \forall v, w \in V.$$

A symmetric bilinear form $a(\cdot,\cdot)$ on $V \times V$ is said to be a *scalar product* on V if

$$a(v, v)>0 \quad \forall v \in V, v \neq 0.$$

The *norm* $\|\cdot\|_a$ associated with a scalar product $a(\cdot,\cdot)$ is defined by

$$\|v\|_a=(a(v, v))^{1/2}, \quad \forall v \in V.$$

Further, if $<\cdot,\cdot>$ is a scalar product with corresponding norm $\|\cdot\|$, then we have *Cauchy's inequality*

$$|<v, w>| \leq \|v\| \, \|w\|. \qquad (1.28)$$

We further recall that if V is a linear space with a scalar product with corresponding norm $\|\cdot\|$, then V is said to be a *Hilbert space* if V is complete, i.e., if every *Cauchy sequence* with respect to $\|\cdot\|$ is convergent. We recall that a sequence $v_1, v_2, v_3, \ldots$, of elements v_i in the space V with norm $\|\cdot\|$ is said to be a Cauchy sequence if for all $\varepsilon>0$ there is a natural number N such that $\|v_i-v_j\|<\varepsilon$ if $i, j>N$. Further, v_i converges to v if $\|v-v_i\| \rightarrow 0$ as $i \rightarrow \infty$. The reader unfamiliar with the concept of completeness may bypass this remark and think of a Hilbert space simply as a linear space with a scalar product.

We now introduce some Hilbert spaces that are natural to use for variational formulations of the boundary value problems we will consider. Let us start with the one-dimensional case. If $I=(a, b)$ is an interval, we define the space of "square integrable functions" on I:

$$L_2(I)=\{v: v \text{ is defined on } I \text{ and } \int_I v^2 dx<\infty\}.$$

The space $L_2(I)$ is a Hilbert space with the scalar product

$$(v, w)=\int_I vw \, dx,$$

and the corresponding norm (the L_2-norm):

$$\|v\|_{L_2(I)}=\left(\int_I v^2 dx\right)^{1/2}=(v, v)^{1/2}.$$

By Cauchy's inequality,

$$|(v, w)| \leq \|v\|_{L_2(I)} \|w\|_{L_2(I)},$$

we see that (v, w) is well-defined, i.e., the integral (v, w) exists, if v and $w \in L_2(I)$.

Remark. To really appreciate the definition of $L_2(I)$ and realize that this space is complete requires some familiarity with the Lebesgue integral. In this book, however, it is sufficient to get an idea of $L_2(I)$ by using the usual Riemann integral; from this point of view we may think of a "typical" function $v \in L_2(I)$ as a piecewise continuous function, possibly unbounded, such that $\int_I v^2 dx<\infty$. □

Example 1.2 We have that the function $v(x)=x^{-\beta}$, $x \in I=(0, 1)$ belongs to $L_2(I)$ if $\beta<\frac{1}{2}$. □

We also introduce the space $H^1(I)=\{v: v \text{ and } v' \text{ belong to } L_2(I)\}$, and we equip this space with the scalar product

$$(v,w)_{H^1(I)}=\int_I (vw+v'w')dx,$$

and the corresponding norm

$$\|v\|_{H^1(I)}=\left(\int_I [v^2+(v')^2]dx\right)^{1/2}.$$

The space $H^1(I)$ thus consists of the functions v defined on I which together with their first derivatives are square-integrable, i.e., belong to $L_2(I)$.

In the case of boundary value problems of the form $-u''=f$ on $I=(a, b)$ with boundary conditions $u(a)=u(b)=0$, we shall use the space

$$H_0^1(I)=\{v \in H^1(I): v(a)=v(b)=0\}$$

with the same scalar product and norm as for $H^1(I)$.

Our introductory boundary value problem

$$(1.29)\qquad \begin{aligned} -u'' &= f \quad \text{on } I=(0,1),\\ u(0) &= u(1)=0, \end{aligned}$$

can now be given the following variational formulation

$$(1.30)\qquad \text{Find } u\in H_0^1(I) \text{ such that } (u',v')=(f,v) \quad \forall v\in H_0^1(I),$$

with $(\cdot,\cdot)$ as in Section 1.1. If we compare (1.30) with the formulation (V) in Section 1.1, we note that the space $H_0^1(I)$ is larger than the space V used in the formulation (V). The space $H_0^1(I)$ is specially tailored for a variational formulation of (1.29) and is in fact the largest space for which a variational formulation of the form (1.30) is meaningful. From a mathematical point of view the "right" choice of function space is essential since this may make it easier to prove the existence of a solution to the continuous problem. From the finite element point of view the formulation (1.30) as opposed to (V) is of interest mainly because the basic error estimate for the finite element method is an estimate in the norm indicated by (1.30) (the $H^1(I)$-norm). Further, using the standard notation $L_2(I)$, $H^1(I)$, $H_0^1(I)$ etc. we may give our boundary value problems variational formulations in a concise way, as will be seen below.

Now let Ω be a bounded domain R^d, $d=2$ or 3, and define

$$L_2(\Omega)=\{v: v \text{ is defined on } \Omega \text{ and } \int_\Omega v^2dx<\infty\},$$

$$H^1(\Omega)=\{v\in L_2(\Omega): \frac{\partial v}{\partial x_i}\in L_2(\Omega),\ i=1,\ldots,d\},$$

and introduce the corresponding scalar products and norms

$$(v,w)=\int_\Omega vw\,dx,\quad \|v\|_{L_2(\Omega)}=(\int_\Omega v^2dx)^{1/2},$$

$$(v,w)_{H^1(\Omega)}=\int_\Omega [vw+\nabla v\cdot\nabla w]dx,$$

$$\|v\|_{H^1(\Omega)}=(\int_\Omega [v^2+|\nabla v|^2]dx)^{1/2}.$$

We also define

$$H_0^1(\Omega)=\{v\in H^1(\Omega): v=0 \text{ on } \Gamma\},$$

where Γ is the boundary of Ω and we equip $H_0^1(\Omega)$ with the same scalar product and norm as $H^1(\Omega)$.

The boundary value problem

$$(D)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

can now be given the following variational formulation:

$$(V)\qquad \text{Find } u\in H_0^1(\Omega) \text{ such that } a(u,v)=(f,v) \quad \forall v\in H_0^1(\Omega),$$

or equivalently

$$(M)\qquad \text{Find } u\in H_0^1(\Omega) \text{ such that } F(u)\leq F(v) \quad \forall v\in H_0^1(\Omega),$$

where

$$F(v)=\frac{1}{2}a(v,v)-(f,v),$$

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v\,dx,\quad (f,v)=\int_\Omega fv\,dx.$$

Remark. The formulation (V) is said to be a *weak formulation* of (D) and the solution of (V) is said to be a *weak solution* of (D). If u is a weak solution of (D) then it is not immediately clear that u is also a classical solution of (D), since this requires u to be sufficiently regular so that Δu is defined in a classical sense. The advantage mathematically of the weak formulation (V) is that it is easy to prove the existence of a solution to (V), whereas it is relatively difficult to prove the existence of a classical solution to (D). To prove the existence of a classical solution of (D) one usually starts with the weak solution of (D) and shows, often with considerable effort, that in fact this solution is sufficiently regular to be also a classical solution. For more complicated, e.g. non-linear problems, it may be extremely difficult or practically impossible to prove the existence of classical solutions whereas existence of weak solutions may still be within reach. □

Problems

1.12 Let $\Omega=\{x\in R^2: |x|\leq 1\}$. Show that the function $v(x)=|x|^\alpha$ belongs to $H^1(\Omega)$ if $\alpha>0$.

1.13 Prove Cauchy's inequality (1.28).

1.14 Consider the problem corresponding to (D) with an inhomogeneous boundary condition, i.e., the problem

$$(1.31)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= u_0 \quad \text{on } \Gamma, \end{aligned}$$

where f and u_0 are given. Show that this problem can be given the following equivalent variational formulations

(V) Find $u \in V(u_0)$ such that $a(u,v)=(f,v) \quad \forall v \in H_0^1(\Omega)$,
(M) Find $u \in V(u_0)$ such that $F(u) \leq F(v) \quad \forall v \in V(u_0)$,

where

$$V(u_0)=\{v \in H^1(\Omega): v=u_0 \text{ on } \Gamma\}.$$

Then formulate a finite element method for (1.31) and prove an error estimate.

1.6 A geometric interpretation of FEM

We shall now give an interpretation of the finite element method in geometric terms in the function space $H_0^1(\Omega)$. We recall that two elements v and w in a linear space with scalar product $<\cdot,\cdot>$ are said to be *orthogonal* if $<v, w>=0$.

Let us for simplicity consider the following variant of our previous problem (1.16):

$$\begin{aligned} -\Delta u+u&=f \quad \text{in } \Omega, \\ u&=0 \quad \text{on } \Gamma, \end{aligned} \qquad (1.32)$$

(cf Problem 2.5 below). The corresponding variational problem reads: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx+\int_\Omega uv\, dx=(f, v) \qquad \forall v \in H_0^1(\Omega),$$

or

$$<u, v>=(f, v) \qquad \forall v \in H_0^1(\Omega), \qquad (1.33)$$

using the notation

$$<u, v>=\int_\Omega (\nabla u \cdot \nabla v+uv) dx.$$

Note that $<\cdot,\cdot>$ is in fact the scalar product in the space $H_0^1(\Omega)$.

