Lecture 2

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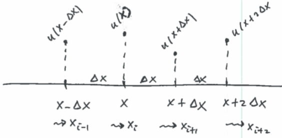
Numerical Methods for Time-Dependent PDEs, Spring 2024

Outline of Lecture 2

- exercises of Lecture 1
- finite difference approximations of derivatives
- method of undetermined coefficients
- finite difference *matrices* and their *eigenvalues*
- non-uniform grids & transformations
- boundary-value models (stationary) ⇒ exercises!
- outlook to Lecture 3

Finite differences [1]

Calculus of finite differences:



constant Δx
~ uniform grid

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forward difference : $\Delta_+[u](x) = u(x+\Delta x) - u(x)$

$\overset{\rightsquigarrow}{\text{forward difference}}$ approximation $\quad D_{1+}u(x) = \dfrac{u(x+\Delta x)-u(x)}{\Delta x} \xrightarrow[\text{matrix}]{} \mathcal{D}_{1+} = \dfrac{1}{\Delta x}\begin{pmatrix} \ddots & \ddots & & \Theta \\ & -1 & 1 & \\ & & -1 & 1 \\ \Theta & & & -1 & 1 \\ & & & & \ddots & \ddots \end{pmatrix}$

$\approx u'(x)$

↑ diagonal

backward difference : $\Delta_-[u](x) = u(x) - u(x-\Delta x)$

$\overset{\rightsquigarrow}{\text{backward difference}}$ approximation $\quad D_{1-}u(x) = \dfrac{u(x)-u(x-\Delta x)}{\Delta x} \xrightarrow[\text{matrix}]{} \mathcal{D}_{1-} = \dfrac{1}{\Delta x}\begin{pmatrix} \ddots & \ddots & & & \\ & -1 & 1 & & \Theta \\ & & -1 & 1 & \\ & & & -1 & 1 \\ \Theta & & & & -1 & 1 \\ & & & & & \ddots & \ddots \end{pmatrix}$

$\approx u'(x)$

↖ diagona

central difference : $\Delta_c[u](x) = u(x+\tfrac{1}{2}\Delta x) - u(x-\tfrac{1}{2}\Delta x)$

$\overset{\rightsquigarrow}{\text{central difference}}$ approximation $\quad D_{1c}u(x) = \dfrac{u(x+\frac{1}{2}\Delta x)-u(x-\frac{1}{2}\Delta x)}{\Delta x} \xrightarrow[\text{matrix}]{} \mathcal{D}_{1c} = \dfrac{1}{2\Delta x}\begin{pmatrix} -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \Theta \\ & & \ddots & \ddots & \ddots & \\ \Theta & & & -1 & 0 & 1 \end{pmatrix}$

OR $\dfrac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}$

$\approx u'(x)$

↖ diagonc

Finite differences [2]

higher-order:

central: $u''(x) \approx \frac{\Delta_c^2[u](x)}{(\Delta x)^2} = \frac{\frac{u(x+\Delta x)-u(x)}{\Delta x} - \frac{u(x)-u(x-\Delta x)}{\Delta x}}{\Delta x}$

$$= \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2}$$

forward: $u''(x) \approx \frac{\Delta_{1+}^2[u](x)}{(\Delta x)^2} = \frac{\frac{u(x+2\Delta x)-u(x+\Delta x)}{\Delta x} - \frac{u(x+\Delta x)-u(x)}{\Delta x}}{\Delta x}$

$$= \frac{u(x+2\Delta x) - 2u(x+\Delta x) + u(x)}{(\Delta x)^2}$$

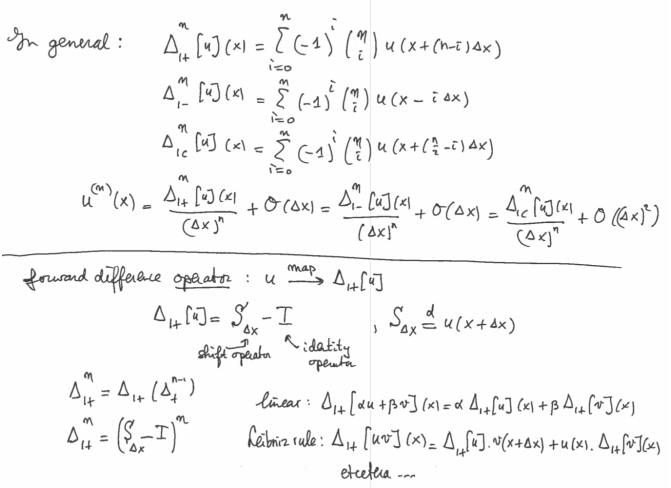
backward: $u''(x) \approx \frac{\Delta_{1-}^2[u](x)}{(\Delta x)^2} = \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{(\Delta x)^2}$

$$\Delta_c\left(\Delta_c[u]\right)(x) = \Delta_c[u]\left(x+\tfrac{1}{2}\Delta x\right) - \Delta_c[u]\left(x-\tfrac{1}{2}\Delta x\right)$$

$$= \left(u(x+\Delta x) - u(x)\right) - \left(u(x) - u(x-\Delta x)\right)$$

$$= u(x+\Delta x) - 2u(x) + u(x-\Delta x)$$

Finite differences [3]



