Lecture 11

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Numerical Methods for Time-Dependent PDEs, Spring 2024

Travelling Waves (TWs)

general PDE in one space dimension $u_t = F(u)$ ← linear or nonlinear operator in $\frac{\partial}{\partial x}$

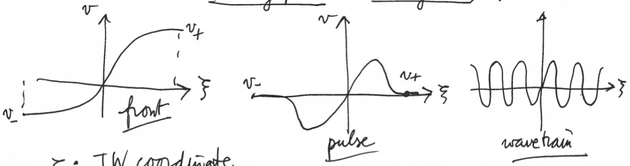
$u(x,t) \in \mathbb{R}^m$, $x \in \mathbb{R}$, $t \geq 0$

A solution $u(x,t)$ of the form $u(x,t) = v(x-ct)$, $x \in \mathbb{R}$, $t \geq 0$ is called a TW with profile v and speed c

If $\lim_{\xi \to -\infty} v(\xi) = v_- \in \mathbb{R}^m$ and $\lim_{\xi \to +\infty} v(\xi) = v_+ \in \mathbb{R}^m$ exist

then u is called a travelling front if $v_- \neq v_+$

and a travelling pulse (or solitary wave) if $v_- = v_+$



ξ: TW coordinate

TWs: Heat equation

Heat equation $\quad u_t = a \cdot u_{xx} \quad , x \in \mathbb{R}, t \geq 0$

$a \in \mathbb{R}^+$

TW "Ansatz" $\quad u(x,t) = v(x - ct) \quad , x \in \mathbb{R}, t \geq 0 \quad \boxed{v' = \frac{dv}{d\xi}}$

$\xi = x - ct \quad \Big\} \Rightarrow -cv' = a v'' \quad , \xi \in \mathbb{R}$

two linearly independent solutions: $v_1(\xi) = 1$ and $v_2(\xi) = e^{-\frac{c}{a}\xi}$

$\xi \in \mathbb{R}$

both are either constant or unbounded

$\Rightarrow$ heat equation has no true bounded TW solutions.

TWs: Advection equation

Advection equation

$u_t + a u_x = 0 \quad , x \in \mathbb{R}, t \geq 0$

$a \in \mathbb{R} \setminus \{0\}$

$u(x,t) = v(x-ct)$

TW ODE : $-cv' + av' = 0 \quad , \xi \in \mathbb{R}$

$\boxed{v' = \frac{dv}{d\xi}}$

v nonconstant $\Rightarrow v' \neq 0 \Rightarrow c = a$

$u(x,t) = v(x-ct)$

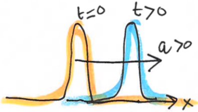
any function v (sufficiently smooth) i.e. $v \in C^1(\mathbb{R}, \mathbb{R})$

is a TW solution for $c = a$

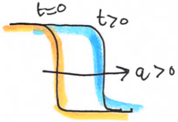
if $u(x,0) = u_0(x) \quad , x \in \mathbb{R}$

then $u(x,t) = u_0(x-ct)$

"all solutions of this PDE are TWs"



or



TWs: Wave equation

wave equation $\quad u_{tt} = a^2 u_{xx}$

BCs: $\begin{cases} u(-\infty,t) = \text{constant} \\ u(+\infty,t) = \text{constant} \end{cases}$

TW "Ansatz": $u(x,t) = v(x-ct)$

$$\underset{(--)}{\overset{}{\Rightarrow}}{}_{+}\; c^2 v'' = a^2 v'' \;\Rightarrow\; v''.(c^2-a^2) = 0$$

1) ⇙ or ⇘ 2)

$v(\xi) = A + B\xi$ $\qquad$ $c = \pm a$ with v arbitrary

$u(x,t) = A + B(x \pm ct)$

BCs can not be satisfied unless $B = 0$

⇓

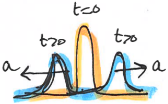
TW solution: $u(x,t) = \text{constant}$

⇓

for any twice differentiable v such that $\lim_{\xi \to \pm\infty} = d_{\pm\infty}$

the solution $u(x,t) = v(x \pm at)$ is a TW solution

(or a travelling pulse if $d_{+\infty} = d_{-\infty}$)



TWs: Burgers [1]

Example 3 (nonlinear) Burgers' equation (with viscosity)

$$\boxed{\underbrace{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}}_{\text{produces shocks}}\underbrace{-\beta\frac{\partial^2 u}{\partial x^2}}_{\text{smooths out the solution}}=0}$$

Assume solution of the form: $u(x,t)=f(x-ct)$

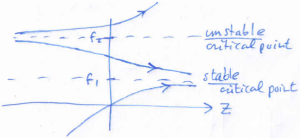
$$\Rightarrow \quad -cf'(z)+f(z)f'(z)-\beta f''(z)=0, \text{ i.e. } \beta f''(z)=\frac{d}{dz}\left[\tfrac{1}{2}f(z)^2-cf(z)\right]$$

then (integrate once) $\beta f'(z)=\frac{1}{2}f(z)^2-cf(z)+\tilde{c}$ or $f'(z)=\frac{1}{2\beta}\left[f^2-2cf+\tilde{c}\right]=\frac{1}{2\beta}(f-f_1)(f-f_2)$

