Lecture 5

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Numerical Methods for Time-Dependent PDEs, Spring 2024

Outline of Lecture 5

- exercises of Lecture 4
- the advection equation: travelling wave solutions
- FTCS
- Lax-Friedrichs & Lax-Wendroff
- upwind/downwind & Beam-Warming
- modified PDE/equation & CFL-condition
- outlook to Lecture 6

The advection equation [1]

$$\boxed{u_t + c\,u_x = 0} \qquad \Rightarrow \qquad u(x,t) = u_0(x-ct)$$
$$\boxed{u(x,0) = u_0(x)}$$

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(1) Method of Characteristics: $\dfrac{dt}{ds} = 1$, $\dfrac{dx}{ds} = c$, $\dfrac{du}{ds} = 0$

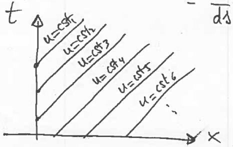
characteristics

$$\Rightarrow \underbrace{\frac{du}{dt}}_{\text{total derivative}} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot \frac{dx}{dt} = 0 \quad \Rightarrow u = \text{constant on each}$$

characteristic curve:

$$x - ct = C$$

$$\frac{dx}{dt} = \frac{dx}{ds} \Big/ \frac{dt}{ds} = c$$



The advection equation [2]

(2) "energy": define $E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u(x,t))^2\, dx$

advection PDE $* u$:
$$u u_t + c u u_x = 0$$
$$\Leftrightarrow u u_t = -c u u_x$$
$$\Leftrightarrow \tfrac{1}{2}(u^2)_t = -\tfrac{c}{2}(u^2)_x$$
$$\Leftrightarrow \frac{1}{2}\int_{-\infty}^{\infty} (u^2)_t\, dx = \underbrace{-\frac{c}{2}\int_{-\infty}^{\infty}(u^2)_x\, dx}$$

$$= -\frac{c}{2}\cdot \lim_{L\to\infty}\int_{-L}^{L}(u^2)_x\, dx$$
$$= -\frac{c}{2}\cdot \lim_{L\to\infty}\left[u^2(L,t) - u^2(-L,t)\right]$$
$$= 0$$

assume $\lim_{L\to\infty} u(\pm L,t) = 0$

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}\int_{-\infty}^{\infty}(u(x,t))^2\, dx\right]$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}(u^2)_t\, dx$$

$$\Rightarrow \frac{dE}{dt} = 0 \text{ and } E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty}(u(x,0))^2\, dx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}(u_0(x))^2\, dx \quad \forall t \geq 0$$

$\Rightarrow$ "Energy" is conserved

The advection equation [3]

(3) Fourier-transform method (see lecture 1) $\Rightarrow u(x,t) = u_0(x-ct)$

"in the x-direction"

(4) Laplace-transform method

"in the t-direction"

take c=1 for example $\underset{\text{apply } \mathcal{L}\text{-transform}}{\Longrightarrow}$ $\quad s\,U(x,s) - u_0(x) + U_x(x,s) = 0$

with $U(x,s) = \mathcal{L}\,(u(x,s))$

solve ODE $\Longrightarrow$ $U(x,s) = e^{-sx} \int_0^x e^{sz} u_0(z)\,dz$

$$= \int_0^x u_0(z)\, e^{-(x-z)s}\, dz$$

apply $\mathcal{L}^{-1}$

$$\Rightarrow \quad u(x,t) = \int_0^x u_0(z) \left[\mathcal{L}^{-1}\left(e^{-(x-z)s}\right)\right]_{s\to t} dz$$

property of $\mathcal{L}$-trafo $\quad = \int_0^x u_0(z)\,.\,\delta(\underbrace{t-(x-z)}_{=z-(x-t)})\,dz = u_0(x-t)$ "shifting property"

The advection equation [4]

