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ex. 1.1

(a) $$\int_0^1 u u_t \, dx = \delta \int_0^1 u u_{xx} \, dx$$

$$\Leftrightarrow \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}(u^2)\,dx = \delta \Big[\underbrace{u u_x \Big|_{x=0}^{x=1}}_{=0} - \int_0^1 (u_x)^2 dx \Big]$$

$$\Leftrightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u\|_2^2 = -\delta \int_0^1 (u_x)^2 dx \leq 0$$ ($\|u\|_2$ can not grow in time)

integrate in time $$\Leftrightarrow \|u\|_2^2 \Big|_{t=0}^{t=t} = \underbrace{\|u(\cdot,t)\|_2^2 - \|u(\cdot,0)\|_2^2 \leq 0}$$

$$\Downarrow$$

$$\|u(\cdot,t)\|_2^2 \leq \|u(\cdot,0)\|_2^2$$

(b) suppose also v is a solution.

Define $w = u - v$ $\overset{\text{linearity}}{\Longrightarrow}$ $\begin{cases} w_t = \delta w_{xx} \\ w(x,0) = 0 \\ w(0,t) = w(1,t) = 0 \end{cases}$

$$\left.\begin{array}{l} \overset{(a)}{\Longrightarrow} \|w(\cdot,t)\|_2^2 \leq 0 \\ \text{but } \|\cdot\|_2^2 \geq 0 \end{array}\right\} \Rightarrow \|w(\cdot,t)\|_2 = 0$$

$\Rightarrow w = 0$ ("almost everywhere")

$\Rightarrow u = v$ (---)

(c) suppose u solves with $u(x,0) = u_0(x)$ and v with $v(x,0) = v_0(x)$

consider $w = u - v$ once more, then w solves with $w(x,0) = u_0(x) - v_0(x)$

$\underset{\text{energy estinate}}{\Longrightarrow}$ $\|w(\cdot,t)\|_2^2 \leq \delta \cdot \|u_0(\cdot) - v_0(\cdot)\|_2^2$ (continuity)

as u_0 and v_0 become closer, then $\|w(\cdot,t)\|_2 \to 0$

Exercise 1.2:

Solve the PDEs:

(a) $yu_y = u$

Rewrite the eq^n as:

$$\frac{1}{u}\,du = \frac{1}{y}\,dy$$

After integration

$$\ln u = \ln y + \ln C$$

or

$$C = u/y$$

(b). $Cu_x - u_y = 0$

Assume $c \neq 0$, the characteristic equation is

$$\frac{dy}{dx} = -\frac{1}{c}$$

with the general solution defined by the equation

$$cy + x = k = \text{constant}$$

we make the transformation with

$$\xi = x \quad , \quad \eta = x + cy$$

So $u(x,y)$ transforms $u(\xi,\eta)$ as:

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}$$

$$\boxed{u_x = u_\xi + u_\eta}$$

and $$u_y = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y}$$

$$= u_\xi (0) + u_\eta \cdot c$$

$$\boxed{u_y = c\, u_\eta}$$

$\Rightarrow$ In (ξ,η) the PDE takes the form:

$$c(u_\xi + u_\eta) - c u_\eta = 0$$

or $c u_\xi = 0$

$\Rightarrow$ integrate $u(\xi,\eta) = g(\eta)$, $c \neq 0$

$\Downarrow$ in (x,y) coordinates

$$u(x,y) = g(x+cy)$$

"$g = u_0$"

Alternative way:

We have the PDE

$$cu_x - u_y = 0$$

and we suppose that we have the initial condition $u(x,0) = u_0(x)$. From the MoCh we have the following system of ODE's

$$\begin{cases} \frac{dx}{dt} = c \\ \frac{dy}{dt} = -1 \end{cases}$$

and the initial curve C, i.e. $t = 0$, given by

$$\begin{cases} x = s \\ y = 0 \\ z = u_0(s) \end{cases}$$

Integrating the system of ODE's yields

$$\begin{cases} x(s,t) = ct + c_1(s) = ct + s \\ y(s,t) = -t + c_2(s) = -t \end{cases}$$

where we used the fact that from the initial curve C we have that $c_1(s) = s$ and $c_2(s) = 0$. Now note that

$$u(x(s,0), y(s,0)) = u(s,0) = u_0(s)$$

So rewriting $t = -y$ and $s = x - ct = x + cy$ we conclude that

$$u(x,y) = u(s(x,y), t(x,y)) = u(x+cy, -y) = u_0(x+cy)$$

* (c) please check the lecture on the wave equation (March/April)

(a special transformation is needed) involving the wave speeds ± 1

Exercise 1.3 :

Given: The potential equation is $\Delta U = 0$, $(x,y) \in \mathbb{R}^2$

with $Z = x + iy \in \mathbb{C}$ and $U(x,y) = \mathrm{Re}(f(z))$

- **for $f(z) = 1$:**

Here $U(x,y) = \mathrm{Re}(f(z)) = 1$

So $U_{xx} = 0$, $U_{yy} = 0$

the Laplace eq^n is :

$$\Delta U = U_{xx} + U_{yy} = 0 . \quad \text{(Satisfied)}$$

- **for $f(z) = z^2$:**

As

$U(x,y) = \mathrm{Re}(f(z))$.