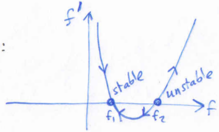
with $f_1+f_2=2c$ and $f_1f_2=\tilde{c}$

$$\Rightarrow f_{1,2}=c\pm\sqrt{c^2-\tilde{c}},\quad 0<\tilde{c}<c^2$$

Note that $f'(z)<0$ if $f_1<f<f_2$ and $f'(z)>0$ if $f<f_1$ or $f>f_2$

$\Rightarrow$ 

in phase plane:



TWs: Burgers [2]

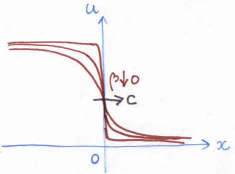
Explicit solution: $\rightsquigarrow \displaystyle\int \frac{df}{(f-f_1)(f-f_2)} = \frac{1}{2\beta}\int dz$

leading to $f(z) = \dfrac{f_1 + f_2 e^{-kz}}{1+e^{-kz}}$ for $k = \dfrac{f_2 - f_1}{2\beta} > 0$ $\quad \odot$

Clearly $f(z) \to f_1$ as $z \to +\infty$, $f(z) \to f_2$ as $z \to -\infty$, $f(0) = \frac{1}{2}(f_1+f_2)$

If we denote $u_\beta(x,t) = f(x-ct)$ for f given by $\odot$, with $c = \frac{1}{2}(f_1+f_2) = f(0)$ then as $\beta \downarrow 0$, $u_\beta(x,t)$ tends to the shock solution of the Riemann problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad u(x,0) = \begin{cases} f_2, & \text{if } x<0 \\ f_1, & \text{if } x>0 \end{cases}$$



Speed of the travelling wave $(\beta \neq 0)$ $c = \frac{1}{2}(f_1+f_2) = f(0)$

$=$ speed of the shock solution for $\beta = 0$

TWs: Burgers [3]

More general version:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} - \beta \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{where } F''(u) > 0$$

$$\Rightarrow f'(z) = \frac{1}{\beta} \left[F(f(z)) - c\, f(z) + C_0 \right]$$

For a travelling wave we must require that $f'(z) \to 0$ as $|z| \to \infty$

and hence $$c = \frac{F(f(\infty)) - F(f(-\infty))}{f(\infty) - f(-\infty)} = \frac{F(f_2) - F(f_1)}{f_2 - f_1}$$

speed = speed of shock solution of the Riemann problem:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0 \qquad (\beta = 0)$$

Example 4 the Korteweg-de Vries equation (KdV)

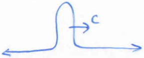
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0$$

$\Rightarrow -c f'(z) + f(z)f'(z) + \beta f'''(z) = 0$, i.e. $\frac{d}{dz}\left[-c f(z) + \frac{1}{2}f(z)^2 + \beta f''(z)\right] = 0$

and $-c f(z) + \frac{1}{2}f(z)^2 + \beta f''(z) = C_0$

Now impose on f the conditions $f(z), f'(z)$ and $f''(z)$ all tend to zero as $|z| \to \infty$

$\Rightarrow$ a solution of the form $f(x-ct)$ in the form of a solitary wave:



Then $-c f(z) + \frac{1}{2} f(z)^2 + \beta f''(z) = 0$

and $-c f(z)f'(z) + \frac{1}{2} f(z)^2 f'(z) + \beta f''(z)f'(z) = 0$

i.e. $\frac{d}{dz}\left[-\frac{1}{2}c f(z)^2 + \frac{1}{6}f(z)^3 + \frac{1}{2}\beta f'(z)^2\right] = 0$

TWs: KDV [2]

then $-\frac{1}{2}c\, f(z)^2 + \frac{1}{6} f(z)^3 + \frac{1}{2}\beta f'(z)^2 = C_1 = 0$ because of solitary wave assumptions

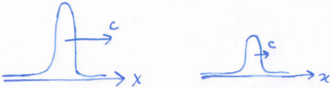
$$\Rightarrow 3\beta f'(z)^2 = 3c\, f(z)^2 - f(z)^3$$

and $f'(z) = \frac{1}{\sqrt{3\beta}} f(z)\sqrt{3c - f(z)}$

then $\int \frac{df}{f\sqrt{3c-f}} = \frac{1}{\sqrt{3\beta}} \int dz$ leads to: $f(z) = 3c\, \mathrm{sech}^2\left(\sqrt{\frac{c}{4\beta}}(z - z_0)\right)$

and $u(x,t) = f(x-ct) = 3c\, \mathrm{sech}^2\left(\sqrt{\frac{c}{4\beta}}(x - ct - z_0)\right)$

the wave speed c is proportional to the wave amplitude $3c$



TWs : KDV [3]