(5) travelling wave assumption ("TW Ansatz")

find $u(x,t)$ of the form $\varphi(\underbrace{x-vt}_{=\xi \text{ (TW-coordinate)}})$ with constant velocity v

$u_t = -v\varphi'$ & $u_x = 1 \cdot \varphi'$

where $'$ denotes differentiation with respect to ξ

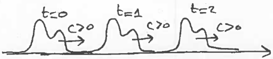
$$\Rightarrow \quad -v\varphi' + c\varphi' = 0 \quad \underset{\varphi' \neq 0}{\Rightarrow} \quad v = c \quad \text{and} \quad u(x,t) = \varphi(x-ct)$$

with $u(x,0) = u_0(x) \Rightarrow u(x,t) = u_0(x-ct)$

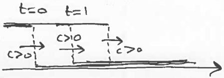
This is a general concept, also applicable to nonlinear PDEs (here, it seems rather trivial)

Solutions of the advection (or transport) equation:

smooth solutions



discontinuous solutions



FTCS [1]

FTCS applied to heat equation $\rightsquigarrow$ stability constraint $\frac{\kappa \cdot \Delta t}{(\Delta x)^2} \le \frac{1}{2}$

? FTCS applied to advection equation $\overset{?}{\rightsquigarrow}$ " $\frac{\Delta t}{\Delta x} \le C$?

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$

$$\Rightarrow u_i^{n+1} = u_i^n - \frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

Taylor expansions: $(u_t)_i^n + (c u_x)_i^n + \frac{\Delta t}{2}(u_{tt})_i^n + c \cdot \frac{(\Delta x)^2}{6}(u_{xxx})_i^n + O((\Delta t)^2) + O((\Delta x)^4)$ $=0$

$$\text{LTE} = \tau_i^n = \underbrace{\frac{\Delta t}{2}(u_{tt})_i^n}_{\text{first order in time}} + \underbrace{c\frac{(\Delta x)^2}{6}(u_{xxx})_i^n}_{\text{second order in space}} + O((\Delta t)^2) + O((\Delta x)^4)$$

as $\Delta t, \Delta x \to 0 \Rightarrow \tau_i^n \to 0 \Rightarrow$ FTCS for advection equation is consistent

FTCS [2]

Von Neumann stability analysis: insert $u_i^n = e^{at} \cdot e^{i\xi_m x}$

$(i = \sqrt{-1})$

$$\Rightarrow \frac{e^{a(t+\Delta t)}\, e^{i\xi_m x} - e^{at}\, e^{i\xi_m x}}{\Delta t} + \frac{c}{2\Delta x}\left[e^{at}\, e^{i\xi_m (x+\Delta x)} - e^{at}\, e^{i\xi_m (x-\Delta x)}\right] = 0$$

divide by $e^{at} e^{i\xi_m x}$ $\Rightarrow \dfrac{e^{a\Delta t} - 1}{\Delta t} + \dfrac{c}{2\Delta x}\left[e^{i\xi_m \Delta x} - e^{i\xi_m \Delta x}\right] = 0$

$\Rightarrow$ amplification factor: $G = e^{a\Delta t} = 1 - \dfrac{i\Delta t}{\Delta x} \sin(\xi_m \Delta x)$

and $|G| = \sqrt{1 + \left(\dfrac{\Delta t}{\Delta x}\right)^2 \sin^2(\xi_m \Delta x)} > 1 \Rightarrow$ unconditionally unstable

(for any $\Delta t, \Delta x$ combination)

Two other ways of interpreting this instability

1) the spectrum $\sigma(D_{1c}) \in i\mathbb{R}$

and $S(EF)$

stability region of EF

$\Rightarrow \sigma(D_{1c}) \cap S(EF) = \emptyset$

FTCS [3]