In general:

$$\Delta_{1+}^{n}[u](x) = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} u(x+(n-i)\Delta x)$$

$$\Delta_{1-}^{n}[u](x) = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} u(x-i\Delta x)$$

$$\Delta_{1c}^{n}[u](x) = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} u(x+(\tfrac{n}{2}-i)\Delta x)$$

$$u^{(n)}(x) = \frac{\Delta_{1+}^{n}[u](x)}{(\Delta x)^{n}} + O(\Delta x) = \frac{\Delta_{1-}^{n}[u](x)}{(\Delta x)^{n}} + O(\Delta x) = \frac{\Delta_{1c}^{n}[u](x)}{(\Delta x)^{n}} + O((\Delta x)^{2})$$

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forward difference operator: $u \xrightarrow{\text{map}} \Delta_{1+}[u]$

$$\Delta_{1+}[u] = S_{\Delta x} - I \quad , \quad S_{\Delta x} \overset{d}{=} u(x+\Delta x)$$

shift operator

identity operator

$$\Delta_{1+}^{n} = \Delta_{1+}(\Delta_{+}^{n-1})$$

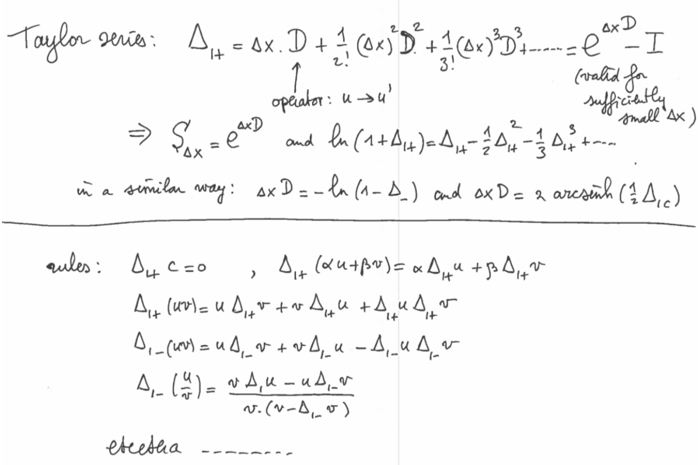
$$\Delta_{1+}^{n} = (S_{\Delta x} - I)^{n}$$

linear: $\Delta_{1+}[\alpha u + \beta v](x) = \alpha \Delta_{1+}[u](x) + \beta \Delta_{1+}[v](x)$

Leibniz rule: $\Delta_{1+}[uv](x) = \Delta_{1+}[u].v(x+\Delta x) + u(x).\Delta_{1+}[v](x)$

etcetera ...

Finite differences [4]



Taylor series: $\Delta_{1+} = \Delta x . \underset{\substack{\uparrow \\ \text{operator: } u \to u'}}{D} + \frac{1}{2!}(\Delta x)^2 D^2 + \frac{1}{3!}(\Delta x)^3 D^3 + \cdots = e^{\Delta x D} - I$ (valid for sufficiently small Δx)

$\Rightarrow \; S_{\Delta x} = e^{\Delta x D}$ and $\ln(1+\Delta_{1+}) = \Delta_{1+} - \frac{1}{2}\Delta_{1+}^2 - \frac{1}{3}\Delta_{1+}^3 + \cdots$

in a similar way: $\Delta x D = -\ln(1-\Delta_{-})$ and $\Delta x D = 2\,\mathrm{arcsinh}(\frac{1}{2}\Delta_{1c})$

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rules: $\Delta_{1+} c = 0$, $\Delta_{1+}(\alpha u + \beta v) = \alpha \Delta_{1+} u + \beta \Delta_{1+} v$

$\Delta_{1+}(uv) = u\,\Delta_{1+} v + v\,\Delta_{1+} u + \Delta_{1+} u\,\Delta_{1+} v$

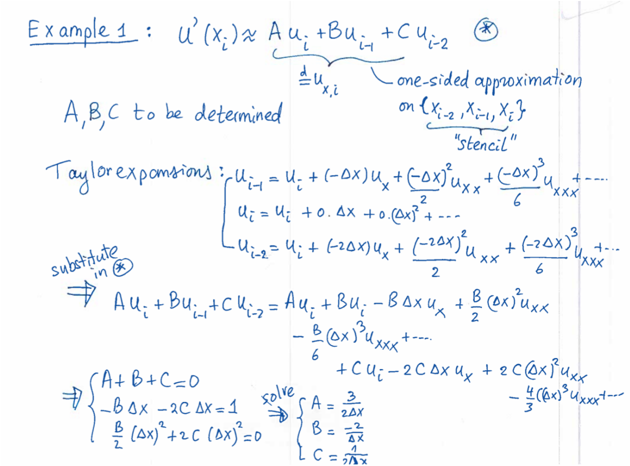
$\Delta_{1-}(uv) = u\,\Delta_{1-} v + v\,\Delta_{1-} u - \Delta_{1-} u\,\Delta_{1-} v$

