1.2 A brief history of fractional derivatives

Most authors on this topic cite a particular date as the birthday of so called "Fractional Calculus" [84, 111]. In a letter dated September 30th 1695, L'Hôpital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$: $\frac{\mathcal{D}^n x}{\mathcal{D}x^n}$. L'Hôpital posed the following question to Leibniz: "what would the result be if $n = 1/2$?". Leibniz's response was: "An apparent paradox, from which one day useful consequences will be drawn." These words were, in fact, the origin of fractional calculus. Following L'Hôpital's and Leibniz's first notion, fractional calculus was primarily a study reserved for the brightest minds in mathematics. Fourier [114], Euler [88] and Laplace [71] are among the many who got interested in fractional calculus and its mathematical consequences.

Many mathematicians found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. Well-known definitions of fractional derivatives, perhaps not yet completely accepted in the calculus community, are the Riemann-Liouville [101, 109, 97], Caputo [22] and Grünwald-Letnikov [101, 109, 97] definitions.

In 1819, the first serious approach to mathematically understand a derivative of an arbitrary order was made by the French mathematician S. F. Lacroix [107]. In his publication, he devoted a few pages to this subject among the total of more than 700 pages. Lacroix started with the monomial $y = x^n$, where n is a positive integer, and worked out the mth derivative for this function:

$$\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!} x^{n-m}. \tag{1.1}$$

Then, he found, realizing the connection between the factorial "!" and the Gamma-function, for the function $y = x^a$ with $a \in \mathbb{R}^+$:

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(a+1/2)} x^{a-1/2}. \tag{1.2}$$

Note that the function denoted by Γ will be defined in section 1.3.1. Lacroix expressed the arbitrary order derivative of the function x^a with $a \in \mathbb{R}$. The special case $a = 1$ reads then:

$$\frac{d^{1/2}}{dx^{1/2}} x = \frac{2\sqrt{x}}{\sqrt{\pi}}, \tag{1.3}$$

since $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(2) = 1$ (see also section 1.3.1).

The concept of fractional derivative has taken almost three hundred years to be formally proposed in a text, since L'Hôpital [111] first described it. The first application in the literature was given by Niels Henrik Abel [12] in 1823. He applied fractional calculus in the solution of an integral equation related with the tautochrone problem [46]. Abel worked out an elegant solution to this problem. It took the attention of Liouville who first gave the formal definition of a fractional derivative. He published a number of memoirs between 1832 and 1855. His starting point was the equality:

$$D^m e^{\lambda x} = \lambda^m e^{\lambda x}. \tag{1.4}$$

After some algebraic calculations, he finally defined the fractional *integral* as follows:

$$D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(u)(x-u)^{\alpha-1}\, du, \quad a \in (-\infty, 0). \tag{1.5}$$

Liouville succeeded to apply his definitions in potential theory. But some of his colleagues believed that these definitions were too narrow.

During the period from 1835 until 1850, there was some controversy between Lacroix's generalization, favored by George Peacock [107], and Liouville's definition. Augustus De Morgan claimed that these two definitions could provide a more general definition. In 1850, William Center [107] found a discrepancy between Lacroix and Liouville. While Lacroix claimed that a fractional derivative of a constant must be unequal to zero, Liouville stated that it is zero because of the property $\Gamma(0) = \infty$.

In 1847, when Riemann was still a student, he published a paper on fractional integrals. He followed Liouville's idea and defined:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(u)(x-u)^{\alpha-1}\, du + \Psi(x), \tag{1.6}$$

where he added a complementary function $\Psi(x)$ (see also [106]).

Most of the mathematical studies regarding fractional calculus were developed prior to the turn of the 20th century. For example, Caputo [101, 109, 97] reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations. Erdélyi and Kober [41] introduced the properties of a generalization of the Riemann-Liouville and Weyl definitions. The topic of fractional calculus wasn't deeply surveyed between the early 1940s and the 1960s of the last century. From that moment, a number of papers were published by Erdélyi, Mikolás [91], Higgins [63], Al-Bassam [4] and others in the 1960's and 1970's. Recently, in the mid 1990's, Kolowankar, as mentioned in [84, 111], again reformulated the Riemann-Liouville fractional derivative in order to differentiate non-differentiable fractal functions.

It is interesting to note that most applications in engineering and sciences have been developed in the past hundred years. Especially, in the last twenty years, numerous applications and physical interpretations of fractional calculus have been described in literature. Among others, we find nowadays fractional derivatives in hydrology (non-Fickian laws), finance (Lévy-flights and non-Markovian models), non-Brownian motions, super- and sub-diffusion (anomalous transport), visco-elasticity, rheology, and electro-physiology of the heart. A few of those will be discussed in section 1.6.

1.3 Special functions in fractional calculus

In this section we discuss two important special functions, that are widely used in fractional calculus.

1.3.1 The Gamma function

The Gamma function $\Gamma(z)$ (see [101, 109, 97]), sometimes denoted by the Euler-Gamma function, plays an important role not only in ordinary calculus and physics, but also in fractional calculus, which allows us to generalize the factorial $n!$ to non-integer values. The Gamma function is often used in probability, statistics and combinatorics. It is defined as follows:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, \mathrm{d}t, \tag{1.7}$$

or, alternatively, by

$$\int_0^1 [-\ln(t)]^{z-1} dt,$$

which is known to converge in the complex plane for $\Re(z) > 0$ (see [101]).