2) re-write the advection equation and its LTE:

$$u_t + cu_x = 0 \Rightarrow (u_t)_t + (cu_x)_t = u_{tt} + c\, u_{xt} = u_{tt} + c(\underbrace{u_t}_{-cu_x})_x$$

$$= u_{tt} - c^2 u_{xx} = 0$$

$$\Rightarrow u_{tt} = c^2 u_{xx}$$

LTE: $$(u_t)_i^n + (cu_x)_i^n = \frac{-\Delta t}{2}(u_{tt})_i^n + \dots$$

$$= \underbrace{\frac{-\Delta t}{2}(u_{xx})_i^n}_{\text{"negative diffusion"}} + \dots$$

"negative diffusion" (unstable)

remember $u_t = +u_{xx}$  (damping) & $u_t = -u_{xx}$  (ill-posed)

Lax-Friedrichs [1]

A minor modification to FTCS:

replace u_i^n by an averaged value $\frac{1}{2}(u_{i-1}^n + u_{i+1}^n)$

$$\Rightarrow \quad u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

is called Lax-Friedrichs method ("LF")

Note: $\frac{1}{2}(u_{i-1}^n + u_{i+1}^n) = u_i^n + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

$$\overset{LF}{\Rightarrow} \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

re-arrange terms: $\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{(\Delta x)^2}{2\Delta t}\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$

Calculate LTE via Taylor expansions $\Rightarrow$ a consistent approximation of $u_t + cu_x = 0$ for $\Delta t, \Delta x \to 0$

Lax-Friedrichs [2]

However, it looks more as a discretization of the advection-diffusion PDE:

$$\underbrace{u_t + cu_x = \varepsilon_{LF}\, u_{xx}}_{} \quad \text{with } \varepsilon_{LF} = \frac{(\Delta x)^2}{2\Delta t}$$

"numerical diffusion" term ⇝ "numerical damping" of the solution

Stable?

Note that Lax-Friedrichs method "=" FTCS "+" numerical diffusion } in the Method of Lines sense

FTCS: unstable; numerical diffusion: stabilizing effect?

or, LF = EF applied to the ODE system $\dot{\vec{U}} = B_{LF}\vec{U}$

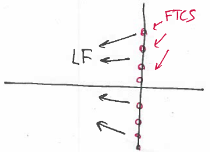
$$\text{with } B_{LF} = \underbrace{\frac{-c}{2\Delta x}\begin{pmatrix} \ddots & \ddots & & \Theta \\ & -1 & 0 & 1 \\ \Theta & & \ddots & \ddots \end{pmatrix}}_{\substack{\text{skew-symmetric} \\ \text{with } \lambda\text{'s} \in i\mathbb{R}}} + \underbrace{\frac{\varepsilon_{LF}}{(\Delta x)^2}\begin{pmatrix} \ddots & \ddots & & \Theta \\ & 1 & -2 & 1 \\ \Theta & & \ddots & \ddots \end{pmatrix}}_{\substack{\text{symmetric} \\ \text{with } \lambda\text{'s} \in \mathbb{R}^-}}$$

Lax-Friedrichs [3]

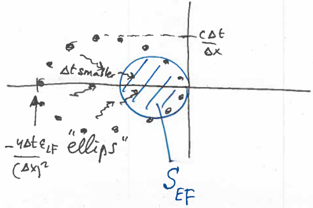
The eigenvectors of both matrices are the same $\Rightarrow$

$$\lambda_p(B_{LF}) = \frac{ic}{\Delta x}\sin(2\pi p\Delta x) - \frac{2\varepsilon_{LF}}{(\Delta x)^2}\left(1-\cos(2\pi p\Delta x)\right),$$

"the eigenvalues of the $D_{1,c}$-matrix for u_x are shifted off the imaginary axis to the left half plane $\mathbb{C}^-$"



Stability is now possible (under certain restrictions)



it can be shown that LF is stable, if $\boxed{\left|\frac{c\Delta t}{\Delta x}\right| \leq 1}$

$\Leftrightarrow$

$\Delta t \cdot \sigma(B_{LF}) \in S_{EF}$

Lax-Wendroff [1]