$\Delta_{1-}\left(\frac{u}{v}\right) = \dfrac{v\,\Delta_{1} u - u\,\Delta_{1-} v}{v.(v - \Delta_{1-} v)}$

etcetera --------

Finite differences [5]

Method of undetermined coefficients:



Example 1 : $$u'(x_i) \approx \underbrace{A u_i + B u_{i-1} + C u_{i-2}}_{\overset{d}{=} u_{x,i}} \qquad \circledast$$

one-sided approximation on $\{\underbrace{x_{i-2}, x_{i-1}, x_i}_{\text{"stencil"}}\}$

A, B, C to be determined

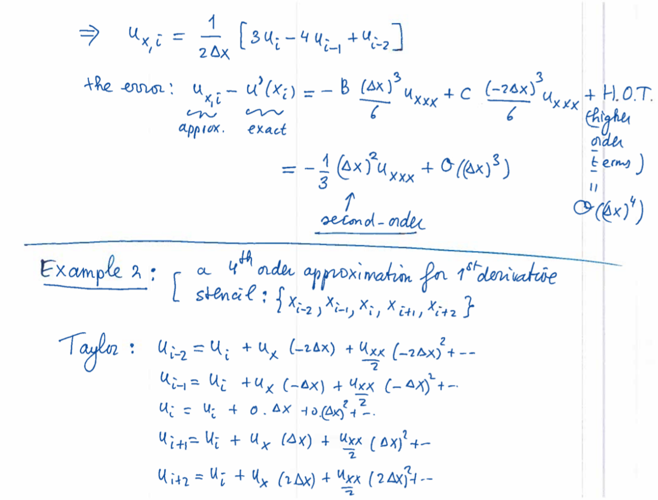
Taylor expansions :
$$\begin{cases} u_{i-1} = u_i + (-\Delta x) u_x + \frac{(-\Delta x)^2}{2} u_{xx} + \frac{(-\Delta x)^3}{6} u_{xxx} + \cdots \\ u_i = u_i + 0 . \Delta x + 0 . (\Delta x)^2 + \cdots \\ u_{i-2} = u_i + (-2\Delta x) u_x + \frac{(-2\Delta x)^2}{2} u_{xx} + \frac{(-2\Delta x)^3}{6} u_{xxx} + \cdots \end{cases}$$

substitute in $\circledast$

$$\Rightarrow \; A u_i + B u_{i-1} + C u_{i-2} = A u_i + B u_i - B \Delta x\, u_x + \frac{B}{2} (\Delta x)^2 u_{xx} - \frac{B}{6} (\Delta x)^3 u_{xxx} + \cdots + C u_i - 2C \Delta x\, u_x + 2C (\Delta x)^2 u_{xx} - \frac{4}{3} (\Delta x)^3 u_{xxx} + \cdots$$

$$\Rightarrow \begin{cases} A + B + C = 0 \\ -B \Delta x - 2C \Delta x = 1 \\ \frac{B}{2} (\Delta x)^2 + 2C (\Delta x)^2 = 0 \end{cases} \xRightarrow{\text{solve}} \begin{cases} A = \frac{3}{2\Delta x} \\ B = \frac{-2}{\Delta x} \\ C = \frac{1}{2\Delta x} \end{cases}$$

Finite differences [6]



$$\Rightarrow \quad u_{x,i} = \frac{1}{2\Delta x}\left[3u_i - 4u_{i-1} + u_{i-2}\right]$$

the error: $\underbrace{u_{x,i}}_{\text{approx.}} - \underbrace{u'(x_i)}_{\text{exact}} = -B\,\frac{(\Delta x)^3}{6}u_{xxx} + C\,\frac{(-2\Delta x)^3}{6}u_{xxx} + \text{H.O.T.}$ (higher order terms) $= \mathcal{O}((\Delta x)^4)$

$$= \underbrace{-\frac{1}{3}(\Delta x)^2 u_{xxx}}_{\uparrow\ \text{second-order}} + \mathcal{O}((\Delta x)^3)$$

Example 2: [a 4th order approximation for 1st derivative
stencil: $\{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$

Taylor:

$$u_{i-2} = u_i + u_x(-2\Delta x) + \frac{u_{xx}}{2}(-2\Delta x)^2 + \dots$$
$$u_{i-1} = u_i + u_x(-\Delta x) + \frac{u_{xx}}{2}(-\Delta x)^2 + \dots$$
$$u_i = u_i + 0\cdot\Delta x + 0\cdot\frac{(\Delta x)^2}{2} + \dots$$
$$u_{i+1} = u_i + u_x(\Delta x) + \frac{u_{xx}}{2}(\Delta x)^2 + \dots$$
$$u_{i+2} = u_i + u_x(2\Delta x) + \frac{u_{xx}}{2}(2\Delta x)^2 + \dots$$

Finite differences [7]