Let V_h be a finite-dimensional subspace of $H_0^1(\Omega)$, e.g. the space of piecewise linear functions of Section 1.4, and consider the following finite element method for (1.32): Find $u_h \in V_h$ such that

$$<u_h, v>=(f, v) \quad \forall v \in V_h \qquad (1.34)$$

Since $V_h \subset H_0^1(\Omega)$ we may choose $v \in V_h$ in (1.33) and on substraction from (1.34), we obtain

$$<u-u_h, v>=0 \quad \forall v \in V_h, \qquad (1.35)$$

i.e. the error $u-u_h$ is orthogonal to V_h with respect to $<\cdot,\cdot>$. We may also express this fact as follows: The finite element solution u_h is the *projection* with respect to $<\cdot,\cdot>$ of the exact solution u *on* V_h, i.e. u_h is the element in V_h closest to u with respect to the $H^1(\Omega)$-norm $\|\cdot\|_{H^1(\Omega)}$, or in other words

$$\|u-u_h\|_{H^1(\Omega)} \leq \|u-v\|_{H^1(\Omega)} \quad \forall v \in V_h \qquad (1.36)$$

This situation is symbolically illustrated in Fig 1.13 where $H_0^1(\Omega)$ is represented by the whole plane while the straight line through the origin represents V_h.

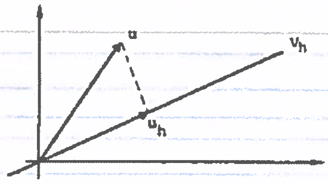

Fig 1.13

According to (1.36), u_h is the best approximation of the exact solution u, in the sense that for no other function $v \in V_h$, is the error $u-v$ smaller when measured in the $H^1(\Omega)$-norm. We have seen that u_h can be found by solving a linear system of equations with right hand side depending on the given function f. Thus, we can compute a best approximation u_h of u, without knowing u itself, knowing only that $-\Delta u+u=f$ in Ω and $u=0$ on Γ. This remarkable fact reflects the ellipticity of the boundary value problem (1.32).

Problem

1.15 Prove that (1.35) and (1.36) are equivalent (cf the proof of Theorem 1.1).

1.7 A Neumann problem. Natural and essential boundary conditions

We shall now consider a problem with another type of boundary condition, namely the following *Neumann problem* (*D*):

(1.37a) $\quad -\Delta u+u=f \quad$ in Ω

(1.37b) $\quad \frac{\partial u}{\partial n}=g \quad$ on Γ,

where again Ω is a bounded domain with boundary Γ and $\frac{\partial}{\partial n}$ denotes the outward normal derivative to Γ. The boundary condition is a *Neumann condition* while the boundary condition $u=u_0$ on Γ considered previously is said to be a *Dirichlet condition*. In mechanics or physics the Neumann condition (1.37b) corresponds to a given force or flow g on Γ.

We can give the problem (1.37) the following variational formulation (V): Find $u\in H^1(\Omega)$ such that

(1.38) $\quad a(u, v)=(f, v)+\langle g, v\rangle \quad \forall v\in H^1(\Omega)$,

where

$$a(u, v)=\int_\Omega[\nabla u\cdot\nabla v+uv]dx,\ (f, v)=\int_\Omega fv\ dx,\ \langle g, v\rangle=\int_\Gamma gv\ ds.$$

This is equivalent to the following minimization formulation (M): Find $u\in H^1(\Omega)$ such that $F(u)\le F(v)$, $\forall v\in H^1(\Omega)$, where

$$F(v)=\frac{1}{2}a(v, v)-(f, v)-\langle g, v\rangle.$$

To see that (1.38) follows from (1.37) we multiply (1.37a) with the test function $v\in H^1(\Omega)$ and integrate over Ω. According to Green's formula (1.18), we then get, since $\frac{\partial u}{\partial n}=g$ on Γ,

$$(f, v)=\int_\Omega(-\Delta u+u)v\ dx=-\int_\Gamma\frac{\partial u}{\partial n}v\ dx+\int_\Omega\nabla u\cdot\nabla v dx+\int_\Omega uv\ dx=$$

$$=-\langle g, v\rangle+\int_\Omega[\nabla u\cdot\nabla v+uv]dx=a(u, v)-\langle g, v\rangle,$$

which proves (1.38).

Let us now also motivate why a solution $u\in H^1(\Omega)$ of the variational problem (1.38) also should satisfy (1.37). Using Green's formula again we find from (1.38) that if u is sufficiently regular, then

$$(f, v)+\langle g, v\rangle=a(u, v)=\int_\Gamma\frac{\partial u}{\partial n}v\ dx+\int_\Omega(-\Delta u+u)v\ dx,$$

so that, rearranging terms,

(1.39) $\quad \int_\Omega(-\Delta u+u-f)v\ dx+\int_\Gamma(\frac{\partial u}{\partial n}-g)v\ ds=0 \quad \forall v\in H^1(\Omega)$.

Now, as (1.39) holds in particular for all v in $H_0^1(\Omega)$ and for these functions the boundary term vanishes, we conclude that (1.37a) holds, i.e.

$$-\Delta u+u-f=0 \quad \text{in } \Omega$$

Thus (1.39) is reduced to

$$\int_\Gamma(\frac{\partial u}{\partial n}-g)v\ ds=0 \qquad \forall v\in H^1(\Omega).$$

But varying now v over $H^1(\Omega)$, which means that v will vary freely on Γ, we finally get

$$\frac{\partial u}{\partial n}-g=0 \qquad \text{on } \Gamma,$$

and (1.37b) follows.

We note that the Neumann condition (1.37b) does not appear explicitly in the variational formulation (V), the solution u of (V) is only required to belong to $H^1(\Omega)$ and is not explicitly required to satisfy (1.37b). This boundary condition is instead implicitly contained in (1.38), by first varying v "inside" Ω we obtain (1.37a) and then (1.37b) by varying v on the boundary Γ. Such a boundary condition, that does not have to be explicitly imposed in the variational formulation, is said to be a *natural boundary condition*. This is in contrast to a so-called *essential boundary condition*, like the Dirichlet condition $u=0$ on Γ in e.g. (1.32), that has to be explicitly satisfied in a variational formulation of the form (1.33).

Let us now formulate a finite element method for the Neumann problem (1.37). Let then T_h be a triangulation of Ω as in Section 1.4 and define

$$V_h=\{v: v \text{ is continuous on } \Omega,\ v|_K \text{ is linear } \forall K\in T_h\}.$$

As parameters to describe the functions in V_h we of course choose the values at the nodes, now including also the nodes on the boundary Γ. Note that the functions in V_h are not required to satisfy any boundary condition and that $V_h \subset H^1(\Omega)$. By starting from (1.38) we now have the following finite element method for (1.37): Find $u_h \in V_h$ such that

(1.40) $\quad a(u_h, v) = (f, v) + \langle g, v \rangle \qquad \forall v \in V_h.$

As in Section 1.4 we see that this problem has a unique solution u_h that can be determined by solving a symmetric, positive definite linear system of equations. We also have the following error estimate

$$\|u-u_h\|_{H^1(\Omega)} \leq \|u-v\|_{H^1(\Omega)} \qquad \forall v \in V_h,$$

and hence as above

$$\|u-u_h\|_{H^1(\Omega)} \leq Ch,$$

if u is regular enough. The function u_h will satisfy the Neumann condition (1.37b) approximatly, i.e., $\frac{\partial u_h}{\partial n}$ will be an approximation to g on Γ (cf Problem 1.16).

Remark. When formulating a difference method for (1.37) one meets severe difficulties due to the boundary condition (1.37b) unless Ω has a very simple shape such as a rectangle. On the other hand, in the finite element formulation, the same boundary condition does not cause any complication. □

Problems

1.16 Show that the problem

$$-u'' = f \qquad \text{on } I = (0, 1),$$
$$u(0) = u'(1) = 0,$$

can be given the following variational formulation: Find $u \in V$ such that

$$(u', v') = (f, v) \qquad \forall v \in V,$$

where $V = \{v \in H^1(I) : v(0) = 0\}$. Formulate a finite element method for this problem using piecewise linear functions. Determine the corresponding linear system of equations in the case of a uniform partition and study in particular how the boundary condition $u'(1)=0$ is approximated by the method.

1.17 Show that the problems (*M*) and (*V*) of this section are equivalent.

1.18 Let Ω be a bounded domain in the plane and let the boundary Γ of Ω be divided into two parts Γ_1 and Γ_2. Give a variational formulation of the following problem:

$$\Delta u = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{in } \Gamma_1,$$
$$\frac{\partial u}{\partial n} = g \qquad \text{on } \Gamma_2,$$

where f, u_0 and g are given functions. Then formulate a finite element method for this problem. Also give an interpretation of this problem in mechanics or physics.

1.19 Consider the finite element method (1.2) for the model problem (1.29). Let $G_i \in H_0^1(I)$ satisfy

(1.41) $\quad (v', G_i') = v(x_i) \qquad \forall v \in H_0^1(I),$

where x_i is a given node, $i = 1, \ldots, M$. Prove that G_i is given by

$$G_i(x) = \begin{cases} (1-x_i)x & \text{for } 0 \leq x \leq x_i, \\ x_i(1-x) & \text{for } x_i \leq x \leq 1. \end{cases}$$

Note that G_i is the *Green's function* for (1.29) associated with a delta function $\delta(x_i)$ at node x_i (G_i satisfies $-G_i'' = \delta(x_i)$ on I, $G_i(0) = G_i(1) = 0$). Further, note that it so happens that $G_i \in V_h$. Now, by choosing $v = e = u - u_h$ in (1.41), show that

$$e(x_i) = (e', G_i') = 0, \qquad i = 1, \ldots, M.$$

Thus, u_h is in fact exactly equal to u at the node points x_i. This somewhat surprising fact is a true one-dimensional effect due to the fact that the Green's function $G_i \in V_h$, and does not exist in higher dimensions. The technique of working with a Green's function in this way is however useful in proving for instance pointwise error estimates (maximum norm estimates) in higher dimensions.