A <u>higher order</u> approximation $O((\Delta t)^2) + O((\Delta x)^2)$ instead of $O(\Delta t) + O((\Delta x)^2)$

Taylor: $u(x, t+\Delta t) = u(x,t) + \Delta t\, u_t(x,t) + \frac{(\Delta t)^2}{2} u_{tt}(x,t) + \dots$

(in t-direction)

replace u_t by $-cu_x$ and u_{tt} by $+c^2 u_{xx}$

$$\Rightarrow u(x, t+\Delta t) = u(x,t) + c\,\Delta t\, u_x(x,t) + \tfrac{1}{2} c^2 (\Delta t)^2 u_{xx}(x,t) + \dots$$

use $u_x(x_i,t) \approx \frac{u_{i+1}-u_{i-1}}{2\Delta x}$ and $u_{xx}(x_i,t) \approx \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2}$

$\sim D_{1c}$ $\qquad \sim D_{2c} (= D_{1+} D_{1-})$

"Central approximations"

and EF in the time-direction

$$\Longrightarrow u_i^{n+1} = u_i^n - \frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{(\Delta t)^2}{2} c^2 \left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right)$$

(still a <u>three-point</u> stencil!)

<u>Lax-Wendroff method</u> (LW)

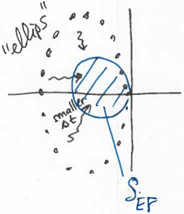
Lax-Wendroff [2]

Write this as an ODE-system (in the MoL-sense)

$$\dot{\vec{U}} = B_{LW}\,\vec{U} \qquad (+\,\mathrm{EF})$$

with $B_{LW} = -\dfrac{c}{2\Delta x}\begin{pmatrix} \ddots & \ddots & & \Theta \\ & -1 \;\; 0 & 1 & \\ \Theta & & \ddots & \ddots \end{pmatrix} + \dfrac{\varepsilon_{LW}}{(\Delta x)^2}\begin{pmatrix} \ddots & \ddots & & \Theta \\ & 1 \;\; -2 & 1 & \\ \Theta & & \ddots & \ddots \end{pmatrix}$ and $\varepsilon_{LW} = \dfrac{c^2\Delta t}{2}$

Note that $\Delta t \cdot \lambda_p(B_{LW}) = i\dfrac{c\Delta t}{\Delta x}\sin(2\pi p\Delta x) - \left(\dfrac{c\Delta t}{\Delta x}\right)^2\left(1-\cos(2\pi p\Delta x)\right)$



ellipses with centre at $-\left(\dfrac{c\Delta t}{\Delta x}\right)^2$ and semi-axis $\left(\dfrac{c\Delta t}{\Delta x}\right)^2$ and $\dfrac{c\Delta t}{\Delta x}$

Stability, if $\boxed{\left|\dfrac{c\Delta t}{\Delta x}\right| \leq 1}$ (then $\Delta t \cdot \lambda_p(B_{LW}) \in S_{EF}$)

Accuracy: 2^{nd} order in space and time

Upwind-downwind [1]

<u>One-sided</u> (non-symmetric) approximations:

$$u_x(x_i,t) \approx \frac{u_i - u_{i-1}}{\Delta x} \quad \text{and} \quad u_x(x_i,t) \approx \frac{u_{i+1} - u_i}{\Delta x}$$

coupled with EF for u_t:

FTBS (↑ backward) $\quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$

FTFS (↑ forward) $\quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$

Both methods are $O(\Delta t) + O(\Delta x)$ "first order"

Note that the exact solution reads: $u(x,t) = u_0(x-ct)$

at the grid point (x_i, t^{n+1}): $u_0(x_i - c(t+\Delta t)) = u_0((x_i - c\Delta t) - ct) = u(x_i - c\Delta t, t)$

$\Rightarrow$ the solution at x_i on the next time level is given by data to the left of x_i if $c>0$ and by data to the right of x_i if $c<0$

This suggests that FTBS might be a better choice for $c>0$ and FTFS for $c<0$.