Set: $u_{x,i} = A u_{i-2} + B u_{i-1} + \underset{\substack{\uparrow \\ \text{suppose}}}{0} \cdot u_i + C \cdot u_{i+1} + D \cdot u_{i+2}$

$$---- \Rightarrow \begin{cases} -2A - B + C + 2D = 1/\Delta x \\ 4A + B + C + 4D = 0 & \text{(eliminate } 2^{nd} \text{ order derivative terms)} \\ -8A - B + C + 8D = 0 & (\quad " \quad 3^{rd} \quad " \quad " \quad " \quad) \\ 16A + B + C + 16D = 0 & (\quad " \quad 4^{th} \quad " \quad " \quad " \quad) \end{cases}$$

$$\Rightarrow A = \frac{2}{4!\Delta x}, B = \frac{-16}{4!\Delta x}, C = \frac{16}{4!\Delta x}, D = -\frac{2}{4!\Delta x} \quad \text{error: } \mathcal{O}((\Delta x)^4)$$

In general: $\begin{cases} \text{given (small) } \Delta x, \text{ mth order derivative} \\ \text{error} = \mathcal{O}((\Delta x)^p) \end{cases}$

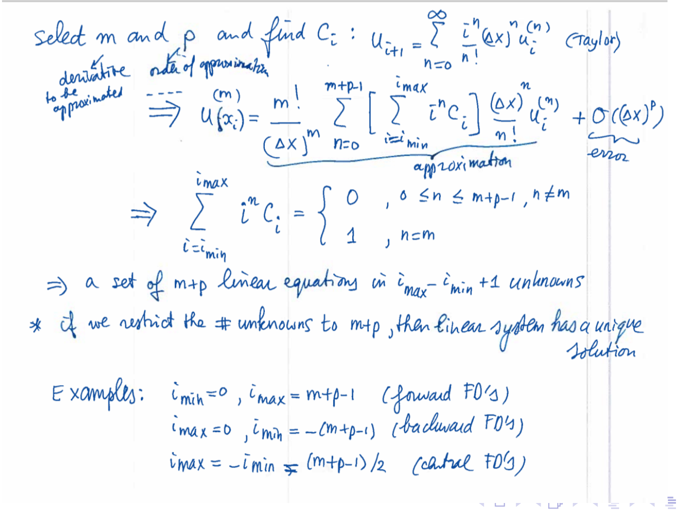
$$\frac{(\Delta x)^m}{m!} u_i^{(m)} = \sum_{i=i_{min}}^{i_{max}} C_i u_i$$

$\vec{C} = (C_{i_{min}}, \dots, C_{i_{max}})$ is called the template

or convolution mask

for the approximation

Finite differences [8]



Select m and p and find C_i : $u_{i+1} = \sum_{n=0}^{\infty} \frac{i^n}{n!} (\Delta x)^n u_i^{(n)}$ (Taylor)

derivative to be approximated   order of approximation

$$\Rightarrow u^{(m)}(x_i) = \underbrace{\frac{m!}{(\Delta x)^m} \sum_{n=0}^{m+p-1} \left[\sum_{i=i_{min}}^{i_{max}} i^n C_i \right] \frac{(\Delta x)^n}{n!} u_i^{(n)}}_{\text{approximation}} + \underbrace{O((\Delta x)^p)}_{\text{error}}$$

$$\Rightarrow \sum_{i=i_{min}}^{i_{max}} i^n C_i = \begin{cases} 0 & , 0 \leq n \leq m+p-1 , n \neq m \\ 1 & , n = m \end{cases}$$

$\Rightarrow$ a set of m+p linear equations in $i_{max} - i_{min} + 1$ unknowns

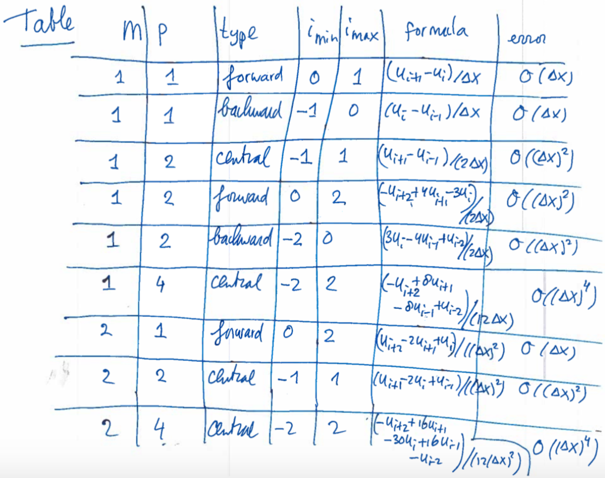
* if we restrict the # unknowns to m+p, then linear system has a unique solution

Examples: $i_{min} = 0$, $i_{max} = m+p-1$ (forward FD's)

$i_{max} = 0$, $i_{min} = -(m+p-1)$ (backward FD's)

$i_{max} = -i_{min} = (m+p-1)/2$ (central FD's)