1.8 Remarks on programming

Let us briefly discuss some of the essential features of a typical computer program implementing a finite element method. To be concrete we consider the Neumann problem of the previous section. Thus, let $T_h = \{K\}$ be a

2. Abstract formulation of the finite element method for elliptic problems

2.1 Introduction. The continuous problem

We shall now give an abstract formulation of the finite element method for elliptic problems of the type that we have studied in Chapter 1. This is not a goal in itself, but makes it possible to give a unified treatment of many problems in mechanics and physics so that we do not have to repeat in principle the same argument in different concrete cases. Further the abstract formulation is very easy to grasp and helps us to understand the basic structure of the finite element method.

Thus, let V be a Hilbert space with scalar product $(\cdot,\cdot)_V$ and corresponding norm $\|\cdot\|_V$ (the V-norm). Suppose that (cf Section 1.5) $a(\cdot,\cdot)$ is a bilinear form on $V\times V$ and L a linear form on V such that

(i) $a(\cdot,\cdot)$ is symmetric,

(ii) $a(\cdot,\cdot)$ is *continuous*, i.e., there is a constant $\gamma>0$ such that

$$(2.1)\qquad |a(v,w)|\leq\gamma\|v\|_V\|w\|_V \qquad \forall v,w\in V,$$

(iii) $a(\cdot,\cdot)$ is *V-elliptic*, i.e., there is a constant $\alpha>0$ such that

$$(2.2)\qquad a(v,v)\geq\alpha\|v\|_V^2 \qquad \forall v\in V.$$

(iv) L is *continuous*, i.e., there is a constant $\Lambda>0$ such that

$$(2.3)\qquad |L(v)|\leq\Lambda\|v\|_V \qquad \forall v\in V.$$

Let us now consider the following abstract minimization problem (*M*): Find $u\in V$ such that

$$(2.4)\qquad F(u)=\min_{v\in V} F(v),$$

where

$$F(v)=\frac{1}{2}a(v,v)-L(v),$$

and consider also the following abstract variational problem (*V*): Find $u\in V$ such that

$$(2.5)\qquad a(u,v)=L(v) \qquad \forall v\in V.$$

Let us now first prove

Theorem 2.1 The problems (2.4) and (2.5) are equivalent, i.e., $u\in V$ satisfies (2.4) if and only if u satisfies (2.5). Moreover, there exists a unique solution $u\in V$ of these problems and the following stability estimate holds

$$(2.6)\qquad \|u\|_V\leq\frac{\Lambda}{\alpha}.$$

Proof Existence of a solution follows from the Lax-Milgram theorem which is variant of the Riesz' representation theorem in Hilbert space theory (see e.g. [Ne], [Ci], cf also Theorem 13.1 below). The reader unfamiliar with these concepts may simply bypass this remark. To prove that (2.4) and (2.5) are equivalent, we argue exactly as in Section 1.1. We first show that if $u\in V$ satisfies (2.4), then also (2.5) holds, and we leave the proof of the reverse implication to the reader. Thus, let $v\in V$ and $\varepsilon\in R$ be arbitrary. Then $(u+\varepsilon v)\in V$ so that since u is a minimum,

$$F(u)\leq F(u+\varepsilon v) \qquad \forall\varepsilon\in R.$$

Using the notation $g(\varepsilon)=F(u+\varepsilon v)$, $\varepsilon\in R$, we thus have

$$g(0)\leq g(\varepsilon) \qquad \forall\varepsilon\in R,$$

so that g has a minimum at $\varepsilon=0$. Hence $g'(0)=0$ if the derivative $g'(\varepsilon)$ exists at $\varepsilon=0$. But

$$g(\varepsilon)=\frac{1}{2}a(u+\varepsilon v,u+\varepsilon v)-L(u+\varepsilon v)$$

$$=\frac{1}{2}a(u,u)+\frac{\varepsilon}{2}a(u,v)+\frac{\varepsilon}{2}a(v,u)+\frac{\varepsilon^2}{2}a(v,v)-L(u)-\varepsilon L(v)$$

$$=\frac{1}{2}a(u,u)-L(u)+\varepsilon a(u,v)-\varepsilon L(v)+\frac{\varepsilon^2}{2}a(v,v),$$

where we used the symmetry of $a(\cdot,\cdot)$. It follows that

$$0=g'(0)=a(u,v)-L(v),$$

which proves (2.5). To prove the stability result we choose $v=u$ in (2.5) and use (2.2) and (2.3) to obtain

$$\alpha\|u\|_V^2 \le a(u,u) = L(u) \le \Lambda\|u\|_V,$$

which proves (2.6) upon division by $\|u\|_V \neq 0$. Finally, the uniqueness follows from the stability estimate (2.6) since if u_1 and u_2 are two solutions so that $u_i \in V$ and

$$a(u_i, v) = L(v) \qquad \forall v \in V,\ i=1,2,$$

then by subtraction we see that $u_1 - u_2 \in V$ satisfies

$$a(u_1 - u_2, v) = 0 \qquad \forall v \in V.$$

Applying the stability estimate to this situation (with $L \equiv 0$, i.e. $\Lambda = 0$) we conclude that $\|u_1 - u_2\|_V = 0$, i.e. $u_1 = u_2$. □

Remark 2.1 Even without the symmetry condition (i) and with only (ii)–(iv) satisfied, one can prove that there exists a unique $u \in V$ such that

$$a(u,v) = L(v) \qquad \forall v \in V,$$

and the stability estimate (2.6) of course holds (cf Example 2.6 below). In this case there is however no associated minimization problem. □

2.2 Discretization. An error estimate

Now let V_h be a finite-dimensional subspace of V of dimension M. Let $\{\varphi_1, \ldots, \varphi_M\}$ be a basis for V_h, so that $\varphi_i \in V_h$ and any $v \in V_h$ has the unique representation

$$v = \sum_{i=1}^{M} \eta_i \varphi_i, \quad \text{where } \eta_i \in R \tag{2.7}$$

We can now formulate the following discrete analogues of the problems (*M*) and (*V*). Find $u_h \in V_h$ such that

$$F(u_h) \le F(v) \qquad \forall v \in V_h, \tag{2.8}$$

or equivalently: Find $u_h \in V_h$ such that

$$a(u_h, v) = L(v) \qquad \forall v \in V_h. \tag{2.9}$$

As in Section 1.2 we see that (2.9) is equivalent to

$$a(u_h, \varphi_j) = L(\varphi_j),\ j = 1, \ldots, M.$$

Using the representation

$$u_h = \sum_{i=1}^{M} \xi_i \varphi_i, \quad \xi_i \in R, \tag{2.10}$$

(2.9) can be written as

$$\sum_{i=1}^{M} a(\varphi_i, \varphi_j)\xi_i = L(\varphi_j),\ j = 1, \ldots, M,$$

or, in matrix form,

$$A\xi = b, \tag{2.11}$$

where $\xi = (\xi_i) \in R^M$, $b = (b_i) \in R^M$ with $b_i = L(\varphi_i)$, and $A = (a_{ij})$ is an $M \times M$ matrix with elements $a_{ij} = a(\varphi_i, \varphi_j)$. From the representation (2.7), we have

$$a(v,v) = a\Big(\sum_{i=1}^{M} \eta_i \varphi_i, \sum_{j=1}^{M} \eta_j \varphi_j\Big) = \sum_{i,j=1}^{M} \eta_i a(\varphi_i, \varphi_j)\eta_j = \eta \cdot A\eta,$$

$$L(v) = L\Big(\sum_{j=1}^{M} \eta_j \varphi_j\Big) = \sum_{j=1}^{M} \eta_j L(\varphi_j) = b \cdot \eta,$$

where the dot denotes the usual scalar product in R^M

$$\xi \cdot \eta = \sum_{i=1}^{M} \xi_i \eta_i.$$

It follows that (2.8) may be formulated as

$$\frac{1}{2}\xi \cdot A\xi - b \cdot \xi = \min_{\eta \in R^M} \Big[\frac{1}{2}\eta \cdot A\eta - b \cdot \eta\Big]. \tag{2.12}$$

We also have, recalling (2.2),

$$\eta \cdot A\eta = a(v,v) \ge \alpha\|v\|_V^2 > 0,$$

if $v \neq 0$, i.e. if $\eta \neq 0$. Since also $a(\varphi_i, \varphi_j) = a(\varphi_j, \varphi_i)$, this proves the following result.

Theorem 2.2 The stiffness matrix A is symmetric and positive definite.

We can now prove the following basic result where the equivalence follows as above.

Theorem 2.3 There exists a unique solution $\xi \in R^M$ to the equivalent problems (2.11) and (2.12), i.e., there exists a unique solution $u_h \in V_h$ to the equivalent problems (2.8) and (2.9). Further, the following stability estimate holds:

$$(2.13) \qquad \|u_h\|_V \leq \frac{\Lambda}{\alpha}$$

Proof Since A is positive definite, A is non-singular, which proves existence and uniqueness. The stability estimate follows by choosing $v = u_h$ in (2.9) which gives, using (2.2) and (2.3),

$$\alpha\|u_h\|_V^2 \leq a(u_h, u_h) = L(u_h) \leq \Lambda\|u_h\|_V,$$

from which (2.13) follows upon division by $\|u_h\|_V \neq 0$.

Remark The stability estimate (2.13) for the finite element solution, which is an analogue of the stability estimate (2.6) for the continuous problem, reflects a very important property of the finite element method. In a certain sense it can be viewed as the theoretical basis for the success of the method. □

Let us now prove the following error estimate.