If we let $u = f$ and $v = f'$ then $-c f(z) + \frac{1}{2} f(z)^2 + \beta f''(z) = 0$

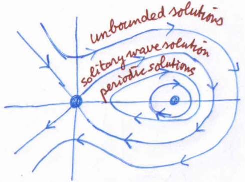
gives $\begin{cases} u' = v \\ v' = \frac{1}{2\beta} u(2c - u) \end{cases}$

the 2-dimensional dynamical system has critical points at $(0,0)$ and $(2c,0)$

the Jacobian of the system is $J(u,v) = \begin{pmatrix} 0 & 1 \\ \frac{c-u}{\beta} & 0 \end{pmatrix}$

then $J(0,0)$ has the eigenvalues $\lambda = \pm\sqrt{\frac{c}{\beta}}$ and $J(2c,0)$ the eigenvalues $\lambda = \pm i\sqrt{\frac{c}{\beta}}$

so that $(0,0)$ is a <u>saddle</u> point and $(2c,0)$ is a <u>center</u>



The <u>homoclinic</u> orbit (from saddle to itself)

= <u>the separatrix</u>

is the solitary wave solution of the KdV PDE model

TWs: Fisher [1]

Example 7 Reaction–diffusion

ex. Fisher's equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1-u)$

$u(x,t) = f(x-ct) = f(z)$, where $f(z)$ approaches constant values as $z \to \pm\infty$

$\Rightarrow -c f'(z) - f''(z) = f(1-f)$

Write as $\begin{cases} f' = g \\ g' = -cg - f(1-f) \end{cases}$ with critical points at $(0,0)$ and $(1,0)$

Jacobian matrix: $J(f,g) = \begin{pmatrix} 0 & 1 \\ 2f-1 & -c \end{pmatrix}$

$\xrightarrow{(0,0)} J = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$ with $\lambda_\pm = \frac{-c \pm \sqrt{c^2-4}}{2}$

$\xrightarrow{(1,0)} J = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$ with $\lambda_\pm = \frac{-c \pm \sqrt{c^2+4}}{2}$

$\Downarrow$

saddle point

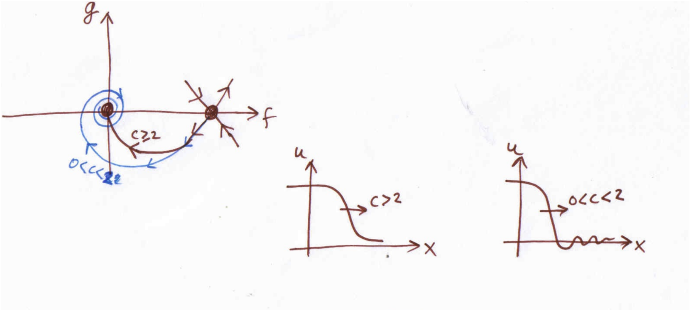
and $\lambda_\pm^{(0,0)} < 0$ if $c > 2$ (then a stable node)

if $0 < c < 2$, $\lambda_\pm^{(0,0)}$ are complex conjugate with negative real part

$\Rightarrow$ stable focus

(f must have finite limits for $z \to \pm\infty$)

TWs: Fisher [2]



TWs: Sine Gordon

Examples the Sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin(u)$$ then $c^2 f''(z) - f''(z) = \sin(f(z))$

or $f''(z) = A \sin(f(z))$, where $A = \frac{1}{c^2-1}$

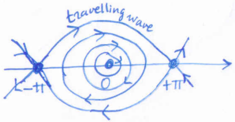
We can rewrite this as $f''(z)f'(z) = A\sin(f(z))f'(z)$ or $\frac{d}{dz}\left[\frac{1}{2}f'(z)^2\right] = -A\frac{d}{dz}\left[\cos(f(z))\right]$

and $f'(z)^2 = c_1 - 2A\cos(f(z)) \Rightarrow f'(z) = \sqrt{c_1 - 2A\cos(f(z))}$

Integration leads to elliptic functions ....

Other way, via dynamical system $f' = g$, $g' = \frac{\sin(f(z))}{c^2-1}$

Critical points at $(n\pi, 0)$; when $c^2>1$ the even integer multiples of π are saddle points while the odd integer multiples of π are centers (when $c^2<1$ it is the other way round ...)



heteroclinic orbit joining saddle $(-\pi,0)$ to saddle $(+\pi,0)$ corresponds to a travelling wave solution

TWs: Schroedinger

Example 6 Nonlinear Schrödinger equation

$$\boxed{i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u|u|^2 = 0}$$

u is complex-valued

The travelling wave solution appears as a standing wave

(the equation arises in applications where it is expressed relative to a frame of reference that is moving with the group velocity of a travelling wave)

$$u(x,t) = f(x)e^{i\beta t} \Rightarrow -\beta f(x) + f''(x) + f(x)^3 = 0$$

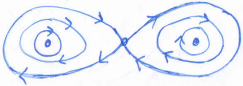
$$\text{or } f''(x) = f(x)(\beta - f(x)^2)$$

explicit integration yields again elliptic functions ----

Instead let $f' = g,\ g' = f(\beta - f^2)$

Critical points at $(0,0)$ and $(\pm\sqrt{\beta},0)$. The origin is a saddle point while the other two are centers