Upwind-downwind [2]

Von Neumann stability analysis shows that FTBS is stable only if $0 < \frac{c\Delta t}{\Delta x} < 1$

Since $\Delta t, \Delta x > 0$, FTBS can only be used for $c > 0$

and FTFS is stable only if $-1 < \frac{c\Delta t}{\Delta x} < 1$

$\Rightarrow$ FTFS for $c < 0$

FTBS can be written as (compare Lax-Friedrichs)

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \underbrace{c\frac{\Delta t \Delta x}{2}}_{\varepsilon_{upw} = \frac{c\Delta x}{2}} \underbrace{\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right)}_{\approx u_{xx}(x_i, t^n)}$$

Note that, for $c < 0$: $\varepsilon_{upw} < 0$

$\Rightarrow$ negative diffusion added to the FD-scheme $\Rightarrow$ unstable for $c < 0$

and stable (conditionally) for $c > 0$

Upwind-downwind [3]

FTFS can be re-written similarly but now with $\varepsilon_{upw} = -\frac{c\Delta x}{2}$

$\Rightarrow$ FTFS is unstable for $c > 0$
and stable for $c < 0$
(conditionally)

The numerical diffusion constants read:

$$\varepsilon_{LF} = \frac{\Delta x}{2\Delta t}$$

$$\varepsilon_{LW} = \frac{c^2 \Delta t}{2}$$

$$\varepsilon_{upw} = \frac{\text{sign}(c)\Delta x}{2}$$

A <u>2nd order upwind method</u> (one-sided approximation):

follow the derivation of Lax-Wendroff and use <u>one-sided</u> FD's for the spatial derivatives: for $c > 0$

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}\left(3u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) + \frac{1}{2}c^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(u_i^n - 2u_{i-1}^n + u_{i-2}^n\right)$$

<u>"Beam-Warming" method</u> (BW)

Upwind-downwind [4]

BW is stable (via von Neumann) for $0 < \frac{c\Delta t}{\Delta x} < 2 \quad (c>0)$

$\Uparrow$

$\equiv$

Theorem 1 (Lax-Richtmeyer equivalence theorem) ⟵ see also previous lecture

Given a (well-posed) linear hyperbolic PDE and its FD approximation that satisfies the consistency condition. Then stability is the necessary and sufficient condition for convergence of the method.

Theorem 2 (see literature for a proof)

There exists NO explicit unconditionally stable FD scheme for solving hyperbolic PDEs

Theorem 3 Satisfying the so-called $\boxed{\text{"CFL-condition"}}$ → see later is necessary for convergence (but not sufficient!)

Modified equation [1]

We may ask ourselves now the following question:

Is there a PDE $v_t = --$ such that our current FD approximation for $u_t + cu_x = 0$ at (x_i, t^n), u_i^n, is actually the "exact" solution for this new PDE, i.e., $u_i^n = v(x_i, t^n)$?

OR, a bit less ambitiously, can we, at least, find a PDE $v_t = --$, let's call it now $\widetilde{PDE}$, that is "better" satisfied by u_i^n than it does for the original PDE ?

(Then, studying the behaviour of $\widetilde{PDE}$ tells us much about the numerical approximation of our original PDE)

Example: upwind method for $c>0$

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \quad \boxed{\circledast} \quad (FTBS)$$

1) insert $v(x, t)$ into the FD equation

Modified equation [2]

2) suppose $v(x,t)$ agrees exactly with u_i^n at the gridpoints

$\Longrightarrow$ $v(x,t)$ satisfies $\boxed{\bullet}$ exactly.
(unlike $u(x,t)$!!)

$$v(x,t+\Delta t) = v(x,t) - \frac{c\Delta t}{\Delta x}\left(v(x,t) - v(x-\Delta x,t)\right)$$
$$\quad u_i^{n+1} \qquad\quad u_i^n \qquad\qquad u_i^n \qquad u_{i-1}^n$$