Theorem 2.4 Let $u \in V$ be the solution of (2.5) and $u_h \in V_h$ that of (2.9) where $V_h \subset V$. Then

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\|u-v\|_V \quad \forall v \in V_h.$$

Proof Since $V_h \subset V$ we have from (2.5) in particular

$$a(u,w) = L(w) \qquad \forall w \in V_h,$$

so that after subtracting (2.9),

$$(2.14) \qquad a(u-u_h, w) = 0 \qquad \forall w \in V_h.$$

For an arbitrary $v \in V_h$, define $w = u_h - v$. Then $w \in V_h$, $v = u_h - w$ and by (2.2) and (2.14), we have

$$\alpha\|u-u_h\|_V^2 \leq a(u_h, u-u_h) = a(u-u_h, u-u_h) + a(u-u_h, w)$$

$$= a(u-u_h, u-u_h+w) = a(u-u_h, u-v) \leq \gamma\|u-u_h\|_V\|u-v\|_V,$$

where the last inequality follows from (2.1). Dividing by $\|u-u_h\|_V$ we obtain the desired estimate. □

From the abstract qualitative estimate of Theorem 2.4 we may obtain a quantitative estimate by choosing a suitable function $v \in V_h$ and estimating $\|u-v\|_V$. Usually one then chooses $v = \pi_h u$ where $\pi_h u \in V_h$ is a suitable interpolant of u (e.g. $\pi_h u$ may be the piecewise linear interpolant $\tilde{u}_h$ of Section 1.3). In Chapter 4 we give estimates for the interpolation error $\|u-\pi_h u\|_V$ in a variety of situations.

2.3 The energy norm

By (2.1) and (2.2) it follows that we may introduce a new norm $\|\cdot\|_a$ on V defined by

$$\|v\|_a^2 = a(v, v), \quad v \in V.$$

This norm is *equivalent* to the norm $\|\cdot\|_V$, i.e., there are positive constants c and C such that

$$(2.15) \qquad c\|v\|_V \leq \|v\|_a \leq C\|v\|_V \qquad \forall v \in V.$$

More precisely, we may choose $c = \sqrt{\alpha}$ and $C = \sqrt{\gamma}$. The scalar product $(\cdot,\cdot)_a$ corresponding to $\|\cdot\|_a$ is given by

$$(v,w)_a = a(v,w).$$

The norm $\|\cdot\|_a$ is referred to as the *energy norm*. The error equation (2.14) may now be written

$$(u-u_h, v)_a = 0 \qquad \forall v \in V_h,$$

from which follows as in Section 1.3 or by the proof of Theorem 2.4, that

$$(2.16) \qquad \|u-u_h\|_a \leq \|u-v\|_a \qquad \forall v \in V_h,$$

or equivalently that u_h is the projection of u onto V_h with respect to the scalar product $(\cdot,\cdot)_a$ (cf Section 1.6). Clearly (2.16) shows that u_h is a best approximation of u in the energy norm.

2.4 Some examples

Let us now consider some concrete examples of the form (2.5). In Chapter 5 further examples from mechanics and physics will be presented. Let Ω be a bounded domain in R^2 or R^3 with boundary Γ. The coordinates in R^2 and R^3 are denoted by $x = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$.

Example 2.1 Let $V=H^1(\Omega)$, $\Omega\subset R^2$,

$$a(v, w)=\int_\Omega [\nabla v\cdot\nabla w+vw]dx,$$

$$L(v)=\int_\Omega fv dx,$$

where $f\in L_2(\Omega)$ in which case (2.5) is a variational formulation of the Neumann problem (1.37) with $g=0$. Let us verify that the conditions (i)–(iv) above are satisfied. Clearly $a(\cdot,\cdot)$ is a symmetric bilinear form on $V\times V$ and L is a linear form. Further,

$$a(v, v)=\|v\|^2_{H^1(\Omega)}$$

and by Cauchy's inequality

$$a(v,w)\le a(v,v)^{1/2}a(w,w)^{1/2}=\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},$$

which proves (2.1) and (2.2) with $\alpha=\gamma=1$. Finally

$$|L(v)|\le\left|\int_\Omega fv\, dx\right|\le\|f\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)},$$

which proves (2.3) with $\Lambda=\|f\|_{L_2(\Omega)}$. □

Example 2.2 Let $V=H_0^1(I)$, $(I=(0, 1))$,

$$a(v, w)=\int_I v'w'dx, \quad L(v)=\int_I fv\, dx,$$

where $f\in L_2(I)$ is given, which corresponds to our introductory boundary value problem (1.30). To verify that (i)–(iv) are satisfied, we first note that $a(\cdot,\cdot)$ is obviously symmetric and bilinear and L is linear and since

$$|a(v,w)|\le\|v'\|_{L_2(I)}\|w'\|_{L_2(I)}\le\|v\|_{H^1(I)}\|w\|_{H^1(I)},$$

we have that $a(\cdot,\cdot)$ is continuous. The continuity of L follows as in Example 2.1 and it thus remains to prove the V-ellipticity (2.2), i.e, the inequality

$$\int_I (v')^2dx\ge\alpha\left(\int_I v^2dx+\int_I (v')^2dx\right)\qquad\forall v\in H_0^1(I), \tag{2.17}$$

for some positive constant α. We shall prove that

$$\int_I v^2dx\le\int_I (v')^2dx\qquad\forall v\in H_0^1(I), \tag{2.18}$$

from which (2.17) follows with $\alpha=\frac{1}{2}$. Since $v(0)=0$ for $v\in H_0^1(I)$, we have

$$v(x)=v(0)+\int_0^x v'(y)dy=\int_0^x v'(y)dy,$$

so that by Cauchy's inequality

$$|v(x)|\le\int_0^1|v'|dy\le\left(\int_0^1 dy\right)^{1/2}\left(\int_0^1(v')^2dy\right)^{1/2}=\left(\int_0^1(v')^2dy\right)^{1/2}.$$

Squaring this inequality and then integrating over I we obtain (2.18). We note that the inequality (2.18) does not hold for $v(x)\equiv 1$, in which case the left hand side is 1 and the right hand side 0. Thus we need e.g a boundary condition of the form $v(0)=0$ for (2.18) to hold in order to control the norm of the function v by the norm of the derivative v', i.e, we need a "fixed point" to start from.

If we choose V_h to consist of piecewise linear functions on I as in Section 1.2, we obtain in this case

$$\|u-u_h\|_{H^1(\Omega)}\le Ch,$$

if u is smooth enough. □

Example 2.3 Let $V=H_0^1(\Omega)$, $\Omega\subset R^2$,

$$a(v,w)=\int_\Omega \nabla v\cdot\nabla w\, dx, \quad L(v)=\int_\Omega fv dx,$$

where $f\in L_2(\Omega)$, in which case (2.5) is a variational formulation of the Dirichlet problem (1.16) for the Poisson equation. We directly see that (i), (ii) and (iv) are satisfied in this case. Thus, only the V-ellipticity, i.e, the inequality

$$\int_\Omega|\nabla v|^2dx=a(v,v)\ge\alpha\|v\|^2_{H^1(\Omega)}=\alpha\left(\int_\Omega(v^2+|\nabla v|^2)dx\right) \tag{2.19}$$

requires comment. To prove (2.19), it is sufficient to prove that there is a constant C such that

$$\int_\Omega v^2dx\le C\int_\Omega|\nabla v|^2dx\qquad\forall v\in H_0^1(\Omega), \tag{2.20}$$

since then (2.19) follows with $\alpha=\frac{1}{C+1}$. The proof of (2.20) is analogous to the proof of (2.18) (cf Problem 2.1 below). With the V_h of Section 1.4 we obtain the error estimate

$$\|u-u_h\|_{H^1(\Omega)}\le Ch,$$

if u is sufficiently smooth. □

Example 2.4 Consider the following boundary value problem

$$\frac{d^4u}{dx^4}=f\qquad\text{for } x\in I=(0, 1), \tag{2.21a}$$

$$(2.21b) \qquad u(0)=u'(0)=u(1)=u'(1)=0,$$

where $f \in L_2(I)$ (cf Problem 1.5). We introduce the space

$$H^2(I)=\{v \in L_2(I): v', v'' \in L_2(I)\},$$

with norm

$$\|v\|_{H^2(I)}=\left(\int_I [v^2+(v')^2+(v'')^2]\,dx\right)^{1/2},$$

and the space

$$H_0^2(I)=\{v \in H^2(I): v(0)=v'(0)=v(1)=v'(1)=0\}$$

with the same norm. The problem (2.21) can now be given the variational formulation: Find $u \in V$ such that

$$a(u,v)=L(v) \qquad \forall v \in V,$$

where $V=H_0^2(\Omega)$,

$$a(v,w)=\int_I v''w''\,dx, \quad L(v)=\int_I fv\,dx.$$

We see that the conditions (i), (ii) and (iv) are satisfied. By (2.18) we have for $v \in H_0^2(I)$

$$\int_I v^2dx \le \int_I (v')^2dx \le \int_I (v'')^2dx,$$

since $v(0)=v'(0)=0$, which proves that

$$\|v\|^2_{H^2(I)} \le 3\int_I (v'')^2dx = 3\,a(v,v),$$

and (iii) holds with $\alpha=\frac{1}{3}$. □

We now introduce some notation that will be used below. We define

$$D^\alpha v=\frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}},$$

where here $\alpha=(\alpha_1, \alpha_2)$, α_i is a non-negative natural number and $|\alpha|=\alpha_1+\alpha_2$. As an example, a partial derivative of order 2 can then be written as $D^\alpha v$ with $\alpha=(2,0)$, $\alpha=(1,1)$ or $\alpha=(0,2)$, which are the α with $|\alpha|=2$. We now define for $k=1, 2, \ldots$,

$$H^k(\Omega)=\{v \in L_2(\Omega): D^\alpha v \in L_2(\Omega),\ |\alpha| \le k\},$$

with norm

$$\|v\|_{H^k(\Omega)}=\Big(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha v|^2dx\Big)^{1/2}.$$

Thus the space $H^k(\Omega)$ consists of all functions v on Ω that, together with the partial derivatives $D^\alpha v$ of order $|\alpha|$ at most k, belong to $L_2(\Omega)$. The space $H^k(\Omega)$ is a Hilbert space with the indicated norm and corresponding scalar product. The spaces $H^k(\Omega)$ are examples of so called *Sobolev spaces* named after the Russian mathematician S. L. Sobolev 1908–, cf [Ad].