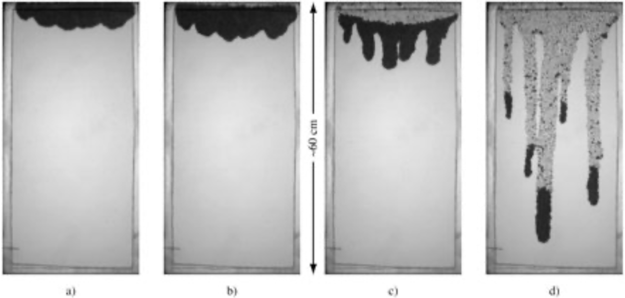
Separatrix is a homoclinic orbit



= solitary wave with $f, f' = 0$ at $x = -\infty$, increases to a point of maximum intensity ($f' = 0, f > 0$) then decreases to $f, f' = 0$ at $x = +\infty$

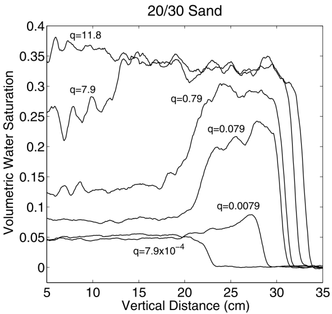
GeoHydro model [1]

Experimental observations by Nicholl & Glass (2005):



GeoHydro model [2]

Experimental measurements by DiCarlo (2004): 



GeoHydro model [3]

Richards  (1904-1993)

Conventional Richards' equation:

$$\frac{\partial S}{\partial t} = \nabla \cdot [K(S)\nabla p] + \frac{\partial}{\partial z}[f(S)], \quad f(S) : \text{fractional flow function}$$

$$p = P(S) : \quad (\text{"}\underline{static}\text{ capillary pressure relation"})$$

$0 \leq S(\vec{x}, t) \leq 1$: effective saturation level; p: water pressure;
$K(S)$: hydraulic conductivity; $P(S)$: equilibrium pressure

GeoHydro model [4]

Majid Hassanizadeh  (Utrecht University)

Nonequilibrium **Richards' equation:**

Replace $p = P(S)$ by a kinetic equation:

$$\mathcal{F}(S, p, S_t, p_t, S_{tt}, p_{tt}) = 0$$

Hassanizadeh & Gray (1990): $\boxed{\tau S_t = p - P(S)}$, with τ a relaxation parameter ($\leftrightarrow$ "dynamic capillary pressure equation")

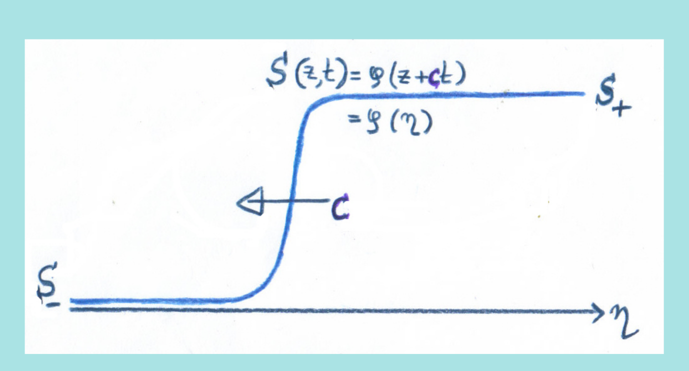
GeoHydro model [5]

Insert into PDE model ⇒

$$S_t = \nabla \cdot [D(S)\nabla S] + [f(S)]_z + \tau\ \nabla \cdot [H(S)\nabla(S_t)]$$

in 1D: $\boxed{S_t = [dD(S)S_z]_z + [f(S)]_z + \tau[H(S)S_{zt}]_z}$

GeoHydro model [6]



GeoHydro model [7]

Assumption:

$S(z,t) = \phi(z + ct) = \phi(\eta),\ \eta \in [-\infty, +\infty]$ and substitute in PDE:

$$c\phi' = [dD(\phi)\phi']' + [f(\phi)]' + [c\tau H(\phi)\phi'']'$$

Integrating $\int_{-\infty}^{\eta}$ and using: $\phi(-\infty) = S_-,\ \phi'(-\infty) = \phi''(-\infty) = 0 \Rightarrow$

$$\boxed{c(\phi - S_-) = dD(\phi)\phi' + f(\phi) - f(S_-) + c\tau H(\phi)\phi''}$$

Take limit $\eta \to \infty$ and using $\phi(+\infty) = S_+,\ \phi'(+\infty) = \phi''(+\infty) = 0$:

$$c = \frac{f(S_+) - f(S_-)}{S_+ - S_-}$$

GeoHydro model [8]

$$\boxed{c(\phi - S_-) = dD(\phi)\phi' + f(\phi) - f(S_-) + c\tau H(\phi)\phi''}$$