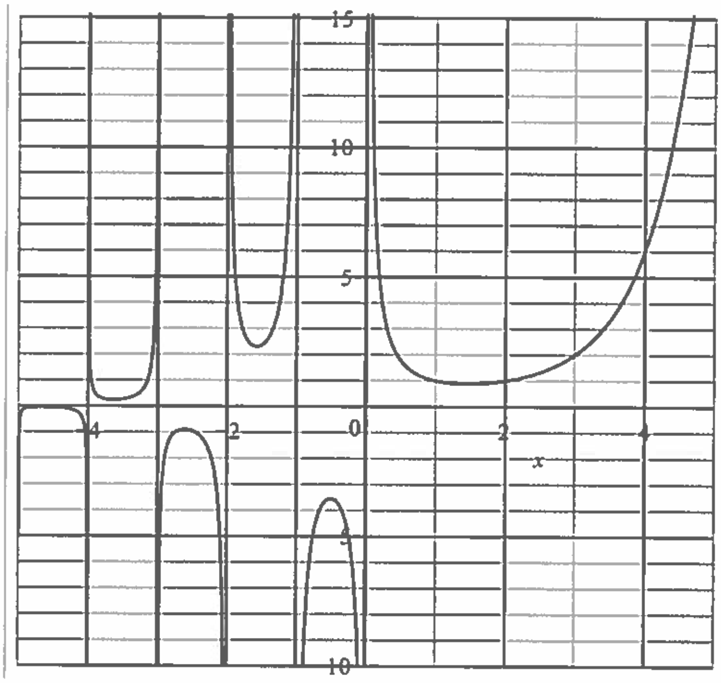

FIGURE 1.1: The Gamma function on the real axis.

An important and widely-used property of the Gamma function is given by the functional equation:

$$\Gamma(z+1) = z\,\Gamma(z), \tag{1.8}$$

which can be easily checked as follows:

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,\mathrm{d}t = [-e^{-t}t^z]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t = z\,\Gamma(z). \tag{1.9}$$

It is obvious that $\Gamma(1) = 1$. We can conclude from (1.8):

$$\begin{cases} \Gamma(2) = 1! = 1, \\ \Gamma(3) = 2! = 2, \\ \Gamma(4) = 3! = 6, \\ \ldots \\ \Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)! = n!. \end{cases} \tag{1.10}$$

The Gamma function has simple poles at the points $z = -n$ for $n = 0,\ 1,\ 2,\ \ldots$. Another interesting formula in which the Gamma function appears is the following limit representation:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}, \quad \Re(z) > 0. \tag{1.11}$$

Further, it can also be shown that $\Gamma(z)$ defines an analytical function on $z \in \mathbb{C} : \Re(z) > 0$ (see [101]). There exist also several alternative definitions for the Gamma function:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}, \tag{1.12}$$

$$\Gamma(z) = \frac{\mathrm{e}^{-\gamma z}}{z}\prod_{n=1}^{\infty}(1+\frac{z}{n})^{-1}\mathrm{e}^{\frac{z}{n}}, \tag{1.13}$$

where γ is the Euler-Mascheroni constant, approximately equal to 0.577216.... Another useful property of the Gamma function is the one given by Euler's reflection formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}. \tag{1.14}$$

From this follows, for example, easily that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This property helps us to calculate fractional order derivatives and integrals more easily. Some other Gamma function values are [101]:

$$\begin{cases} \Gamma(-2) = \infty \\ \Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi} \approx 2.363271 \\ \Gamma(-1) = \infty \\ \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \approx 0.886226 \\ \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \approx 1.329340 \\ \Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \approx 3.323350, \quad etcetera. \end{cases} \tag{1.15}$$

The Gamma function is also strongly related to the so-called Beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.16}$$

Note that the function $\Gamma(x)$ is not a unique solution of (1.8) (for positive values of $x = \Re(z)$). Other solutions of this functional equation are, e.g.:

$$\cos(2m\pi x)\Gamma(x),\ m \in \mathbb{N},$$

or

$$\frac{1}{\Gamma(1-x)}\frac{d}{dx}\ln\left(\frac{\Gamma(\frac{1}{2}-\frac{1}{2}x)}{\Gamma(1-\frac{1}{2}x)}\right).$$

With an extra condition on the function that must satisfy (1.8), Bohr and Mollerup in 1922 [20], proved that the Gamma function $\Gamma(x)$ *is* the unique solution of the functional equation. They added the extra constraint on $f(x)$, that it must be *logarithmically convex*. Their theorem is known as the Bohr-Mollerup theorem.

1.3.2 The Mittag-Leffler function

Another important formula in fractional calculus is the Mittag-Leffler (ML) function [93]. The Mittag-Leffler function shows its importantance in physics, biology, engineering and applied sciences. The Mittag-Leffler function is a fundamental solution of fractional differential equations and fractional order integral equations. The most noticeable general property of the Mittag-Leffler function is dealing with the Laplace transform and asymptotic expansions of these functions. The Mittag-Leffler function has two definitions in the literature:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\quad \alpha \in \mathbb{C},\ \Re(\alpha) > 0, \tag{1.17}$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\quad \alpha,\beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0. \tag{1.18}$$

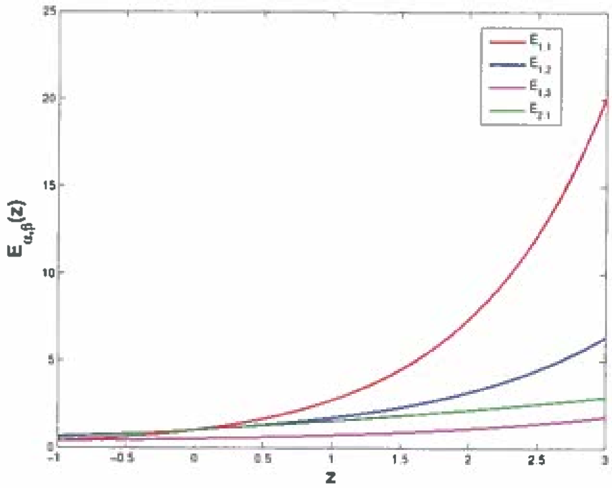

FIGURE 1.2: The Mittag-Leffler function (1.18) for different α and β values. The exponential function can be recognized, if we choose $\alpha = 1$ and $\beta = 1$: $E_{1,1} = e^x$.

Its graph is displayed in Figure 1.2 for different α and β values. Definition (1.17), which can be seen as a generalization of the exponential function, was studied by Mittag-Leffler in 1903.