Example 2.5 Let us now consider a fourth-order problem in a two-dimensional domain Ω, namely the *biharmonic problem*

$$(2.22a) \qquad \Delta\Delta u=f \quad \text{in } \Omega,$$

$$(2.22b) \qquad u=\frac{\partial u}{\partial n}=0 \quad \text{on } \Gamma,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ. This problem gives a formulation of the Stokes equations in fluid mechanics (cf Problem 5.3) and also models the displacement of a thin elastic plate, clamped at its boundary, under a transversal load (cf Problem 5.4). To give a variational formulation of (2.22), we introduce the space

$$H_0^2(\Omega)=\{v \in H^2(\Omega): v=\frac{\partial v}{\partial n}=0 \text{ on } \Gamma\}.$$

Now we multiply (2.22a) with $v \in H_0^2(\Omega)$ and integrate over Ω. By Green's formula as $v=\frac{\partial v}{\partial n}=0$ on Γ, we have

$$\int_\Omega fv\,dx=\int_\Omega \Delta\Delta u\, v\,dx=$$

$$=\int_\Gamma \frac{\partial}{\partial n}(\Delta u)v\,ds-\int_\Omega \nabla(\Delta u)\cdot\nabla v\,dx=$$

$$=-\int_\Omega \nabla(\Delta u)\nabla v\,dx=-\int_\Gamma \Delta u\frac{\partial v}{\partial n}ds+\int_\Omega \Delta u\Delta v\,dx=\int_\Omega \Delta u\Delta v\,dx.$$

We are thus led to the following variational formulation of the biharmonic problem (2.22): Find $u \in V$ such that

$$a(u,v)=L(v) \qquad \forall v \in V,$$

where $v = H_0^2(\Omega)$ and

$$a(u, v) = \int_\Omega \Delta u \, \Delta v \, dx, \quad L(v) = \int_\Omega fv \, dx$$

Again we see directly that (i), (ii) and (iv) are satisfied in this case and the V-ellipticity (iii) can easily be proved using the hints of Problem 2.2 below. In Chapter 3 below we shall construct finite element spaces $V_h \subset H_0^2(\Omega)$. □

Example 2.6 Consider the following problem in a domain $\Omega \subset R^2$.

$$(2.23a) \qquad -\mu \Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \quad \text{in } \Omega,$$

$$(2.23b) \qquad u = 0 \text{ on } \Gamma,$$

where μ and the β_i are constants with $\mu > 0$. This is an example of a stationary *convection-diffusion problem*, the Laplace term corresponds to diffusion with diffusion coefficient μ and the first order derivatives correspond to convection in the direction $\beta = (\beta_1, \beta_2)$. Let us here assume that $\mu = 1$ and that the size of $|\beta|$ is moderate (for convection-diffusion problems with $|\beta|/\mu$ large, see Chapter 9). By multiplying (2.23a) by a test function $v \in V = H_0^1(\Omega)$, integrating over Ω and using Green's formula for the Laplace term as usual, we are led to the following variational formulation of (2.23). Find $u \in V$ such that

$$(2.24) \qquad a(u, v) = L(v) \quad \forall v \in V,$$

where

$$a(v, w) = \int_\Omega \left(\nabla v \cdot \nabla w + \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2} + v\right) w\right) dx, \quad L(v) = \int_\Omega fv \, dx.$$

It is clear that $a(\cdot, \cdot)$ is V-elliptic since if $v \in V$, we have by Green's formula,

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) dx = \int_\Gamma v^2 (\beta_1 n_1 + \beta_2 n_2) ds -$$

$$- \int_\Omega \left(v \, \beta_1 \frac{\partial v}{\partial x_1} + v \, \beta_2 \frac{\partial v}{\partial x_2}\right) dx = - \int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) dx,$$

i.e.,

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\right) v \, dx = 0,$$

so that

$$a(v, v) = \int_\Omega [\, |\nabla v|^2 + v^2] dx = \|v\|^2_{H^1(\Omega)}.$$

Existence of a unique weak solution of (2.23) now follows from Remark 2.1. Starting from (2.24) we may formulate the following finite element method for (2.23). Find $u_h \in V_h$ such that

$$(2.25) \qquad a(u_h, v) = L(v) \quad \forall v \in V_h,$$

where V_h is a finite dimensional subspace of V. If $\{\varphi_1, \ldots, \varphi_M\}$ is a basis for V_h we have as above that (2.25) is equivalent to the linear system $A\xi = b$ where $A = (a_{ij})$, $a_{ij} = a(\varphi_i, \varphi_j)$, and $b = (b_i)$, $b_i = (f, \varphi_i)$. Note that in this case the matrix A is *not* symmetric.

By the V-ellipticity it follows that solutions of (2.25) are unique and thus A is non-singular so that $A\xi = b$ admits a unique solution, i.e., there exists a unique solution u_h of (2.25). By the same argument as in the proof of Theorem 2.3, we also have the error estimate (here $\alpha = 1$):

$$\|u - u_h\|_{H^1(\Omega)} \leq \gamma \|u - v\|_{H^1(\Omega)} \quad \forall v \in V_h. \quad \square$$

Example 2.7 Let u be the temperature in a heat conducting body occupying the domain $\Omega \subset R^3$. We have in the stationary case the following relations:

$$(2.26a) \qquad -q_i = k_i(x) \frac{\partial u}{\partial x_i} \quad \text{in } \Omega, \; i = 1, 2, 3, \text{ (Fourier's law)},$$

$$(2.26b) \qquad \operatorname{div} q = f \quad \text{in } \Omega \text{ (conservation of energy)},$$

where the q_i denotes the heat flow in the x_i-direction, $k_i(x)$ is the heat conductivity at x in the x_i-direction and f(x) is the heat production at x. If $k_i(x) \equiv 1$, $x \in \Omega$, $i = 1, 2, 3$, i.e., if the heat conductivity is constant and equal in all directions, then eliminating q in (2.26), we obtain Poisson's equation $-\Delta u = f$ in Ω. With the k_i non-constant, (2.26) is an example of a partial differential equation with *variable coefficients*. However, the coefficients k_i are not assumed to depend on the solution u. If this was the case and the heat conductivities k_i depended on the temperature u, then (2.26) would be an example of a *non-linear* partial differential equation, see Chapter 13 below.

Let us now give a variational formulation of (2.26) which in the usual way can be used to formulate a finite element method for (2.26). This shows that the presence of the variable coefficients k_i do not introduce any difficulties. We complement (2.26a, b) with the following boundary conditions:

$$(2.26c) \qquad u = 0 \quad \text{on } \Gamma_1,$$

$$(2.26d) \qquad -q \cdot n = g \quad \text{on } \Gamma_2,$$

element methods of so-called *equilibrium* type (for such a method the equilibrium condition $\operatorname{div} q+f=0$ will be satisfied exactly in the discrete model). Methods of this type may in certain cases have advantages as compared to the conventional finite element methods, so-called *displacement methods*, that we have studied above (in a displacement method for (2.26) the compatibility relation (2.26a) is satisfied exactly). Hint: First show that $p\in H_f$ is a solution of (2.32) if and only if

$$\int_\Omega p\cdot q\,dx=0 \qquad \forall q\in H_0,$$

where $H_0=\{q\in H,\ \operatorname{div} q=0 \text{ in } \Omega\}$.

2.10 Solve Problem 2.3 with the following alternative boundary conditions:

$$u(0)=-u''(0)+\gamma u'(0)=0,\ u(1)=u''(1)+\gamma u'(1)=0,$$

where γ is a positive constant. Also give a mechanical interpretation of the boundary conditions.

2.11 Consider the Neumann problem

(2.33a) $\quad -\Delta u=f$ in Ω,

(2.33b) $\quad \dfrac{\partial u}{\partial n}=g$ on Γ,

(2.33c) $\quad \int_\Omega u\,dx=0.$

Note that if u satisfies (2.33a, b), then so does u+c for any constant c, and that the condition (2.33c) is added to give uniqueness. Give a variational formulation of (2.33) using the space

$$V=\{v\in H^1(\Omega):\ \int_\Omega v\,dx=0\},$$

and prove that the conditions (i)–(iv) are satisfied.

3. Some finite element spaces

3.1 Introduction. Regularity requirements

We shall now present some commonly used finite element spaces V_h. These spaces will consist of piecewise polynomial functions on subdivisions or "triangulations" $T_h=\{K\}$ of a bounded domain $\Omega\subset R^d$, $d=1, 2, 3$, into elements K. For $d=1$, the elements K will be intervals, for $d=2$, triangles or quadrilaterals and for $d=3$ tetrahedrons for instance.