Re-write as a Liénard-type system of ODEs:

$$\begin{cases} \phi' = \psi \\ \psi' = \frac{c(\phi - S_-) + f(S_-) - f(\phi) - dD(\phi)\psi}{c\tau H(\phi)} \end{cases}$$

Critical points of ODE system: $\boxed{(\phi, \psi) = (S_-, 0) \text{ and } (\phi, \psi) = (S_+, 0)}$

Example (special case):

$D(S) = d, \; f(S) = S^2;$ linear stability analysis of critical points $\rightsquigarrow$

GeoHydro model [9]

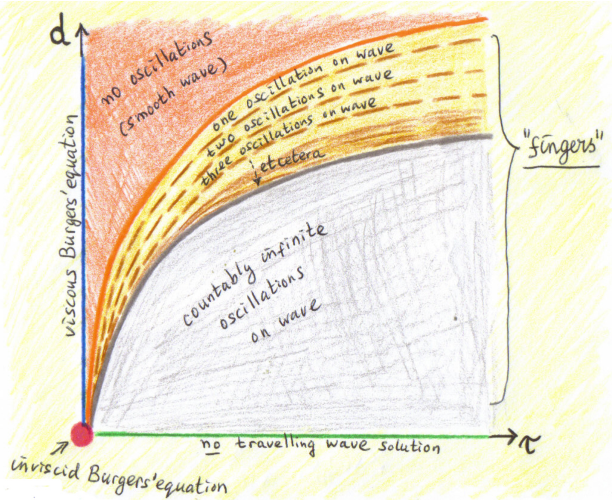
Eigenvalues:

$$\boxed{\lambda_\pm = \frac{-d}{2\tau c S_\pm^2} \pm \sqrt{\frac{d^2}{4\tau^2 c^2 S_\pm^4} + \frac{3c - \frac{2cS_-}{S_\pm} - 4S_\pm + 2\frac{S_-^2}{S_\pm}}{\tau c S_\pm^2}}}$$

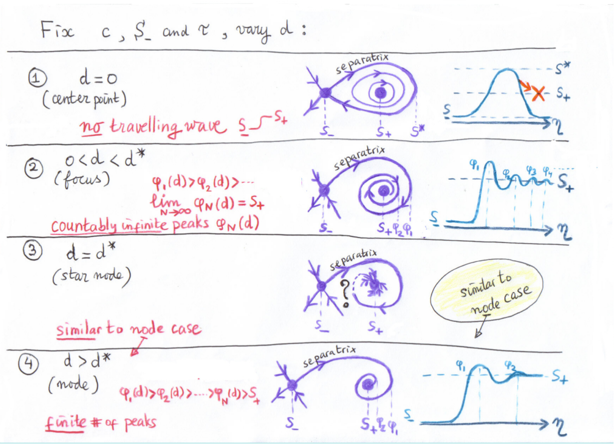
Four cases:

I. Stable focus ("spiral point"), if argument in $\sqrt{\quad} < 0$
II. Stable star node, if argument in $\sqrt{\quad} = 0$
III. Stable (regular) node, if argument in $\sqrt{\quad} > 0$
IV. $d = 0$: center point

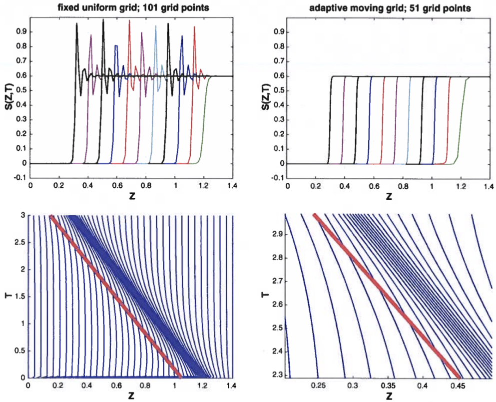
GeoHydro model [10]



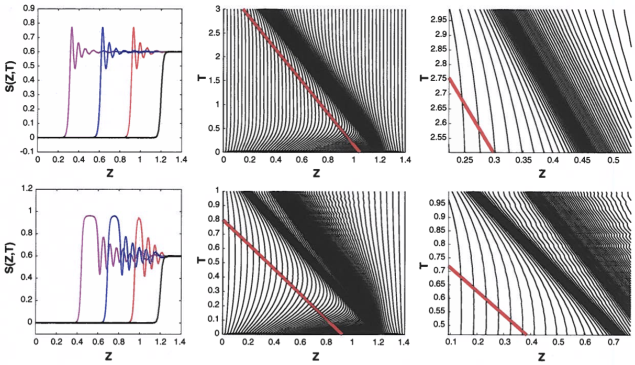
GeoHydro model [11]