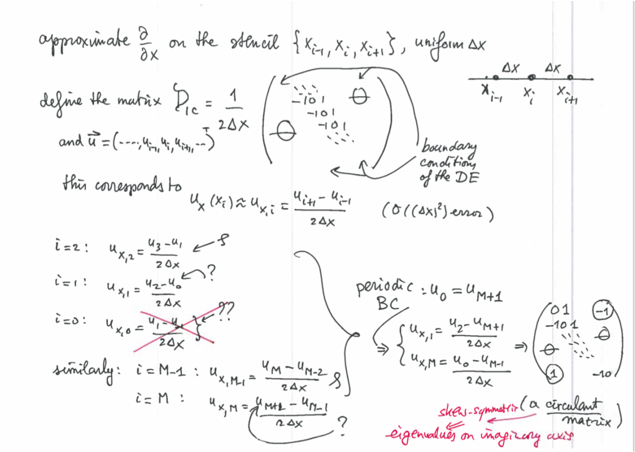
Finite differences [9]



Table

m	p	type	i_{min}	i_{max}	formula	error
1	1	forward	0	1	$(u_{i+1}-u_i)/\Delta x$	$O(\Delta x)$
1	1	backward	−1	0	$(u_i-u_{i-1})/\Delta x$	$O(\Delta x)$
1	2	central	−1	1	$(u_{i+1}-u_{i-1})/(2\Delta x)$	$O((\Delta x)^2)$
1	2	forward	0	2	$(-u_{i+2}+4u_{i+1}-3u_i)/(2\Delta x)$	$O((\Delta x)^2)$
1	2	backward	−2	0	$(3u_i-4u_{i-1}+u_{i-2})/(2\Delta x)$	$O((\Delta x)^2)$
1	4	central	−2	2	$(-u_{i+2}+8u_{i+1}-8u_{i-1}+u_{i-2})/(12\Delta x)$	$O((\Delta x)^4)$
2	1	forward	0	2	$(u_{i+2}-2u_{i+1}+u_i)/((\Delta x)^2)$	$O(\Delta x)$
2	2	central	−1	1	$(u_{i+1}-2u_i+u_{i-1})/((\Delta x)^2)$	$O((\Delta x)^2)$
2	4	central	−2	2	$(-u_{i+2}+16u_{i+1}-30u_i+16u_{i-1}-u_{i-2})/(12(\Delta x)^2)$	$O((\Delta x)^4)$

Finite difference matrices [1]



approximate $\frac{\partial}{\partial x}$ on the stencil $\{x_{i-1}, x_i, x_{i+1}\}$, uniform Δx

define the matrix $D_{1c} = \frac{1}{2\Delta x}\begin{pmatrix} \ddots & \ddots & \ddots & & \Theta \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 & 1 \\ \Theta & & & \ddots & \ddots & \ddots \end{pmatrix}$

and $\vec{u} = (\cdots, u_{i-1}, u_i, u_{i+1}, \cdots)^T$

boundary conditions of the DE

this corresponds to

$$u_x(x_i) \approx u_{x,i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} \qquad (\mathcal{O}((\Delta x)^2)\text{ error})$$

$i=2$: $u_{x,2} = \frac{u_3 - u_1}{2\Delta x} \leftarrow \ell$

$i=1$: $u_{x,1} = \frac{u_2 - u_0}{2\Delta x} \leftarrow ?$

$i=0$: $\cancel{u_{x,0} = \frac{u_1 - u_{-1}}{2\Delta x}} \leftarrow ??$

similarly: $i = M-1$: $u_{x,M-1} = \frac{u_M - u_{M-2}}{2\Delta x}$ ℓ

$i = M$: $u_{x,M} = \frac{u_{M+1} - u_{M-1}}{2\Delta x}$?

periodic BC : $u_0 = u_{M+1}$

$$\Rightarrow \begin{cases} u_{x,1} = \frac{u_2 - u_{M+1}}{2\Delta x} \\ u_{x,M} = \frac{u_0 - u_{M-1}}{2\Delta x} \end{cases} \Rightarrow \begin{pmatrix} 0 & 1 & & -1 \\ -1 & 0 & 1 & \Theta \\ \Theta & \ddots & \ddots & \ddots \\ 1 & & -1 & 0 \end{pmatrix}$$

skew-symmetric ← (a circulant matrix)

⇓

eigenvalues on imaginary axis

Finite difference matrices [2]

forward FD: $u_{x,i} = \dfrac{u_{i+1} - u_i}{\Delta x}$ $\rightsquigarrow$ $\mathcal{D}_{1+} = \dfrac{1}{\Delta x}\begin{pmatrix} -1 & 1 & 0 & \cdots & & \textcircled{0} \\ 0 & -1 & 1 & \cdots & & \\ & & -1 & 1 & & \Theta \\ & \Theta & & \ddots & \ddots & \\ \textcircled{1} & & & & -1 & 1 \end{pmatrix}$

"upwind"

$(\mathcal{O}(\Delta x)$ error$)$

backward FD: $u_{x,i} = \dfrac{u_i - u_{i-1}}{\Delta x}$ $\rightsquigarrow$ $\mathcal{D}_{1-} = \dfrac{1}{\Delta x}\begin{pmatrix} 1 & & & & \textcircled{-1} \\ -1 & 1 & & & \\ & -1 & 1 & & \Theta \\ & \Theta & \ddots & \ddots & \\ \textcircled{0} & & & -1 & 1 \end{pmatrix}$