This formula gives the Taylor series of the exponential function in the case $\alpha = 1$. The ML-function can be found in the exact solution of the following time-fractional fractional differential equation:

$$D_t^{\alpha} u(t) = -\lambda u(t), \; u(0) = u_0. \tag{1.19}$$

Its solution reads (using elements from section 1.4):

$$u(t) = E_{\alpha}(-\lambda t^{\alpha}). \tag{1.20}$$

In Figure 1.3 these solutions are displayed for several values of the parameter α. Definition

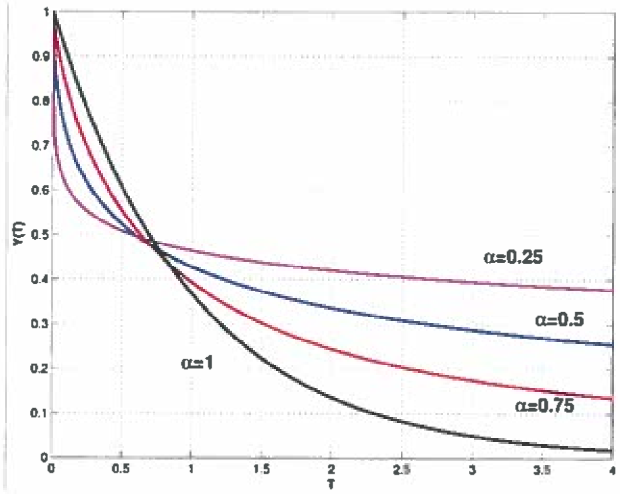

FIGURE 1.3: The solution (1.3) of the fractional differential equation (1.19) for several α values, $\lambda = 1$ and $u_0 = 1$. We see again, that for $\alpha = 1$, we obtain the solution $u(t) = e^{-t}$

(1.18) is another, more general, version of (1.17). It was studied by Wiman [124] in 1905 and Agarwal [3] in 1953.

Here, we mention only a few basic properties of the Mittag-Leffler functions:

$$\begin{cases} E_{\alpha,\beta}(z) = zE_{\alpha,\beta+\alpha}(z) + \frac{1}{\Gamma(\beta)}, \\ E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z), \\ \frac{d^m}{dz^m}\left[z^{\beta-1} E_{\alpha,\beta}(z^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^{\alpha}), \;\; \Re(\beta - m) > 0, \;\; m = 0, 1, ... \\ \frac{d}{dz} E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta-1) E_{\alpha,\beta}(z)}{\alpha z}. \end{cases}$$

More details and properties concerning these functions can be found in [56].

1.4 Definitions of fractional derivatives

In fractional calculus, many definitions of fractional derivatives exist, depending on the starting point of the theory, on the underlying function space and on the type of application, respectively. A complete overview will not be given (for this we refer to [101, 109, 12, 35]), but we would like to mention the four most relevant definitions connected to the underlying thesis.

We start with a well-known, seemingly, contradictory example that can be explained later on with two different points of view in defining fractional derivatives. We call it a "mysterious contradiction" and it can be worked out easily by applying, in a straightforward way, the Taylor series of the exponential function in two ways. For this, consider the monomial $y(x) = x^k$ and calculate its nth derivative. Following the rule that connects the factorial and the Gamma-function from section 1.3.1, we recognize immediately:

$$\frac{d^n y}{dx^n} = \frac{k!}{(k-n)!} x^{k-n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} x^{k-n} \ , k \geq n. \tag{1.21}$$

There seems to be no objection to replace the integer n by the real number $\alpha > 0$:

$$\frac{d^\alpha y}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \ , k \geq \alpha \in \mathbb{R}^+. \tag{1.22}$$

Furthermore, working out the Taylor series of the exponential function, we find:

$$y(x) = e^x \Rightarrow \frac{d^n y}{dx^n} = e^x, \tag{1.23}$$

which suggests that we can take:

$$\frac{d^\alpha y}{dx^\alpha} = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}, \ \alpha \in \mathbb{R}^+. \tag{1.24}$$

On the other hand, using similar properties and calculations, we obtain the following:

$$y(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{1.25}$$

$$\Rightarrow \frac{d^n y}{dx^n} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{k!}{(k-n)!} x^{k-n} = \sum_{k=0}^{\infty} \frac{x^{k-n}}{(k-n)!} = \sum_{k=0}^{\infty} \frac{x^{k-n}}{\Gamma(k-n+1)}. \tag{1.26}$$

From this would directly follow:

$$\frac{d^\alpha y}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{1.27}$$

Comparing equations (1.24) and (1.27), we might conclude that defining a fractional derivative would not be possible in a consistent way. In the following sections we will see that it is possible to explain the confusing difference in these two expressions. For this, we first need to define how we can calculate a fractional integral.

1.4.1 Fractional order integrals

Fractional order integrals can be defined using the well-known concept of repeated integration. This is done by using Cauchy's iterated integral formula. Next, we define the half-Schwartz space[1]

$$\bar{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}^-) : ||f||_{k,l} < \infty, \ \ \forall \, k, l \in \mathbb{Z}^+\}, \tag{1.28}$$

where

$$||f||_{k,l} = \sup_{x \in \mathbb{R}^-} |x^k \frac{d^l f}{dx^l}|.$$

[1]This means, in fact, that functions f must satisfy the following property: $P(x)\frac{d^l f}{dx^l} \to 0$, for $x \to -\infty$ for all polynomials $P(x)$ and all integer lth order derivatives.