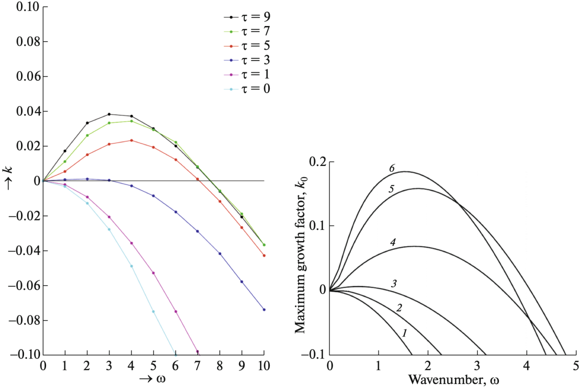
GeoHydro model [12]



GeoHydro model [13]

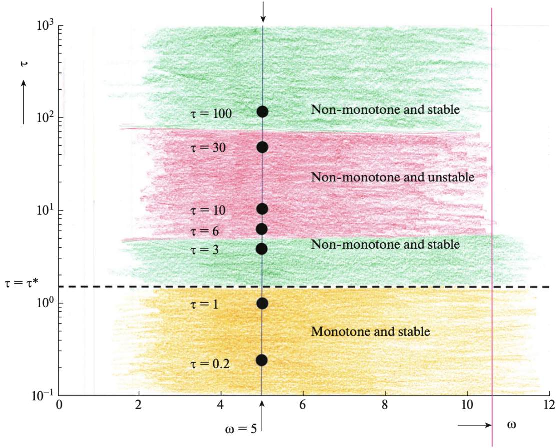


The 2D case [1]



0.10
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−0.10
→ k
0 1 2 3 4 5 6 7 8 9 10
→ ω
τ = 9
τ = 7
τ = 5
τ = 3
τ = 1
τ = 0
0.2
0.1
0
−0.1
Maximum growth factor, k_0
0 1 2 3 4 5
Wavenumber, ω
1
2
3
4
5
6

The 2D case [2]



10^3
τ
10^2
10^1
$\tau = \tau^*$
10^0
10^{-1}
0
2
4
6
8
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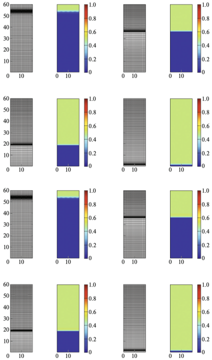
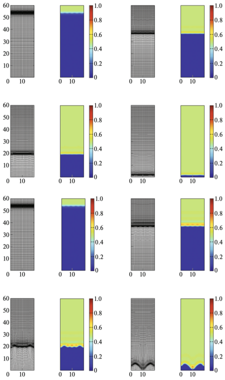
12
$\omega = 5$
ω
$\tau = 100$
$\tau = 30$
$\tau = 10$
$\tau = 6$
$\tau = 3$
$\tau = 1$
$\tau = 0.2$
Non-monotone and stable
Non-monotone and unstable
Non-monotone and stable
Monotone and stable

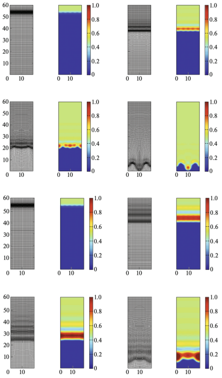
The 2D case [3]



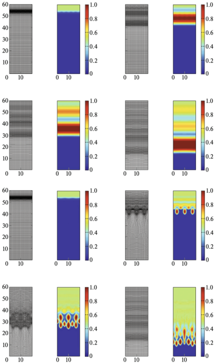
The 2D case [4]



The 2D case [5]



The 2D case [6]



Golden Gate Bridge waves [1]

Historical accounts of travelling wave behaviour in the Golden Gate Bridge in San Francisco motivated McKenna and Walter [21] to study travelling wave solutions in a nonlinear beam equation

$$u_{tt} + u_{xxxx} + f(u) = 0, \tag{1}$$

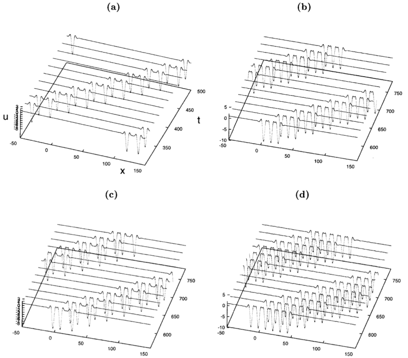
where the highly idealized nonlinearity f is chosen to model the effect that the cable holds the beam up but the constant force of gravity holds it down. (See also [17] for some general issues surrounding using such simple nonlinear beam equations as suspension-bridge models.) This led to the beam equation with restoring force of the general shape

$$f(u) = (u^+ - 1), \quad \text{where} \quad u^+ = \begin{cases} u, & u > 0, \\ 0, & u < 0. \end{cases} \tag{2}$$