3) expand these terms in Taylor series about (x,t):

$$\begin{cases} v(x,t+\Delta t) = v(x,t) + \Delta t\, v_t + \frac{1}{2}(\Delta t)^2 v_{tt} + \dots \\ v(x-\Delta x,t) = v(x,t) - \Delta x\, v_x + \frac{1}{2}(\Delta x)^2 v_{xx} + \dots \end{cases}$$

$$\Longrightarrow\ 0 = v(x,t+\Delta t) - v(x,t) + \frac{c\Delta t}{\Delta x}\left(v(x,t) - v(x-\Delta x,t)\right)$$
$$= \Delta t\, v_t + \frac{1}{2}(\Delta t)^2 v_{tt} + \dots + \frac{c\Delta t}{\Delta x}\left(\Delta x\, v_x - \frac{1}{2}(\Delta x)^2 v_{xx} \dots\right)$$

divide by Δt and re-arrange Δx

$$\Longrightarrow\ \left[v_t + \frac{1}{2}\Delta t\, v_{tt} + \frac{1}{6}(\Delta t)^2 v_{ttt} + \dots\right] + c\left[v_x - \frac{1}{2}\Delta x\, v_{xx} + \frac{1}{6}(\Delta x)^2 v_{xxx} \dots\right] = 0$$

Modified equation [3]

re-write this as : $v_t + c v_x = \frac{1}{2}(c\Delta x\, v_{xx} - \Delta t\, v_{tt})$

(this $\widetilde{\text{PDE}}$ is satisfied by $v(x,t)$) $\quad - \frac{1}{6}\left(c(\Delta x)^2 v_{xxx} + (\Delta t)^2 v_{ttt}\right) + \text{H.O.T.}$ (higher order terms)

4) let us neglect H.O.T.'s and define $\frac{\Delta t}{\Delta x} = \lambda$ (fixed)

$\Rightarrow$ the terms on the righthand side are $O(\Delta t) + O((\Delta t)^2)$

5) for small Δt, we can truncate the series to get a PDE that is quite well satisfied by u_i^n.

[note: dropping all terms on the righthandside gives the original advection PDE]

6) when we keep the $O(\Delta t)$ terms, but neglect the $O((\Delta t)^2)$ terms:

$$v_t + c v_x = \frac{1}{2}(c\Delta x\, v_{xx} - \Delta t\, v_{tt}) \qquad (**)$$

7) derive a slightly different equation:

a) differentiate w.r.t. t : $v_{tt} + c v_{xt} = \frac{1}{2}(c\Delta x\, v_{xxt} - \Delta t\, v_{ttt})$

b) " " x : $v_{tx} + c v_{xx} = \frac{1}{2}(c\Delta x\, v_{xxx} - \Delta t\, v_{ttx})$

8) Combine these two (using $v_{tx} = v_{xt}$) :

$$v_{tt} = c^2 v_{xx} - \frac{c}{2}\left[c\Delta x\, v_{xxx} - \Delta t\, v_{ttx}\right] + \frac{1}{2}\left[c\Delta x\, v_{xxt} - \Delta t\, v_{ttt}\right]$$

Modified equation [4]

$$= c^2 v_{xx} + O(\Delta t) \quad \text{when } \frac{\Delta t}{\Delta x} = \lambda \text{ fixed}$$

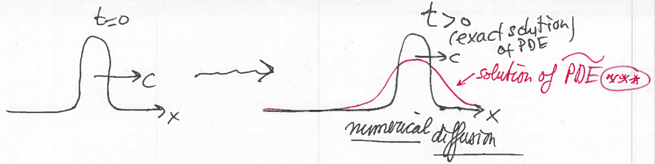
9) insert 8) into (**) $\Rightarrow$ $v_t + c v_x = \frac{1}{2}(c \Delta x v_{xx} - c^2 \Delta t v_{xx}) + O((\Delta t)^2)$ drop this H.O.T. again

$$\Rightarrow \boxed{v_t + c v_x = \frac{1}{2} c \Delta x \left(1 - \frac{c \Delta t}{\Delta x}\right) v_{xx}} \quad (***)$$

Conclusion: the u_i^n from the 1^{st} order upwind for $u_t + c u_x = 0$ can be viewed as a 2^{nd} order approximation of the exact solution of (***).