We will need to satisfy either $V_h\subset H^1(\Omega)$ or $V_h\subset H^2(\Omega)$, corresponding to second order or fourth order boundary value problems, respectively. Since the space V_h consists of piecewise polynomials, we have

(3.1) $\quad V_h\subset H^1(\Omega)\Leftrightarrow V_h\subset C^0(\bar\Omega)$,

(3.2) $\quad V_h\subset H^2(\Omega)\Leftrightarrow V_h\subset C^1(\bar\Omega)$,

where $\bar\Omega=\Omega\cap\Gamma$ and

$$C^0(\bar\Omega)=\{v: v \text{ is a continuous function defined on } \bar\Omega\},$$

$$C^1(\bar\Omega)=\{v\in C^0(\bar\Omega):\ D^\alpha v\in C^0(\bar\Omega),\quad |\alpha|=1\}.$$

Thus, $V_h\subset H^1(\Omega)$ if and only if the functions $v\in V_h$ are continuous, and $V_h\subset H^2(\Omega)$ if an only if the functions $v\in V_h$ *and* their first derivatives are continuous. The equivalence (3.1) depends on the fact that the functions v in V_h are polynomials on each element K so that if v is continuous across the common boundary of adjoining elements, then the first derivatives $D^\alpha v$, $|\alpha|=1$, exist and are piecewise continuous so that $v\in H^1(\Omega)$. On the other hand, if v is not continuous across a certain inter-element boundary, i.e. $v\notin C^0(\bar\Omega)$, then the derivatives $D^\alpha v$, $|\alpha|=1$, do not exist as functions in $L_2(\Omega)$ and thus $v\notin H^1(\Omega)$ (if v is discontinuous across an element side S, then $D^\alpha v$, $|\alpha|=1$, would be a δ-function supported by S which is not a square integrable function). In a similar way we realize that (3.2) holds.

To define a finite element space V_h we will have to specify

(a) the triangulation $T_h=\{K\}$ of the domain Ω,

(b) the nature of the functions v in V_h on each element K (e.g. linear, quadratic, cubic, etc),

(c) the parameters to be used to describe the functions in V_h.

3.2 Some examples of finite elements

Let us now consider some examples. We first consider the case when Ω is a domain in the plane R^2 with polygonal boundary Γ. Let $T_h=\{K\}$ be a given triangulation of Ω according to Section 1.4 into triangles K. We shall use the following notation for $r=0, 1, 2, \ldots$,

$$P_r(K)=\{v: v \text{ is a polynomial of degree} \leq r \text{ on } K\}.$$

Thus, $P_1(K)$ is the space of linear functions defined on K, i.e., functions of the form

$$v(x)=a_{00}+a_{10}x_1+a_{01}x_2, \quad x\in K,$$

where the $a_{ij}\in R$. We see that $\{\psi_1, \psi_2, \psi_3\}$, where

$$\psi_1(x)=1, \quad \psi_2(x)=x_1, \quad \psi_3(x)=x_2,$$

is a basis for $P_1(K)$, and that $\dim P_1(K)=3$, where $\dim W$ denotes the dimension of the linear space W.

Further, $P_2(K)$ is the space of *quadratic* functions on K, i.e., functions of the form

$$v(x)=a_{00}+a_{10}x_1+a_{01}x_2+a_{20}x_1^2+a_{11}x_1x_2+a_{02}x_2^2, \quad x\in K,$$

where the $a_{ij}\in R$. We see that $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ is a basis for $P_2(K)$ and that $\dim P_2(K)=6$. In general we have

$$P_r(K)=\{v: v(x)=\sum_{0\leq i+j\leq r} a_{ij}x_1^i x_2^j \text{ for } x\in K, \text{ where } a_{ij}\in R\},$$

and

$$\dim P_r(K)=\frac{(r+1)(r+2)}{2}.$$

Example 3.1 Let

(3.3) $\qquad V_h=\{v\in C^0(\bar{\Omega}): v|_K\in P_1(K), \ \forall K\in T_h\}$,

i.e., V_h is the space of continuous piecewise linear functions that we have met in Section 1.7. As parameters, or *global degrees of freedom*, to describe the functions in V_h, we choose

(3.4) $\qquad$ the values at the node points of T_h

(including the node points on Γ). Let us now convince ourselves that this is a legitimate choice and show that a function $v\in V_h$ is uniquely determined by the values (3.4). This is of course intuitively quite obvious but let us anyway carry out the argument in detail here, since it will be a model to be used in more complicated situations below. We then first notice that if $K\in T_h$ is a triangle with vertices a^i, $i=1, 2, 3$, then the degrees of freedom for K corresponding to (3.4), i.e., *the element degrees of freedom*, are

(3.5) $\qquad$ the values at the vertices a^i, $i=1, 2, 3$.

To show that a function $v\in V_h$ is uniquely determined by the degrees of freedom (3.4) it is sufficient to show:

Theorem 3.1 Let $K\in T_h$ be a triangle with vertices $a^i=(a_1^i, a_2^i)$, $i=1, 2, 3$. A function $v\in P_1(K)$ is uniquely determined by the degrees of freedom (3.5), i.e., given the values α_i, $i=1, 2, 3$, there is a uniquely determined function $v\in P_1(K)$ such that

(3.6) $\qquad v(a^i)=\alpha_i \qquad i=1, 2, 3.$

Proof Since $v(x)=c_1x_1+c_2x_2+c_3$ for some constants $c_i\in R$, (3.6) is equivalent to the linear system of equations

(3.7) $\qquad c_1a_1^i+c_2a_2^i+c_3=\alpha_i, \qquad i=1, 2, 3,$

in the unknowns c_i. This system has a unique solution for given α_i if and only if the determinant $\det B$ of the coefficient matrix

$$B=\begin{bmatrix} a_1^1 & a_2^1 & 1 \\ a_1^2 & a_2^2 & 1 \\ a_1^3 & a_2^3 & 1 \end{bmatrix}$$

is different from zero. However by basic linear algebra

(3.8) $\qquad \det B/2=\text{area of } K,$

and thus $\det B\neq 0$. Hence B is non-singular, which proves the desired result. Since this argument will be used below, we also give a somewhat different version of this proof. We notice first that

$$\dim P_1(K) = \text{number of degrees of freedom } (=3),$$

i.e. (3.7) has the same number of unknowns as equations. In this case it follows, again by basic linear algebra, that $\det B \neq 0$ if and only if solutions of (3.7) are unique, or in other words if the only solution of (3.7) with $u_i = 0$, $i = 1, 2, 3$, is given by $c_i = 0$, $i = 1, 2, 3$, or formally:

(3.9) If $v \in P_1(K)$ and $v(a^i) = 0$, $i = 1, 2, 3$, then $v \equiv 0$.

In fact it is easy to prove (3.9) directly without using (3.8), which shows that we do not have to be able to compute $\det B$ in order to prove that $\det B \neq 0$. As we shall see below, this latter method of proof makes it possible to easily prove analogues of Theorem 3.1 for higher order polynomials in which case a direct computation of the determinant of the corresponding coefficient matrix could be very complicated. □

We can now determine the (nodal) basis functions for $P_1(K)$ associated with the degrees of freedom (3.5), i.e. the functions $\lambda_i \in P_1(K)$, $i = 1, 2, 3$, such that (see Fig 3.1):

$$\lambda_i(a^j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad i, j = 1, 2, 3.$$

A function $v(x) \in P_1(K)$ then has the representation

(3.10) $$v(x) = \sum_{i=1}^{3} v(a^i)\lambda_i(x) \quad x \in K.$$

To determine the basis functions λ_i, we have to solve the system of equations (3.7) for three special choices of right hand side, namely, (1, 0, 0), (0, 1, 0) and (0, 0, 1).

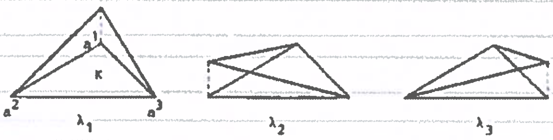

Fig 3.1

The basis function λ_1, say, can also be determined as follows. Let

$$d_1x_1 + d_2x_2 + d_3 = 0,$$

be the equation for the straight line through the vertices a^2 and a^3. Then

$$\lambda_1(x) = \gamma(d_1x_1 + d_2x_2 + d_3),$$

where the constant γ is chosen so that $\lambda_1(a^1) = 1$. In the same way we may determine λ_2 and λ_3. If the triangle K has vertices at (1, 0), (0, 1) and (0, 0), then $\lambda_1 = x_1$, $\lambda_2 = x_2$ and $\lambda_3 = 1 - x_1 - x_2$. The notation λ_1, λ_2 and λ_3 for the nodal basis functions for $P_1(K)$ will be kept below.

Given the choice of global degrees of freedom in (3.4), it is natural to describe the space V_h given by (3.3) alternatively as

(3.11) $$V_h = \{v : v|_K \in P_1(K),\ \forall K \in T_h, \text{ and } v \text{ is continuous at the nodes}\}.$$

We then view a function $v \in V_h$ as a piecewise linear function taking on certain values at the nodes of T_h. Let us be careful and check that (3.11) defines the same space as (3.3) above. We need to check if a function $v \in V_h$ according to (3.11) is continuous, i.e. if $v \in C^0(\bar{\Omega})$. Clearly, it is sufficient to check that v is continuous across all interelement sides. Thus, let K_1 and K_2 be two triangles in T_h having the common side S with the end points N_1 and N_2, say. Suppose now $v \in V_h$ according to (3.11) and let $v_i = v|_{K_i} \in P_1(K_i)$, $i = 1, 2$, be the restrictions of v to the K_i. Then the function $w = v_1 - v_2$ defined on S vanishes at the end points N_1 and N_2 and since w is linear on S it follows that in fact w vanishes on S. Hence, v is continuous across S and we obtain the desired conclusion that $v \in C^0(\bar{\Omega})$.