Here the function $u(x,t)$ represents the displacement of the beam from the unloaded state. The natural equilibrium is at $u \equiv 1$ and solutions of the form $u(x,t) = 1 + y(x - ct)$ were found explicitly for the piecewise nonlinearity. Such solutions satisfy the ODE

$$y'''' + c^2 y'' + f(y+1) = 0, \tag{3}$$

Golden Gate Bridge waves [2]



Extended KdV5 [1]

In recent years, there has been considerable interest in the numerical treatment of partial differential equations (PDEs) describing nonlinear wave phenomena, and particular types of solitary waves. In this study, we focus our attention on an extended fifth-order Korteweg–de Vries (KdV) equation that can be used to model water waves with surface tension. The model is described by the PDE

$$u_t + \tfrac{2}{15} u_{xxxxx} + (\mu u - b) u_{xxx} + (3u + 2\mu u_{xx}) u_x = 0 \tag{1}$$

for $-\infty < x < \infty$, $t > 0$. In 1997, Champneys and Groves [2] studied the global existence properties of solitary wave solutions to Eq. (1), which can also be written in conservative form:

$$u_t + [\tfrac{2}{15} u_{xxxx} - b u_{xx} + (\tfrac{3}{2}) u^2 + \mu((\tfrac{1}{2})(u_x)^2 + (u u_x)_x)]_x = 0. \tag{2}$$

This is a special case of a more general class of Hamiltonian evolution equations studied by Kichenassamy and Olver [10]. For $\mu = 0$ this reduces to the usual fifth-order KdV equation introduced by Kawahara [9]. The extended form (1) may be derived via a regular Hamiltonian perturbation theory from an exact Euler equation formulation for water waves with surface tension [4]. Looking for traveling-wave solutions $u(x - ct)$, integrating once, setting the constant of integration to be zero, one arrives at the following ODE (the prime standing for $\mathrm{d}/\mathrm{d}\xi$ with $\xi := x + at$),

$$\tfrac{2}{15} u'''' - b u'' + a u + \tfrac{3}{2} u^2 + \mu[\tfrac{1}{2}(u')^2 + (u u')'] = 0, \tag{3}$$

where $a := -c$. Nonzero values of μ can be scaled to plus or minus unity. For the rest of this work we shall take the sign value that is significant for water waves in the presence of surface tension and hence set $\mu = 1$. Physically, $u(x, t)$ represents the height of the free surface of a 2D slab of fluid of finite depth. The parameter b and dimensionless wave speed a are related to the difference between the Bond and Froude numbers respectively from their critical values $(\frac{1}{3}, 1)$.

Extended KdV5 [2]

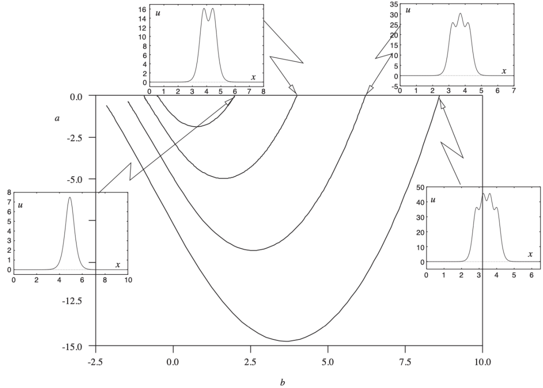


Fig. 1. The region of existence of the embedded solitons for Eq. (3) with $\mu = 1$.

Extended KdV5 [3]

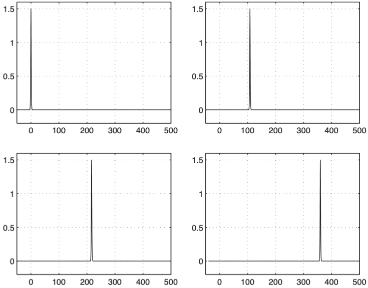


Fig. 4. Numerical solutions for the moving-uniform mesh method with $\rho = 0$ (no initial perturbation) at $t = 0, 90, 180, 300$; $N = 40\,001, b = 0$.

Extended KdV5 [4]

Invariant 1 (conservation of mass for the water waves): $I_1 := \int_{-\infty}^{\infty} u \, \mathrm{d}x$.
Invariant 2 (conservation of the horizontal momentum): $I_2 := \int_{-\infty}^{\infty} u^2 \, \mathrm{d}x$.
Invariant 3 (conservation of the energy): $I_3 := \int_{-\infty}^{\infty} [\frac{1}{15} u_{xx}^2 + \frac{1}{2} u^3 + \frac{1}{2} u_x^2 \{\frac{b}{2} - u\}] \, \mathrm{d}x$.