For functions in $\tilde{S}(\mathbb{R})$, we define the sequence of integral operators $\mathcal{J}^0, \mathcal{J}^1, \mathcal{J}^2, \ldots$ in the following way:

$$\begin{aligned}
&\mathcal{J}^0 f(x) = f(x),\\
&\mathcal{J}^1 f(x) = \textstyle\int_{-\infty}^{x} f(s)\, ds,\\
&\mathcal{J}^2 f(x) = \textstyle\int_{-\infty}^{x} \mathcal{J}^1 f(s)\, ds, \qquad (1.29)\\
&\ldots\ldots\\
&\mathcal{J}^n f(x) = \textstyle\int_{-\infty}^{x} \mathcal{J}^{n-1} f(s)\, ds.
\end{aligned}$$

When applying Cauchy's iterated integral formula to a function f on the interval $(-\infty, x]$, we obtain the nth order integral:

$$I^n f(x) := \frac{1}{(n-1)!} \int_{-\infty}^{x} (x-s)^{n-1} f(s)\, ds. \qquad (1.30)$$

Note that, it can be checked easily, that $\mathcal{J}^n f = I^n f,\ n \in \mathbb{N}$. With this result, it is straightforward to define the fractional integral of order $\alpha \in \mathbb{R}^-$, making use of the fact that $n! = \Gamma(n+1)$ for $n \in \mathbb{N}$:

$$\mathcal{J}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}}\, ds. \qquad (1.31)$$

An important property of this integral is related to a basic property in semi-group theory:

$$\begin{cases} \mathcal{J}^\alpha \mathcal{J}^\beta = \mathcal{J}^\beta \mathcal{J}^\alpha = \mathcal{J}^{\alpha+\beta}, \ \forall \alpha, \beta \geq 0 \\ \mathcal{J}^0 = \mathcal{I} \end{cases}. \qquad (1.32)$$

This result states that one can interchange the order of fractional integration arbitrarily. The property does not hold, however, in general, for fractional derivatives (see the next sections).

1.4.2 The Riesz derivative

Using formula (1.31) for the fractional integral, we are now able to define, for functions $f \in \tilde{S}(\mathbb{R})$, the so-called Riesz fractional derivative:

$$\mathcal{D}_R^\alpha f(x) := \mathcal{J}^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \ m = \lceil \alpha \rceil, \ f \in \tilde{S}(\mathbb{R}).$$

This derivative can be written as a convolution with a weakly singular kernel:

$$\begin{aligned}
\mathcal{J}^{m-\alpha} f(x) &= \tfrac{1}{\Gamma(m-\alpha)} \textstyle\int_{-\infty}^{x} \tfrac{f(s)}{(x-s)^{1+\alpha-m}}\, ds, \ m = \lceil \alpha \rceil, \ f \in \tilde{S}(\mathbb{R}) \\
&= \chi_+^{m-\alpha} * f(x), \ \text{where } \chi_+^{m-\alpha}(x) := \tfrac{1}{\Gamma(m-\alpha)} x^{m-\alpha-1} H(x).
\end{aligned} \qquad (1.33)$$

It can be checked that the Riesz derivative is consistent with the non-fractional (ordinary) derivative. For this, we can use the following basic steps:

$$\begin{aligned}\mathcal{D}_R^k f &= \mathcal{J}^{m-k}(\tfrac{d^m}{dx^m} f), \; m > k \\ &= \chi_+^{m-k} * (\tfrac{d^m}{dx^m} f) \\ &= \tfrac{d^k}{dx^k}[(\tfrac{d^{m-k}}{dx^{m-k}} \chi_+^{m-k}) * f] \\ &= \tfrac{d^k}{dx^k}[\delta * f] \\ &= \tfrac{d^k}{dx^k} f.\end{aligned} \tag{1.34}$$

Here, δ, χ and H are the Dirac-delta function, the indicator function and the Heaviside step function, respectively.

1.4.3 The Riemann-Liouville and Caputo derivative

Two other related types of fractional derivatives, named Caputo and Riemann-Liouville derivative, respectively, are defined by:

$$\mathcal{D}_C^\alpha f(x) := \mathcal{J}_0^{m-\alpha}(\frac{d^m}{dx^m} f(x)), \;\; x > 0, \;\; m = \lceil \alpha \rceil, \tag{1.35}$$

$$\mathcal{D}_{RL}^\alpha f(x) := \frac{d^m}{dx^m}(\mathcal{J}_0^{m-\alpha}(f(x))), \;\; x > 0, \;\; m = \lceil \alpha \rceil. \tag{1.36}$$

Note that we changed the lower limit in the fractional order integral (1.31) from $-\infty$ to 0. The underlying integral operator now reads:

$$\mathcal{J}_0^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} \, ds, \;\; x > 0. \tag{1.37}$$

An interesting difference between the two derivatives (1.35) and (1.36) can be directly observed by applying them to a constant function:

$$\mathcal{D}_C^\alpha(constant) = 0, \tag{1.38}$$

whereas

$$\mathcal{D}_{RL}^\alpha(constant) \sim x^{-\alpha} \neq 0. \tag{1.39}$$

It is obvious then that the fractional derivative of a function depends on which definition is being used. However, there exist also correspondences between the Caputo and Riemann-Liouville derivatives. An important result connecting these two derivatives is given by the following result:

Theorem [101]

Let $f \in L^1([0,\infty)) \cap C^m([0,\infty))$ and $m-1 < \alpha \leq m$ for given $m \in \mathbb{N}$. Then:

$$\mathcal{D}_{RL}^\alpha f(x) = \mathcal{D}_C^\alpha f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(1+k-\alpha)} x^{k-\alpha} \;\; , x > 0, \tag{1.40}$$

where the notation $f^{(k)}(0^+) = \lim_{x \downarrow 0} f^{(k)}(x)$ is being used.

As a corollary to this theorem we find then:

$$\text{if } f^{(k)}(0^+) = 0, \; k = 0, 1, ..., m-1, \text{ then } \mathcal{D}_{RL}^\alpha = \mathcal{D}_C^\alpha, \tag{1.41}$$

which connects the Caputo and Riemann-Liouville derivative, for special types of functions. The consistency of Caputo's definition will be discussed in Chapter 2.