Example 3.2 Let us now show how to construct a space V_h using piecewise quadratic functions v, i.e. $v|_K \in P_2(K)$. Let us first specify the element degrees of freedom. Let $K \in T_h$ be a triangle with vertices a^i, $i = 1, 2, 3$, and denote the midpoints of the sides of K by a^{ij}, $i < j$, $i, j = 1, 2, 3$, see Fig 3.2.

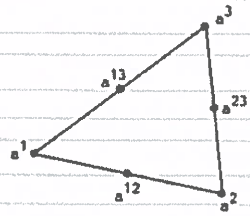

Fig 3.2

We shall prove

Theorem 3.2 A function $v \in P_2(K)$ is uniquely determined by the following degrees of freedom:

$$(3.12) \quad \begin{aligned} &v(a^i), && i=1,2,3,\\ &v(a^{ij}), && i<j,\ i,j=1,2,3. \end{aligned}$$

Proof Since dim $P_2(K)$ is equal to the number of degrees of freedom $(=6)$, it is (see the proof of Theorem 3.1) sufficient to prove that if $v \in P_2(K)$ and

$$(3.13) \quad v(a^i)=0, \quad v(a^{ij})=0, \quad i<j, \quad i,j=1,2,3,$$

then $v \equiv 0$. To this end, consider the side a^2a^3. Along this side the function v has a quadratic variation and v vanishes at the three distinct points a^2, a^{23} and a^3. Thus, (cf Problem 3.1) v vanishes identically on a^{23} which means (cf Problem 3.3) that we can "factor out" the function λ_1 and write

$$v(x)=\lambda_1(x)w_1(x), \qquad x \in K,$$

where $w_1 \in P_1(K)$ and λ_i, $i=1,2,3$, are the basis functions for $P_1(K)$ according to Example 3.1. In the same way we see that v also vanishes along the side a^1a^3 which means that we may also factor out the function λ_2, so that

$$v(x)=\lambda_1(x)\lambda_2(x)w_0, \qquad x \in K,$$

where now w_0 has degree zero, i.e. $w_0 \equiv \gamma =$constant. If we now finally take $x=a^{12}$, we see that

$$0=v(a^{12})=\gamma\lambda_1(a^{12})\lambda_2(a^{12})=\gamma\,\frac{1}{2}\cdot\frac{1}{2},$$

so that $\gamma=0$ and hence $v \equiv 0$ and the proof is complete. □

A function $v \in P_2(K)$ has the representation

$$(3.14) \quad v=\sum_{i=1}^{3} v(a^i)\lambda_i(2\lambda_i-1)+\sum_{\substack{i,j=1\\ i<j}}^{3} v(a^{ij})4\lambda_i\lambda_j.$$

To see this, by Theorem 3.2 it is sufficient to check that the right hand side, RH, and left hand side, LH, of (3.14) take the same values at the node points a^i and a^{ij}, since the difference $LH-RH \in P_2(K)$. From (3.14) it is clear what the nodal basis functions for $P_2(K)$ corresponding to the degrees of freedom (3.12) are: the basis function corresponding to a particular degree of freedom, the value at the vertex a^i for instance, is of course the function $\psi \in P_2(K)$ such that $\psi(a^i)=1$ and ψ vanishes at the other five points a^i, a^{ij} (see Fig 3.3).

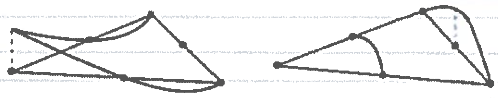

Fig 3.3 Different basis functions for $P_2(K)$

Let us also show that if $v_i \in P_2(K_i)$, $i=1,2$, where K_1 and K_2 are two triangles with the common side S, and v_1 and v_2 take the same values at the end points and the mid point of S, then v_1 and v_2 agree on S. But this follows immediately from the fact that $w=v_1-v_2$ varies quadratically along S and w vanishes at three distinct points on S so that $w \equiv 0$ on S.

Defining now

$$V_h=\{v \in C^0(\bar{\Omega}): v|_K \in P_2(K),\ \forall K \in T_h\},$$

we have seen that the global degrees of freedom of the functions $v \in V_h$ can be chosen as follows:

(i) the values of v at the nodes of T_h,
(ii) the values of v at the mid points of all the sides of the triangles in T_h.

The corresponding global basis functions have the following form:

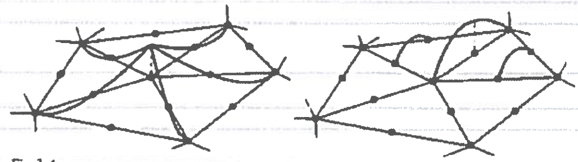

Fig 3.4

Example 3.3 We now define a space V_h using piecewise cubic functions, i.e. functions v such that $v|_K \in P_3(K)$, $\forall K \in T_h$. Let K be a triangle with vertices a^i, $i=1,2,3$, and define (see Fig 3.5):

$$a^{iij}=\frac{1}{3}(2a^i+a^j), \quad i,j=1,2,3, \quad i \neq j,$$

$$a^{123}=\frac{1}{3}(a^1+a^2+a^3).$$

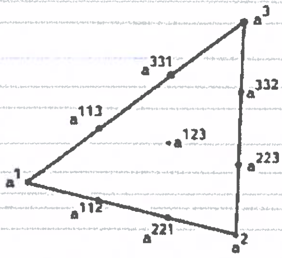

Fig 3.5

We have

Theorem 3.3 A function $v \in P_3(K)$ is uniquely determined by the following degrees of freedom:

$$(3.15) \qquad v(a^i),\ v(a^{iij}),\ i, j=1, 2, 3, \quad i \neq j, \qquad v(a^{123}).$$

Proof Since dim $P_3(K)$ is equal to the number of degrees of freedom (=10), it is sufficient to show that if $v \in P_3(K)$ and

$$(3.16) \qquad v(a^i)=v(a^{iij})=v(a^{123})=0,\ i, j=1, 2, 3, \quad i \neq j,$$

then $v \equiv 0$. Observe that if v has a cubic variation along the side a^2a^3 then $v \equiv 0$ on a^2a^3. In the same way it follows that v vanishes on the sides a^1a^3 and a^1a^2 and hence

$$v(x)=\gamma\lambda_1(x)\lambda_2(x)\lambda_3(x),$$

where γ is a constant. If we now choose $x=a^{123}$, we get from (3.16)

$$0=v(a^{123})=\gamma\,\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3},$$

so that $\gamma=0$ and thus $v \equiv 0$. □

Now let $v_i \in P_3(K_i)$, $i=1, 2$, where K_1 and K_2 are two triangles with common side S and suppose that v_1 and v_2 take the same values at the end points and the two points a^{iij} of S. Since $v_1 - v_2$ varies cubically on S it follows that $v_1 \equiv v_2$ on S (see Fig 3.6).

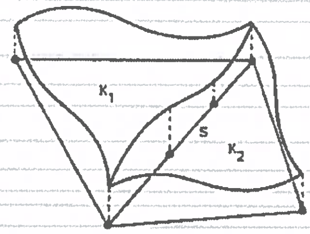

Fig 3.6

We can now introduce the space

$$V_h=\{v \in C^0(\bar{\Omega}) : v|_K \in P_3(K),\ \forall K \in T_h\},$$

with the following degrees of freedom:

(i) the values of v at the nodes of T_h
(ii) the values of v at the points a^{iij} on the sides of T_h,
(iii) the values of v at the center of gravity for all $K \in T_h$

Example 3.4 There is another way of choosing the degrees of freedom for $P_3(K)$, where K is a triangle with vertices a^i, $i=1, 2, 3$, and center of gravity a^{123}. We have

Theorem 3.4 A function $v \in P_3(K)$ is uniquely determined by the following degrees of freedom:

$$(3.17) \qquad v(a^i),\quad i=1, 2, 3, \qquad \frac{\partial v}{\partial x_j}(a^i),\quad i=1, 2, 3,\ j=1, 2, \qquad v(a^{123}).$$

Proof Since again dim $P_3(K)$ is equal to the number of degrees of freedom, it suffices to prove that if $v \in P_3(K)$ and

$$(3.18) \qquad v(a^i)=\frac{\partial v}{\partial x_j}(a^i)=v(a^{123})=0,\ i=1, 2, 3,\ j=1, 2,$$

then $v \equiv 0$. It follows from (3.18) that

$$\frac{\partial v}{\partial s}(a^i)=\frac{\partial v}{\partial x_1}(a^i)s_1+\frac{\partial v}{\partial x_2}(a^i)s_2=0,\ i=1, 2, 3.$$

where $\frac{\partial v}{\partial s}$ is the derivative in a direction $s=(s_1, s_2)$. In particular we then have

$$\frac{\partial v}{\partial s}(a^2) = \frac{\partial v}{\partial s}(a^3) = 0,$$

where s is the direction from a^2 to a^3. Together with the fact that $v(a^2)=v(a^3)$ this shows that v vanishes along the side a^2a^3 since v varies as a cubic polynomial along this side. In the same way see that v vanishes on a^1a^2 and a^1a^3 and the argument is then completed as in the proof of Theorem 3 3. □

We further note that if $v_i \in P_3(K_i)$, $i=1, 2$, where K_1 and K_2 are two triangles with the common side S with endpoints N_j, $j=1, 2$, and v_1 and v_2 agree together with the first derivatives $\frac{\partial v_1}{\partial x_i}(N_j)$ and $\frac{\partial v_2}{\partial x_i}(N_j)$, $i, j=1, 2$, then $v_1 \equiv v_2$ on S.