- PDE: (***) is called the modified equation or modified PDE
- Solutions of (***) move with speed c, but are also diffused ("smeared out")

Modified equation [5]



remark 1 (special case) if $c\Delta t = \Delta x$ then $\frac{1}{2}(c\Delta x - c^2\Delta t) = 0$

and (***) becomes $v_t + c v_x = 0$

(in fact, this holds for all terms in the "H.O.T." expression that we have neglected!)

remark 2 the numerical diffusion coefficient > 0 only if $0 < \frac{c\Delta t}{\Delta x} < 1$

("stable solution")

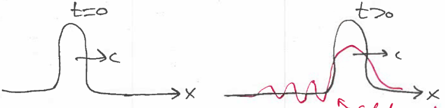
What about Lax-Wendroff?

Following the above recipe gives the modified PDE:

$$v_t + c v_x = -\frac{1}{6} c (\Delta x)^2 \left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) v_{xxx}$$

LW produces a 3rd order approximation of this PDE

(third order derivative)



solution of PDE for LW

numerical dispersion

$\Rightarrow$ "a train of oscillations" behind the solitary wave.

However, this "train" is limited, since (retaining one more term in recipe):

$$v_t + c v_x = -\frac{1}{6} c (\Delta x)^2 \left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) v_{xxx} - \underset{>0}{\varepsilon} v_{xxxx}$$

Modified equation [7]

$\rightsquigarrow$ a higher-order "dissipation" (diffusion) causes the "train" to be damped (partially)

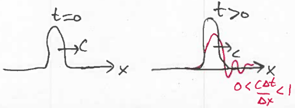
$\Rightarrow$ only a limited number of oscillations are seen in practice

(remember: $+u_{xx}$ damping, $-u_{xx}$ unstable, $+u_{xxxx}$ even more unstable, $-u_{xxxx}$ even more damping, etc...)

Beam-Warming method $\xRightarrow{\ldots}$ $v_t + c v_x = \frac{1}{6} c (\Delta x)^2 \left(2 - \frac{3c\Delta t}{\Delta x} + \left(\frac{c\Delta t}{\Delta x}\right)^2\right) v_{xxx}$

similar as for LW, but now the oscillations move ahead of the wave, instead of behind, if $0 < \frac{c\Delta t}{\Delta x} < 1$

If $1 < \frac{c\Delta t}{\Delta x} < 2$, the oscillations are behind the wave, as in LW



$\boxed{\text{Theorem}}$ Given "any" (consistent) three-point explicit FD scheme for $u_t + cu_x = 0$ of the form:

$$u_i^{n+1} = \alpha u_{i-1}^n + \beta u_i^n + \gamma u_{i+1}^n .$$

Then a necessary (not sufficient!) condition for stability (and therefore, for convergence) is the "CFL-condition"

$$\left| c \frac{\Delta t}{\Delta x} \right| \leq 1$$

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$\boxed{\text{Proof}}$ (see literature)

In other words: a necessary condition for stability is $\Delta t < \frac{\Delta x}{c}$.
Physically: timestep Δt must be < the time for a wave to travel across one grid interval ("time = $\frac{\text{distance}}{\text{speed}} = \frac{\Delta x}{c}$")

Note that, for example, Beam-Warming falls in another category of methods (CHECK)

Outlook to Lecture 6

- prepare exercises of Lecture 5 (see webpage!)
- nonlinear hyperbolic PDEs
- wave equation
- more on the CFL-condition
- finite volumes