1.4.4 The Grünwald-Letnikov derivative

The Grünwald-Letnikov definition of a fractional derivative can be viewed as a derivative which finds its roots in the definition of a first derivative in terms of a limit:

$$f'(x) = \lim_{h_1 \to 0} \frac{f(x) - f(x - h_1)}{h_1} \approx \frac{f(x) - f(x - h_1)}{h_1}, \text{ if } 0 < h_1 \ll 1. \tag{1.42}$$

In a similar way, we can define the second derivative of a function:

$$f''(x) = \lim_{h_2 \to 0} \frac{\lim_{h_1 \to 0} \frac{f(x) - f(x - h_1)}{h_1} - \lim_{h_1 \to 0} \frac{f(x-h_2) - f(x - h_1 - h_2)}{h_1}}{h_2}. \tag{1.43}$$

It is clear that one can extend this derivation to third, fourth and higher integer derivatives as well. These formulas in ordinary calculus may be used in the construction of approximations, i.e., numerical discretizations of derivatives in terms of finite differences. If we take $h = h_1 = h_2$ in (1.43), we see that we obtain for the ordinary second derivative:

$$f''(x) = \lim_{h \to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2} \approx \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}, \tag{1.44}$$

if $0 < h \ll 1$. Using the principle of mathematical induction one can extend this idea to the nth derivative in the following way:

$$f^{(n)}(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh), \; n \in \mathbb{N}, \tag{1.45}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. We have set the upper limit of the sum to n in (1.45), but there is no harm in setting it to ∞, because the binomial coefficients will be zero for m larger than n. Next, replacing the "!"-terms with values in terms of the Gamma function, we define the Grünwald-Letnikov fractional derivative:

$$\mathcal{D}^{\alpha}_{GL} f(x) := \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\alpha + 1)}{m!\Gamma(\alpha - m + 1)} f(x - mh). \tag{1.46}$$

Note that the derivative of integer order n is obtained if $\alpha = n$ and the n-fold integral if $\alpha = -n$. This observation naturally leads to the idea of a generalization of the notions of differentiation and integration by allowing α to be an arbitrary real of even complex number. We have encountered four different definitions of fractional derivatives. An important result from [101] connects these definitions and states that they are *equivalent* for a special class of functions. This result is summarized as follows:

$$\text{For } f \in C^{m+1}_{0^-}(\mathbb{R}^{\geq 0}) := \{f \in C^{m+1}([0, \infty)) \;\&\; f(x) = 0 \text{ for } x \leq 0\}, \tag{1.47}$$

the Riesz (section 1.4.2), Riemann-Liouville, Caputo (both in section 1.4.3) and Grünwald-Letnikov fractional derivatives are equivalent:

$$\mathcal{D}^{\alpha}_{RL} f(x) = \mathcal{D}^{\alpha}_{C} f(x) = \mathcal{D}^{\alpha}_{R} f(x) = \mathcal{D}^{\alpha}_{GL} f(x). \tag{1.48}$$

It is interesting to note that the mentioned "mysterious contradiction" at the beginning of this section can now be explained. In fact, the difference between equations (1.24) and (1.27) is due to the fact that in the calculations, almost unnoticeably, two different definitions have been used, which do not coincide for the exponential function. This difference is similar to the fact that the Caputo derivative of a constant function is zero, while it is a non-zero function in terms of the Riemann-Liouville sense. In the next section some other properties of fractional derivatives are given to show differences with ordinary calculus.

1.4.5 Properties of fractional derivatives

In this section, some additional properties of fractional derivatives are discussed (see also [74]).

Basic properties:

1. For $\alpha > 0$ and any $n \in \mathbb{Z}^+$: $\frac{d^n}{dx^n}[D^{\alpha}_{RL}f(x)] = D^{n+\alpha}_{RL}f(x)$
   and: $\frac{d^n}{dx^n}[D^{-\alpha}_{RL}f(x)] = D^{n-\alpha}_{RL}f(x)$.
2. For $\alpha,\ \beta > 0$: $D^{\alpha}_{RL}[D^{-\beta}_{RL}f(x)] = D^{\alpha-\beta}_{RL}f(x)$.
3. $D^{\alpha}_{C}[D^{-\beta}_{C}f(x)] = D^{\alpha-\beta}_{C}f(x)$ does <u>not</u> hold for all $\alpha,\ \beta > 0$.
4. $D^{-\alpha}_{RL}[D^{\alpha}_{RL}f(x)] = f(x) - \sum_{k=1}^{n}\left(x^{\alpha-k}\frac{[D^{\alpha-k}_{RL}f(x)]_{x=0}}{\Gamma(\alpha-k+1)}\right)$, where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}^+$.
5. $D^{-n}_{RL}[D^{n}_{RL}f(x)] = f(x) - \sum_{k=0}^{n-1}\frac{x^k}{k!}f^{(k)}(0)$.
6. $D^{n}_{RL}c = \frac{x^{-\alpha}c}{\Gamma(1-\alpha)}$, where $\alpha > 0$ and c is an arbitrary constant.
7. $D^{\alpha}_{C}f(x) = D^{\alpha}_{RL}f(x) - \sum_{k=0}^{n-1}\frac{x^k}{k!}f^{(k)}(0)$, where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}^+$.

Linearity of the operator:
As for integer-order differentiation, fractional differentiation defines a linear operator as well:

$$D^{\alpha}\Big(\lambda f(x) + \mu g(x)\Big) = \lambda D^{\alpha}f(x) + \mu D^{\alpha}g(x), \tag{1.49}$$

where D^{α} denotes the fractional operator.

This can easily be checked. For example, for the Grünwald-Letnikov fractional operator while $nh = x$:

$$\begin{aligned} D^{\alpha}_{GL}\Big(\lambda f(x) + \mu g(x)\Big) &= \lim_{h\to 0} h^{-\alpha}\sum_{r=0}^{n}(-1)^r\binom{\alpha}{r}(\lambda f(x-rh) + \mu g(x-rh)), \\ &= \lambda \lim_{h\to 0} h^{-\alpha}\sum_{r=0}^{n}(-1)^r\binom{\alpha}{r}f(x-rh) + \\ &\quad +\mu \lim_{h\to 0} h^{-\alpha}\sum_{r=0}^{n}(-1)^r\binom{\alpha}{r}g(x-rh) \\ &= \lambda D^{\alpha}_{GL}f(x) + \mu D^{\alpha}_{GL}g(x). \end{aligned}$$