"downwind"

$(\mathcal{O}(\Delta x)$ error$)$

FD matrix for $\dfrac{\partial^2 u}{\partial x^2}$ at $x = x_i$ with periodic BCs

$$\mathcal{D}_{2c} = \mathcal{D}_{1+}\mathcal{D}_{1-} = \mathcal{D}_{1-}\mathcal{D}_{1+} = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & & \textcircled{1} \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \Theta \\ \Theta & & \ddots & \ddots & \ddots \\ \textcircled{1} & & & 1 & -2 \end{pmatrix}$$

(a symmetric matrix)

$\Downarrow$

real eigenvalues

$\mathcal{D}_{2c}$ with homogeneous Dirichlet BCs: $u_0 = u_M = 0$

$$u_{xx,i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \quad (\mathcal{O}((\Delta x)^2) \text{ error})$$

$i=1$: $(u_2 - 2u_1 + \overset{=0}{u_0})/(\Delta x)^2$

$i=M-1$: $(\overset{=0}{u_M} - 2u_{M-1} + u_{M-2})/(\Delta x)^2$

$$\Rightarrow \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & & \Theta \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ \Theta & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$$

Finite difference matrices [3]

$$\frac{\partial^4}{\partial x^4} \rightsquigarrow \mathcal{D}_{4c} = \left(\mathcal{D}_{2c}\right)^2 = \frac{1}{(\Delta x)^4}\begin{pmatrix} \\ \uparrow \\ \\ \end{pmatrix} \qquad \left(\mathcal{O}\left((\Delta x)^2\right) \text{ error}\right)$$

<u>exercise</u>

$$\frac{\partial^3}{\partial x^3} \rightsquigarrow \mathcal{D}_{3c} = \underset{\substack{\uparrow \\ \partial/\partial x}}{\mathcal{D}_{1+}}\,\underset{\substack{\uparrow \\ \partial/\partial x}}{\mathcal{D}_{1-}}\,\underset{\substack{\uparrow \\ \partial/\partial x}}{\mathcal{D}_{1c}} = \frac{1}{(\Delta x)^3}\begin{pmatrix} 0 & -1 & \frac{1}{2} & & & \boxed{\begin{matrix} -1 & 2 \\ 0 & -1 \end{matrix}} \\ 1 & 0 & -1 & \frac{1}{2} & & \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots & \\ \boxed{\begin{matrix} 1 & 0 \\ -2 & 1 \end{matrix}} & & & -\frac{1}{2} & 1 & 0 \end{pmatrix} \qquad \left(\mathcal{O}\left((\Delta x)^2\right) \text{ error}\right)$$

periodic BCs

etcetera ....

Eigenvalues of FD matrices [1]

tridiagonal matrix A (size: $M-1 \times M-1$)

$$A = \begin{pmatrix} a & b & & & & \\ b & a & b & & \Theta & \\ & b & a & b & & \\ & & \ddots & \ddots & \ddots & \\ \Theta & & & b & a & b \\ & & & & b & a \end{pmatrix}$$

$\Rightarrow \lambda_j(A) = a + 2b\cos\left(\frac{j\pi}{M}\right)$, $j = 1, 2, \dots, M-1$

& eigenvectors : $\vec{v}_j(A) = \begin{pmatrix} v_j^1 \\ \vdots \\ v_j^{M-1} \end{pmatrix}$

with $v_j^k = \sin\left(\frac{kj\pi}{M}\right)$, $k, j \in \{1, \dots, M-1\}$

special case : $a = -\frac{2}{(\Delta x)^2}$, $b = \frac{1}{(\Delta x)^2}$ $\quad (A = D_{2c})$

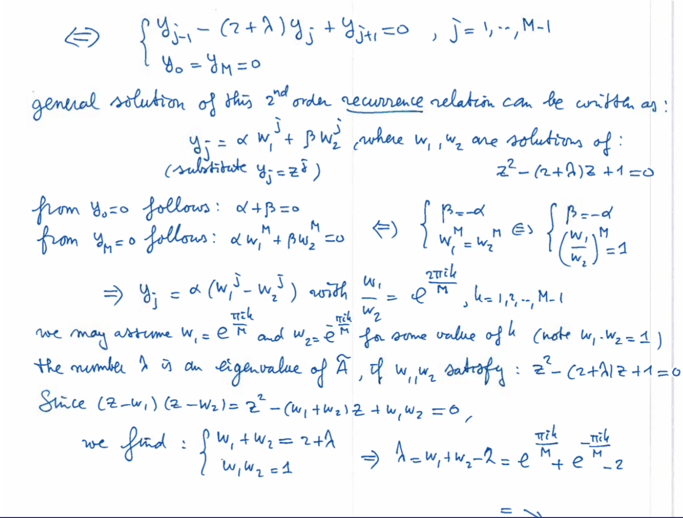
$\Rightarrow \lambda_j(D_{2c}) = \frac{2}{(\Delta x)^2}\left(\cos\left(\frac{j\pi}{M}\right) - 1\right)$, $j = 1, \dots, M-1$

how to find the λ's ?