The corresponding finite element space $V_h \subset C^0(\bar{\Omega})$ is given by

$$V_h = \{v : v|_K \in P_3(K), \quad \forall K \in T_h, \text{ and } v \text{ and } \frac{\partial v}{\partial x_i}, i=1, 2, \text{ are continuous at the nodes}\},$$

with the following degrees of freedom:

(i) the values of v and $\frac{\partial v}{\partial x_i}$, $i=1, 2$, at the nodes of T_h.

(ii) the values of v at the center of gravity of each $K \in T_h$. □

Example 3.5 Let us now consider a finite element space V_h satisfying the condition $V_h \subset C^1(\bar{\Omega})$. We will then work with functions that are polynomials of degree five on each triangle; with polynomials of lower degree, special constructions are required to satisfy the C^1-condition.

Theorem 3.5 Let K be a triangle with vertices a^i, $i=1, 2, 3$ and let a^{ij} be the midpoint on the side a^ia^j, $i, j=1, 2, 3$, $i<j$ (see Fig 3.2). A function $v \in P_5(K)$ is uniquely determined by the following degrees of freedom:

$$D^\alpha v(a^i), \; i=1, 2, 3, \; |\alpha| \leq 2,$$

$$(3.19) \qquad \frac{\partial v}{\partial n}(a^{ij}), \; i, j=1, 2, 3, \; i<j,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary of K.

Proof Since dim $P_5(K)$ is equal to the number of degrees of freedom (=21), it is sufficient as usual to prove that if all the degrees of freedom according

to (3.19) are zero, then $v \equiv 0$. To see this, we first note that if s denotes the direction of the side a^2a^3, then

$$(3.20) \qquad v(a^i) = \frac{\partial v}{\partial s}(a^i) = \frac{\partial^2 v}{\partial s^2}(a^i) = 0 \qquad i=2, 3.$$

Since v is a polynomial on the side a^2a^3 of degree at most 5, it follows that v vanishes on a^2a^3. Further, $\frac{\partial v}{\partial n}$ is a polynomial of degree at most 4 on a^2a^3 and

$$(3.21) \qquad \frac{\partial v}{\partial n}(a^{23}) = \frac{\partial v}{\partial n}(a^i) = \frac{\partial}{\partial s}\left(\frac{\partial v}{\partial n}\right)(a^i) = 0, \qquad i=2, 3,$$

which is only possible if $\frac{\partial v}{\partial n} \equiv 0$ on a^2a^3. Thus, both v and $\frac{\partial v}{\partial n}$ vanish on a^2a^3 which means that we may factor $(\lambda_1(x))^2$ out of $v(x)$ (check this in the special case when a^2a^3 lies on the x_2-axis). Therefore

$$v(x) = (\lambda_1(x))^2 p_3(x), \qquad x \in K,$$

where $p_3 \in P_3(K)$. In the same way we see that we may also factor out $(\lambda_i(x))^2$, $i=2, 3$, and thus

$$v = \gamma \lambda_1^2 \lambda_2^2 \lambda_3^2,$$

where $\gamma \in R$. But $v \in P_5(K)$ and the only possibility then is that $\gamma = 0$ so that $v \equiv 0$ on K. □

Now let $v_i \in P_5(K_i)$, $i=1, 2$, where K_1 and K_2 are two triangles with common side S and suppose that

$$D^\alpha v_1 = D^\alpha v_2 \quad \text{at the endpoints of S, } |\alpha| \leq 2,$$

$$\frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n} \quad \text{at the midpoint of S,}$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the normal direction to S. Then we have the relations (3.20) and (3.21) for the difference $w=v_1-v_2$ and it follows that

$$(3.22) \qquad w = \frac{\partial w}{\partial n} = 0 \text{ on S.}$$

But if $w=0$ on S we also have that

$$(3.23) \qquad \frac{\partial w}{\partial s} = 0 \text{ on S,}$$

where $\frac{\partial}{\partial s}$ denotes differentiation in the direction tangential to S. By (3.22) and (3.23) we see the function v defined by $v|_{K_i} = v_i$ varies continuously across S as do its first derivatives.

We may now define the space $V_h \subset C^1(\bar{\Omega})$ as follows

$$V_h = \{v : v|_K \in P_5(K),\ \forall K \in T_h,\ D^\alpha v \text{ is continuous at the nodes for } |\alpha| \leq 2 \text{ and } \frac{\partial v}{\partial n} \text{ is continuous at the mid points of each side}\}.$$

with the degrees of freedom of (3.19).

Example 3.6 Let us now construct a three-dimensional finite element. We then assume that Ω is the union of a collection $T_h = \{K\}$ of non-overlapping tetrahedrons K such that no vertex of one tetrahedron lies on a side of another tetrahedron. As above, for $r = 1, 2, \ldots$, and $K \in T_h$, we define

$$P_r(K) = \{v : v \text{ is a polynomial to degree} \leq r \text{ on } K, \text{ i.e. } v \text{ has the form } v(x) = \sum_{i+j+m \leq r} a_{ijm} x_1^i x_2^j x_3^m,\ a_{ijm} \in R\}$$

For $r = 1$ a function $v \in P_1(K)$ is uniquely determined by the values $v(a^i)$, $i = 1, \ldots, 4$, where the a^i are the vertices of K. We can then introduce the space

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_K \in P_1(K),\ \forall K \in T_h\},$$

and as global degress of freedom we may take the values at the nodes of T_h points. □

Example 3.7 Let us also consider some rectangular finite elements that can be used for example if $\Omega \subset R^2$ is a square. Let then K be a rectangle with vertices a^i, $i = 1, \ldots, 4$, and with sides parallel to the coordinate axis in R^2. Define

$$Q_1(K) = \{v : v \text{ is } \textit{bilinear} \text{ on } K, \text{ i.e. } v(x) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2,\ x \in K, \text{ where the } a_{ij} \in R\}$$

It is easy to see (prove this!) that a function $v \in Q_1(K)$ is uniquely determined by the values $v(a^i)$, $i = 1, \ldots, 4$. Further, if K_1 and K_2 are two rectangles with the common side S and the functions $v_i \in Q_1(K_i)$ agree at the endpoints of S then $v_1 - v_2 \equiv 0$ on S since $v_1 - v_2$ varies linearly on S. We may now define

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_K \in Q_1(K),\ \forall K \in T_h\}$$

assuming that $T_h = \{K\}$ is a subdivision of Ω into non-overlapping rectangles such that no vertex of any rectangle lies on a side of another rectangle. The values at the nodes may be used as global degrees of freedom

We can also use polynomials of higher degree on each rectangle. For example we may choose

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_K \in Q_2(K),\ \forall K \in T_h\},$$

where $Q_2(K)$ is the set of *biquadratic* functions on K, i.e.,

$$Q_2(K) = \{v : v(x) = \sum_{i,j=0}^{2} a_{ij} x_1^i x_2^j,\ x \in K, \text{ where the } a_{ij} \in R\}.$$

and use as global degrees of freedom

(i) the values at the nodes of T_h,
(ii) the values at the midpoints of the sides of T_h,
(iii) the values at the midpoint of each rectangle $K \in T_h$

Since the use of rectangular elements requires very special geometry of Ω it is of interest to also consider more general quadrilateral elements. The simplest such element is presented in Problem 12.3 below in connection with so-called isoparametric finite elements

3.3 Summary

We have not yet given a formal definition of what we mean by a "finite element". To fill this gap define a *finite element* to mean a triple (K, P_K, Σ), where

- K is a geometric object, for example a triangle,
- P_K is a finite-dimensional linear space of functions defined on K,
- Σ is a set of degrees of freedom,

such that a function $v \in P_K$ is uniquely determined by the degrees of freedom Σ. From Example 3.1 we have that (K, P_K, Σ), where

- K is a triangle,
- $P_K = P_1(K)$,
- Σ is the values at the vertices of K,

is a finite element. In Fig 3.7 below we have collected some of the most common finite elements (cf [Ci]). The various degrees of freedom are denoted as follows

· function values
○ values of the first derivatives,
◯ values of the second derivatives,
/ value of the normal derivative,
↗ value of the mixed derivative $\frac{\partial^2 v}{\partial x_1 \partial x_2}$.

Finally, Fig 3.8 indicates in the case of two dimensions the *support* of certain basis function $v \in V_h$, i.e., the points x such that $v(x) \neq 0$. The different cases correspond to a value at a node, the midpoint of a side or a point in the interior of an element. Clearly the support is always small and if φ and ψ are two basis functions associated with the nodes N_1 and N_2, then the supports of the functions φ and ψ overlap only if N_1 and N_2 belong to the same element.

Degrees of freedom Σ Geometry	Function space P_K	Degree of continuity of corresponding FEM-space V_h
3	$P_1(K)$	C^0
6	$P_2(K)$	C^0
10	$P_3(K)$	C^0
10	$P_3(K)$	C^0
4	$Q_1(K)$	C^0
9	$Q_2(K)$	C^0
16	$Q_3(K)$	C^1
2	$P_1(K)$	C^0
3	$P_2(K)$	C^0
4	$P_3(K)$	C^1
21	$P_5(K)$	C^1
18	$P_5'(K)$ (see Problem 3.7)	C^1
4	$P_1(K)$	C^0
10	$P_2(K)$ (See Problem 3.4)	C^0

Fig 3.7 Some common finite elements

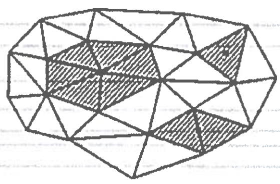

Fig 3.8 The support of different basis functions