Note that we have used the fractional version of the binomial coefficient:

$$\binom{\alpha}{r} := \frac{\Gamma(\alpha+1)}{r!\,\Gamma(\alpha+1-r)}. \tag{1.50}$$

Similarly, we can show the linearity property for the Riemann-Liouville fractional derivative of order α $(1 \leq \alpha < m)$:

$$D^{\alpha}_{RL}\Big(\lambda f(x) + \mu g(x)\Big) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x (x-\tau)^{m-\alpha-1}(\lambda f(\tau) + \mu g(\tau))d\tau,$$

$$
\begin{aligned}
&= \frac{\lambda}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-\alpha-1} f(\tau) d\tau \\
&\quad + \frac{\mu}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-\tau)^{m-\alpha-1} g(\tau) d\tau \\
&= \lambda D^{\alpha}_{RL} f(x) + \mu D^{\alpha}_{RL} g(x).
\end{aligned}
$$

It may be obvious that one can show this property for the Caputo and Riesz derivative as well.

The Leibniz rule:
For two functions, $\varphi(x)$ and $\psi(x)$, the Leibniz rule for the n-th order derivative of the product $\varphi(x)\psi(x)$ reads:

$$
\frac{d^n}{dx^n}(\varphi(x)\psi(x)) = \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)}(x)\psi^{(n-k)}(x). \tag{1.51}
$$

Replacing the integer n in (1.51) by the real-valued parameter α, means that the integer order derivative $\psi^{(n-k)}(x)$ in the formula is replaced by, for example, the Grünwald-Letnikov fractional order derivative $D^{\alpha-k}_{GL}\psi(x)$. Then equation (1.51) becomes:

$$
D^{\alpha}_{GL}\big(\varphi(x)\psi(x)\big) = \sum_{k=0}^{[\alpha]} \binom{\alpha}{k} \varphi^{(k)}(x) D^{\alpha-k}_{GL}\psi(x), \tag{1.52}
$$

where formula (1.50) has been used.

The chain rule:
Unfortunately, the chain rule, $[f \circ g(x)]' = f'(g(x))g'(x)$ for ordinary derivatives, can *not* be applied to fractional derivatives. In general,

$$
D^{\alpha}[f \circ g(x)] \neq f'(g(x))D^{\alpha}g(x) \neq D^{\alpha}f(g(x))g'(x) \neq D^{\alpha}f(g(x))D^{\alpha}g(x). \tag{1.53}
$$

Counter examples can be found in references [101, 109, 97].

To conclude this section: one way of understanding fractional derivatives is to see them as a *nonlinear* interpolation of ordinary derivatives in terms of a convolution with a weakly-singular kernel. For more detailed information about other properties of fractional derivatives, we refer to [69].

1.5 Why fractional derivatives?

We can also pose this question as "why don't we use the classical derivative?". Actually, this question is a modern counterpart of similar fundamental questions throughout the history of mathematics. For example, nowadays, nobody would ask: why do we need rational numbers or even real numbers, let alone, complex numbers? Or: why can't we work merely with the integers? Moreover, taking "1/2" powers (square roots) of numbers is no point of discussion anymore at all. During the last centuries, the use of the "fractional" versions of numbers and powers of numbers was understood, justified and clarified. Such a justification could also be made for fractional integrals and fractional derivatives. For example, fractional space derivatives and fractional Laplacian operators may be used to describe so-called Lévy processes of particles, whereas Brownian motions can be connected to the traditional Laplace operator. These aspects are illustrated in Figures 1.4, 1.5 and 1.6 and are also explained in much more detail in [30, 105, 67, 24] (for fractional derivatives and Lévy flights)) and in [130, 45, 96, 95] (for fractional Laplacians).

In Lévy flight processes the particles may, now and then, jump irregularly to other positions (see Figure 1.4). The effect of changing the order of the fractional space derivative, from values between one and two up to the traditional value of two, is depicted in Figure 1.5. Furthermore, Lévy flights (Figure 1.6, bottom) are of interest in epidemic modeling. They could model the diffusion process extended with an extra feature for the sudden travel of individuals by real flights from one city to another. The spreading of diseases in ancient times (Figure 1.6, top) does not need fractional derivatives: a traditional reaction-diffusion model suffices for such cases. To clarify the situation a little bit more: in Figure 1.7 the picture in the left panel shows solutions of the heat equation (normal diffusion) compared to those of a fractional heat equation, based on Caputo's definition (see previous sections), in the right panel. The differences in the decaying behavior of the solution are obvious from the figures, but, note that they may also depend on the particular choice of fractional derivative.

The use of fractional operators in the time direction (fractional time and time-fractional derivatives), is, due to the fundamental differences between space and time, high-lighted from an even more theoretical perspective in physics: what does 'time' mean and could time be even fractional? This fundamental question is still unsolved, and has been discussed in literature extensively (see, e.g., [66, 64, 65, 14, 51]). It must be remarked that the integer order derivative is a local operator which is, with regard to the above discussion, inappropriate for many interesting applications. In such cases, the effect of a larger neighborhood or larger time span can not be modeled easily without fractional order derivatives. In the next section, a series of interesting applications will be given to illustrate this.

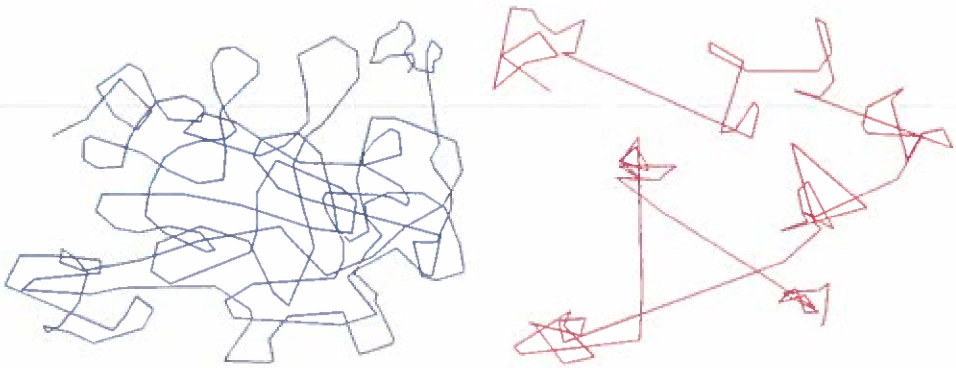

FIGURE 1.4: Brownian motions (left panel) vs Lévy flights (right panel).