define $\tilde{A} = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \Theta \\ \Theta & & \ddots & \\ & & 1 & -2 \end{pmatrix}$ (the factor $\frac{1}{(\Delta x)^2}$ can be added at the end of the calculation)

$$\tilde{A}\vec{y} = \lambda\vec{y}$$

Eigenvalues of FD matrices [2]



$$\Longleftrightarrow \begin{cases} y_{j-1}-(2+\lambda)y_j+y_{j+1}=0, \quad j=1,\dots,M-1 \\ y_0=y_M=0 \end{cases}$$

general solution of this 2nd order recurrence relation can be written as:

$y_j=\alpha w_1^j+\beta w_2^j$, where w_1, w_2 are solutions of:

(substitute $y_j=z^j$) $\quad z^2-(2+\lambda)z+1=0$

from $y_0=0$ follows: $\alpha+\beta=0$

from $y_M=0$ follows: $\alpha w_1^M+\beta w_2^M=0$ $\quad \Longleftrightarrow \begin{cases}\beta=-\alpha \\ w_1^M=w_2^M\end{cases} \Longleftrightarrow \begin{cases}\beta=-\alpha \\ \left(\frac{w_1}{w_2}\right)^M=1\end{cases}$

$$\Rightarrow y_j=\alpha(w_1^j-w_2^j) \text{ with } \frac{w_1}{w_2}=e^{\frac{2\pi ik}{M}}, k=1,2,\dots,M-1$$

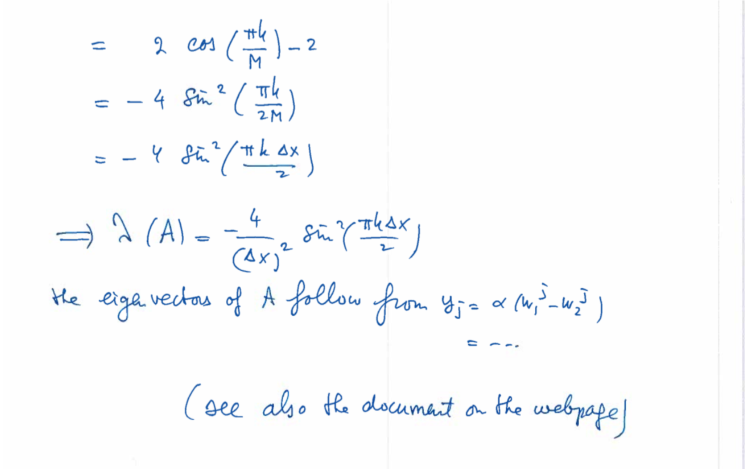
we may assume $w_1=e^{\frac{\pi ik}{M}}$ and $w_2=e^{-\frac{\pi ik}{M}}$ for some value of k (note $w_1 \cdot w_2=1$)

the number λ is an eigenvalue of $\hat{A}$, if w_1, w_2 satisfy: $z^2-(2+\lambda)z+1=0$

Since $(z-w_1)(z-w_2)=z^2-(w_1+w_2)z+w_1w_2=0$,

we find: $\begin{cases} w_1+w_2=2+\lambda \\ w_1w_2=1\end{cases} \Rightarrow \lambda=w_1+w_2-2=e^{\frac{\pi ik}{M}}+e^{-\frac{\pi ik}{M}}-2$

Eigenvalues of FD matrices [3]



$$= 2\cos\left(\frac{\pi k}{M}\right) - 2$$

$$= -4\sin^2\left(\frac{\pi k}{2M}\right)$$

$$= -4\sin^2\left(\frac{\pi k\,\Delta x}{2}\right)$$

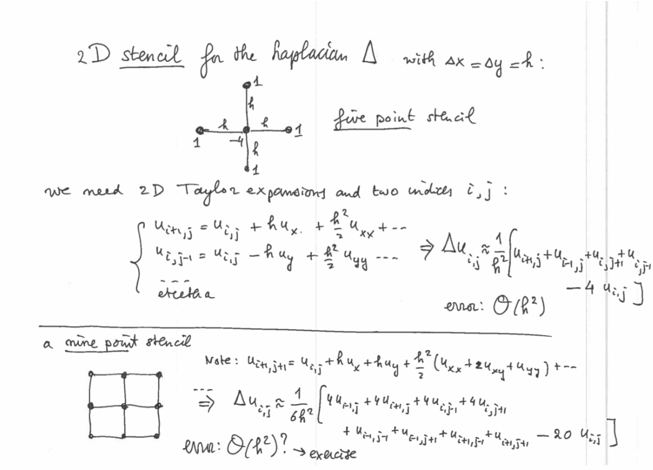
$$\Longrightarrow \lambda(A) = -\frac{4}{(\Delta x)^2}\sin^2\left(\frac{\pi k\,\Delta x}{2}\right)$$

the eigenvectors of A follow from $y_j = \alpha\,(w_1^j - w_2^j)$

$= \ldots$

(see also the document on the webpage)

Higher dimensions [1]