gives $(\mathrm{d}/\mathrm{d}t) \int_{-\infty}^{\infty} u \, \mathrm{d}x = 0$. Since (1) is a Hamiltonian PDE, it also conserves I_2. Further, if we name the Hamiltonian $\mathscr{H} = I_3$, Eq. (2) is equivalent with

$$u_t = \mathscr{J}^{-1} \frac{\delta \mathscr{H}}{\delta u}, \tag{13}$$

with a skew-symmetric operator $\mathscr{J}^{-1} = \partial/\partial x$, i.e. $\mathscr{J}^{-1} = -\mathscr{J} = \mathscr{J}^{\mathrm{T}}$. The variational derivative in (13) is given by

$$\frac{\delta \mathscr{H}}{\delta u} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \frac{\partial \mathscr{H}}{\partial u_{kx}} = \frac{\partial \mathscr{H}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial \mathscr{H}}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial \mathscr{H}}{\partial u_{xx}} - \cdots. \tag{14}$$

If we define the Poisson brackets

$$\{\mathscr{T}, \mathscr{S}\} = \int_\Omega \frac{\delta \mathscr{T}}{\delta u} \mathscr{J} \frac{\delta \mathscr{S}}{\delta u},$$

then it follows that $\{\mathscr{T}, \mathscr{S}\} = -\{\mathscr{S}, \mathscr{T}\}$. To show that $\mathscr{H}$ is invariant, we work out the following relations (the function f is the integrand of I_3):

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f \, \mathrm{d}x = \int_\Omega \left[\frac{\partial f}{\partial u} u_t + \frac{\partial f}{\partial u_x} u_{tx} + \frac{\partial f}{\partial u_{xx}} u_{txx} + \cdots\right] \mathrm{d}x = \int_\Omega \frac{\delta \mathscr{H}}{\delta u} u_t \, \mathrm{d}x = \{\mathscr{H}, \mathscr{H}\} = 0. \tag{15}$$

Extended KdV5 [5]

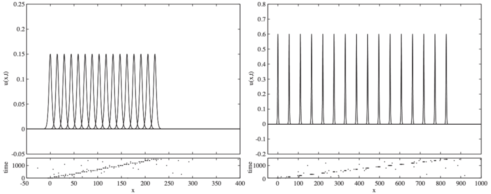


Fig. 7. Numerical solution with the adaptive mesh method in two different cases $b = -0.45$ (left) and $b = -0.30$ (right), at $t = 0, 100, 200, \ldots, 1500$.

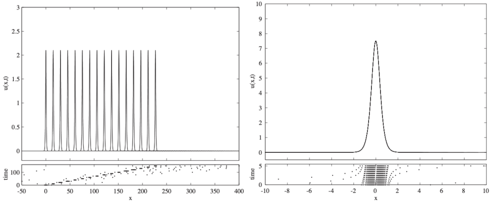


Fig. 8. Numerical solution with the adaptive mesh method in two different cases $b = 0.2$ (left) at $t = 0, 100, 200, \ldots, 1500$, and $b = 2$ (right), at $t = 0, 0.5, 1, \ldots, 5$.

Extended KdV5 [6]

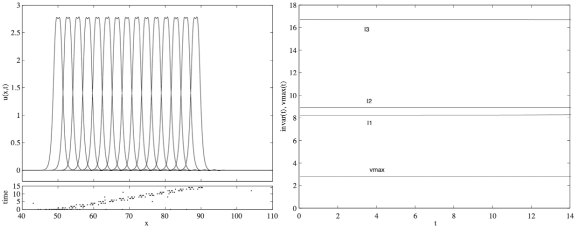


Fig. 9. Numerical solution (left), invariants I_1, I_2, I_3 and soliton amplitude $v_{\max}$ (right) with the adaptive mesh method in the case $b = 0, a = -2.747732$, at $t = 0, 1, 2, \ldots, 14$.

Extended KdV5 [7]

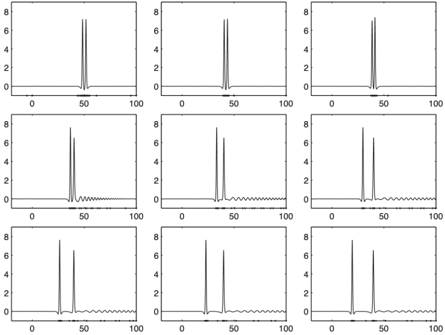
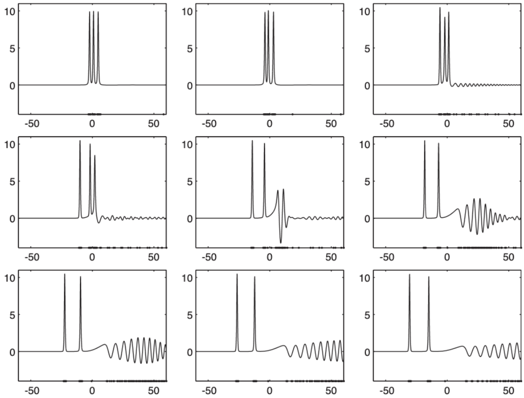


Fig. 11. “Double soliton” solution with the adaptive mesh method in the case $b = 1.5$, $a = 2$, at $t = 0, 5, 6, \ldots, 12$.

Extended KdV5 [8]



;. 13. "Triple soliton" solution with the adaptive mesh method in the case $b = 3$, $a = 2$, at $t = 9, 10, \ldots, 17$.