1.6 Application areas

In this section, several applications of fractional calculus and fractional differential equations will be discussed. We will do this in a kaleidoscopic way: neither in chronological order, nor in order of increasing importance or impact. It shows the richdom of application areas of fractional order models, which mainly have been developed in the last three decennia.

<u>The Tautochrone problem</u>

The Tautochrone problem [46] was studied first by Abel in 1823. This problem is also important from a historical point of view, because it was one of the first applications of fractional calculus to a 'real-life' model. The question here is to find the curve for which the time taken

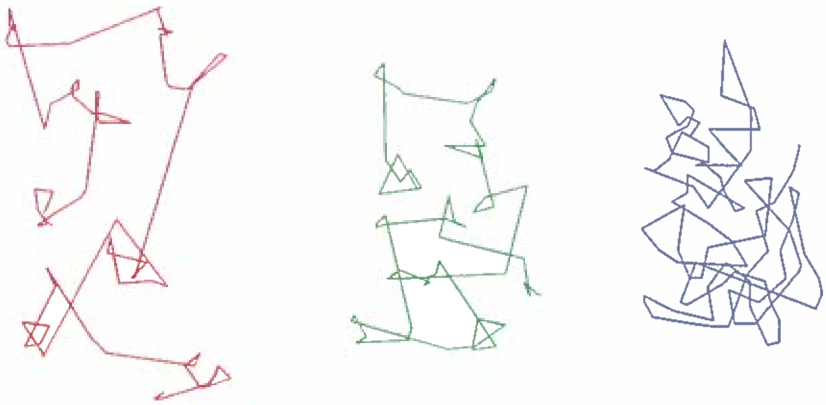

FIGURE 1.5: The variation of the fractional order α to illustrate the effect on the motion of particles: fractional diffusion with $\alpha = 1.5$ (left), $\alpha = 1.75$ (middle) and regular diffusion with $\alpha = 2$ (right).

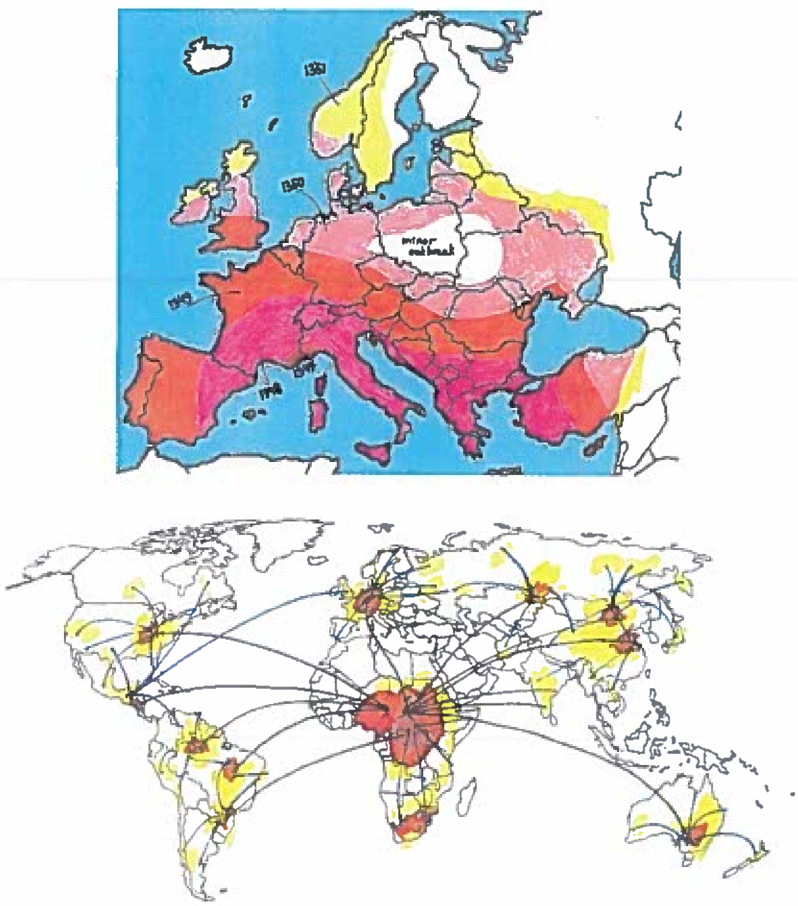

FIGURE 1.6: Diffusion processes in ancient epidemic modeling based on Brownian motions (top) versus Lévy flight processes in modern epidemics (bottom).

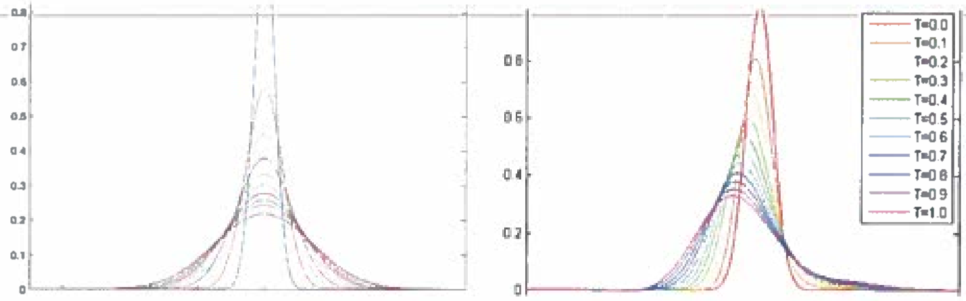

FIGURE 1.7: Solutions of the heat equation at $t = 0, 0.1, 0.2, ..., 1.0$ (left panel) versus the fractional heat equation of order $\frac{3}{2}$ (right panel).

by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point. One can derive that this curve must be a cycloid, and the time is depending on the square root of the radius and the acceleration of gravity. The problem that describes this feature can be modeled by an integral equation of the form:

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{y(s)}{(x-s)^{1-\alpha}}\, ds = h(x), \quad 0 < \alpha < 1.$$

Abel solved this integral equation in an elegant way to obtain:

$$y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{h(s)}{(x-s)^{\alpha}}\, ds = \mathcal{D}^{\alpha}_{RL} h(x).$$

Indeed, we recognize the Riemann-Liouville fractional derivative as the solution for this model.