2D stencil for the Laplacian Δ with $\Delta x = \Delta y = h$:

five point stencil

we need 2D Taylor expansions and two indices i, j:

$$\begin{cases} u_{i+1,j} = u_{i,j} + h u_x + \frac{h^2}{2} u_{xx} + \cdots \\ u_{i,j-1} = u_{i,j} - h u_y + \frac{h^2}{2} u_{yy} \cdots \\ \cdots \\ \text{etcetera} \end{cases} \Rightarrow \Delta u_{i,j} \approx \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4 u_{i,j}\right]$$

error: $\mathcal{O}(h^2)$

---

a nine point stencil

Note: $u_{i+1,j+1} = u_{i,j} + h u_x + h u_y + \frac{h^2}{2}(u_{xx} + 2u_{xy} + u_{yy}) + \cdots$

$$\overset{\cdots}{\Rightarrow} \Delta u_{i,j} \approx \frac{1}{6h^2}\left[4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20 u_{i,j}\right]$$

error: $\mathcal{O}(h^2)$? → exercise

Higher dimensions [2]

3D: seven point stencil $\Delta x = \Delta y = \Delta z = h$

$$\Delta u_{i,j,k} \approx \frac{1}{h^2}\left[u_{i+1,j,k} + \cdots + u_{i,j,k-1} - 6\,u_{i,j,k}\right]$$

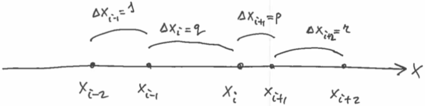
error: $O(h^2)$

dD $$\Delta u_{i_1,i_2,\dots,i_d} \approx \frac{1}{h^2}\left[\underbrace{u_{i_1+1,i_2,\dots,i_d} + \cdots + u_{i_1,i_2,i_{d-1}}}_{2d \text{ terms}} - 2d \cdot u_{i_1,i_2,\dots,i_d}\right]$$

mixed derivative: (in 2D)

$$\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{i,j} \approx \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}}{4h^2}$$

Non-uniform grids [1]



if $p = q = r = s = \cdots = \Delta x$, then uniform grid

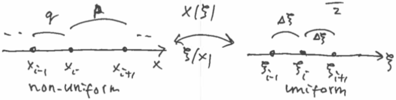
example: $\dfrac{\partial u}{\partial x}$ at $x = x_i$ (central FD, stencil: $\{x_{i-1}, x_i, x_{i+1}\}$

Taylor:

$$u_{i+1} = u_i + p\, u_x + \frac{p^2}{2} u_{xx} + \cdots$$
$$u_i = u_i + 0 \;.\; + 0 \; + \cdots$$
$$u_{i-1} = u_i - q\, u_x + \frac{q^2}{2} u_{xx} \; \cdots$$

$$\cdots \Rightarrow \quad u_{x,i} = \frac{u_{i+1} - u_{i-1}}{p+q} \quad \text{with error:} \; -\underbrace{\frac{p^2-q^2}{2(p+q)}}_{= \frac{p-q}{2}} u_{xx} - \frac{p^3+q^3}{6(p+q} u_{xxx} - \cdots$$

mapping:



Non-uniform grids [2]

define $\Delta\xi = H = \frac{1}{M}$ ← number of gridpoints

$$p = \Delta x_{i+1} = x_{i+1} - x_i = x(\xi_{i+1}) - x(\xi_i)$$
$$= \Delta\xi . x_\xi + \frac{(\Delta\xi)^2}{2} x_{\xi\xi} + \mathcal{O}((\Delta\xi)^3)$$
$$= H . x_\xi + \tfrac{1}{2} H^2 . x_{\xi\xi} + \mathcal{O}(H^3)$$

$$q = \Delta x_{i-1} = x_i - x_{i-1} = x(\xi_i) - x(\xi_{i-1})$$
$$\overset{\cdots}{=} H . x_\xi - \tfrac{1}{2} H^2 x_{\xi\xi} + \mathcal{O}(H^3)$$

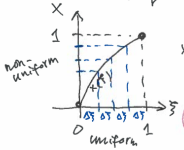
$$\Rightarrow p - q = H^2 x_{\xi\xi} + \mathcal{O}(H^4)$$
$$pq = H^2 x_\xi^2 + \mathcal{O}(H^4)$$

! (check!)

$$\overset{\cdots}{\Longrightarrow} u_{x,i} = \frac{u_{i+1} - u_{i-1}}{p+q} \text{ with error: } -\frac{H^2}{6}\left[3 x_{\xi\xi} u_{xx} + x_\xi^2 u_{xxx}\right] + \mathcal{O}(H^4)$$

second order

properties of mapping (↔ properties of non-uniform grid)



$x(\xi) = \xi$: uniform grid in x

$x_\xi > 0$: mapping is non-singular

x_ξ and $x_{\xi\xi}$ play an important role!

how to choose $x(\xi)$?

→ lecture on adaptive grids

Outlook to Lecture 3

- prepare exercises of Lecture 2 (see webpage!)
- Method-of-Lines (horizontal vs vertical)
- time-integration methods
- stability regions