Tensile and flexural strength of disorder materials

A relatively new application of fractional differential equations in mechanics is the size effect of some building material which aggregates like concrete [84]. This kind of material does not follow the classical rules of solid mechanics and fractional calculus is a significant tool to model this problem.

Fractional advection-dispersion equation

The fractional order form of the advection-dispersion equation [16] models particles which encounter very large transitions. These models play a role in transport in inhomogenous materials.

Bloch equations in MRI

In [102] the authors describe time-fractional Bloch equations and its application in magnetic resonance imaging (MRI). Adding the time-fractional term provides more flexibility in modeling the relaxation process.

Liquid containing gas bubbles

Time-fractional derivatives have been utilized in [52] to model nonlinear wave processes in a liquid containing gas bubbles. Solitary wave solutions are found that follow fractional order rules in the model.

Nonlinear optics

In [128] fractional calculus is entering the field of nonlinear optics to describe unconventional regimes, like disorder biological media and soft-matter. To model this behaviour, a fractional Schrödinger equation is analyzed which also connects the traditional model to Lévy processes.

Long-distance optical communications

Fractional differentiation is of importance to model long-distance optical communications [126]. Caputo fractional derivatives are used in the modeling of optical pulse propagation in nonlinear fibers.

Neurodynamics

A fractional cable equation simulates the electrodiffusion of ions in neurodynamics. Researchers [129] have found that in nerve cells molecular diffusion can be replaced by anomalous subdiffusion.

Acoustics

The study of a fractional Burgers' equation in nonlinear acoustics is presented in [81]. The motivation to use fractional derivatives here comes from an elementary model of shock waves in brass wind instruments, that proves to be useful in musical acoustics.

Econometrics

Long-memory processes and fractional integration in econometrics provide new insights in the analysis of population characteristics [10]. In economics and finance the long-memory volatility is important to describe this behaviour.

Dielectric media

In [115] electromagnetic fields in dielectric media follow a fractional power-law dependence in a wide frequency range. This is modeled by differential equations with time derivatives of non-integer order.

Mechanics of solids

Applications of fractional calculus to nonlinear problems in hereditary mechanics of solids can be found in [108]. Fractional viscoelastic equations are used for modeling damped vibrations in elastic bodies.

Crime modeling

An interesting application arises in the modeling of crime with Lévy flights in [29]. Here, the movement of crime agents is related to a biased Brownian motion with a fractional Laplacian operator. Fractional derivatives are applied to model the formation of hotspots of criminal activity.

Fractional-order epidemic models

In [53] a system of fractional order reaction-diffusion equations is proposed to model the superdiffusive spread of modern epidemics due to Lévy flights. Theoretical analysis and numerical simulations show the potential of fractional derivatives to represent epidemic fronts.

This list of applications is just a brief overview of the recent emergence of fractional models in many different areas. For more applications, we would like to refer to [103, 33, 106, 103].

1.7 Numerical methods for fractional differential equations

Just as for ordinary differential equation models, analytical solutions can be constructed of fractional order differential equations only for special cases. Therefore, numerical methods to obtain approximate solutions need to be developed. Since fractional models in applications have only become of interest in the last half century, their numerical analysis has started up relatively recently. In the last couple of years, several articles and books have appeared which treat approximation techniques for this kind of models. Most of these are based on finite difference types of discretizations that approximate the integro-differential operators with weakly singular kernels. To illustrate this, we give a straightforward, first order, discretization of the Caputo derivative $\mathcal{D}_C^\alpha$ $(1 < \alpha < 2)$ of a function $u(x)$ at the grid point x_i:

$$\begin{aligned} \mathcal{D}_C^\alpha u|_{x=x_i} & \\ &= \tfrac{1}{\Gamma(2-\alpha)} \int_0^{x_i} \tfrac{u''(s)}{(x_i-s)^{\alpha-1}} ds \\ &\approx \tfrac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \{x_{j+1}^{2-\alpha} - x_j^{2-\alpha}\} \tfrac{u_{i-j+1} - 2u_{i-j} + u_{i-j-1}}{h^2} \\ &= \tfrac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \{\tfrac{j^{2-\alpha}-(j-1)^{2-\alpha}}{h^{\alpha-2}}\} \{\tfrac{u_{i-j+1} - 2u_{i-j} + u_{i-j-1}}{h^2}\} \\ &= \tfrac{1}{\Gamma(3-\alpha)h^\alpha} \sum_{j=1}^{i-1} \{j^{2-\alpha} - (j-1)^{2-\alpha}\} \{u_{i-j+1} - 2u_{i-j} + u_{i-j-1}\}. \end{aligned} \tag{1.54}$$

In formula (1.54) we see that the resulting finite-difference matrix does not have a tridiagonal form anymore, which would be the case for a three-point approximation of the second derivative in the normal heat equation (see also (1.43)). This is typical for numerical approximations of fractional order equations. An overview of numerical methods can be found, for instance, in [12, 35, 35] and references therein. Here, we would like to mention just a few important articles on this subject. For instance, in [85, 86] the authors realized that fractional derivative models are strongly related to Abel-Volterra equations. They developed a framework for a discretized fractional calculus and convolution quadrature formulas in terms of fractional linear multistep methods. Meerschaert and co-authors extended the principle of finite difference techniques to many different fractional partial differential equations (see [15, 80, 110, 90]). Another important class of numerical methods was investigated and applied in [87, 23, 25, 25, 24].

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