$$f(z) = z^2 = (x+iy)(x+iy) = x^2 + i\,2xy + i^2 y^2$$

$$f(z) = (x^2 - y^2) + i\,2xy$$

So

$$U(x,y) = (x^2 - y^2)$$

$U_x = 2x$, $U_{xx} = 2$, $U_y = -2y$, $U_{yy} = -2$

the Laplace eq^n is :

$$\Delta U = U_{xx} + U_{yy} = 2 - 2 = 0 . \quad \text{(Satisfied)}$$

- for $f(z) = \log(z - z_0)$; $z_0 \neq 0$

Let $z - z_0 = w \in \mathbb{C}$

and $w = x + iy$

Let $x + iy = re^{i\theta}$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

so

$\log(w) = \log(re^{i\theta}) = \log(r) + i\theta\log(e)$

$u(x,y) = Re\left(f(z)\right)$ i-e

$Re\left(\log(z-z_0)\right) = \log(r)$

$$u(x,y) = \log\left(x^2+y^2\right)^{1/2}$$

$u_x = \dfrac{x}{x^2+y^2}$, $u_{xx} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$, $u_y = \dfrac{y}{x^2+y^2}$, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$\Delta u = u_{xx} + u_{yy} = \dfrac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} = 0$ (satisfied)

Exercise 1.4 :

- **for $u(x,y) = \sin(cx)e^{-c^2y}$:**

To check $u(x,y)$ Solve PDE: $u_{xx} - u_y = 0$; we need to find first, the derivatives u_x, u_{xx}, u_y

$$u_x = e^{-c^2y}\cos(cx)\cdot c \quad , \quad u_{xx} = -c^2\sin(cx)e^{-c^2y}$$

$$u_y = -c^2 e^{-c^2y}\sin(cx).$$

Hence

$$u_{xx} - u_y = -c^2\sin(cx)e^{-c^2y} + c^2e^{-c^2y}\sin(cx)$$

$$u_{xx} - u_y = 0 \quad ; \quad (\text{Solved})$$

- **for $u(x,y) = \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty} u_0(\xi)\, e^{-(x-\xi)^2/4y}\, d\xi$:**

first we find the derivatives u_{xx} & u_y involved in the PDE: $u_{xx} - u_y$

$$u_x = \frac{1}{\sqrt{4\pi y}}\left(\int_{-\infty}^{\infty} u_0(\xi)\, e^{-(x-\xi)^2/4y}\cdot (x-\xi)/2y \; d\xi\right)$$

$$u_{xx} = \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty}\left(u_0(\xi)e^{-(x-\xi)^2/4y}\left(\frac{x-\xi}{2y}\right)^2 + e^{-(x-\xi)^2/4y}\left(\frac{1}{2y}\right)\right)d\xi$$

$$u_y = \frac{1}{2\sqrt{4\pi}\, y^{3/2}}\int_{-\infty}^{-\infty} u_0(\xi)e^{-(x-\xi)^2/4y}\, d\xi + \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty} u_0(\xi)\, e^{-(x-\xi)^2/4y}\cdot\left(\frac{(x-\xi)^2}{4y^2}\right)d\xi$$

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So

$$u_{xx} - u_y = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} u_0(\xi) e^{-(x-\xi)^2/4y} \frac{(x-\xi)^2}{4y^2} d\xi + \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi y}} e^{-(x-\xi)^2/4y} \cdot \frac{1}{2y} d\xi$$

$$-\frac{1}{2\sqrt{4\pi y}} \int_{-\infty}^{\infty} u_0(\xi) e^{-(x-\xi)^2/4y} d\xi + \frac{1}{\sqrt{4\pi y}} \int_{-\infty} u_0(\xi) e^{-(x-\xi)^2/4y} \left(+ \frac{(x-\xi)^2}{4y^2} \right) d\xi$$

$$\Rightarrow u_{xx} - u_y = 0$$

Hence $u(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} u_0(\xi)\, e^{-(x-\xi)^2/4y} d\xi$

Solves the given PDE: $u_{xx} - u_y = 0$

Exercise 1.5 :

(a)- As $u(x,y)$ and $v(x,y)$ are the solutions of the system

$$\begin{cases} u_x + v_y = 0 & \Rightarrow u_x = -v_y \Rightarrow u_{xx} = -v_{xy} \quad (*) \\ v_x + u_y = 0 & \Rightarrow u_y = -v_x \Rightarrow u_{yy} = -v_{xy} \quad (**) \end{cases}$$

To check whether $u(x,y)$ & $v(x,y)$ solve the PDE: $u_{xx} - u_{yy} = 0$

we subsitute (*) & (**) in given PDE:

$u_{xx} - u_{yy} = -v_{xy} + v_{xy} = 0$ (Solved)

(b)- $$\begin{cases} u_x + v_y = 0 & \Rightarrow u_{xx} = -v_{xy} \\ v_x - u_y = 0 & \Rightarrow u_{yy} = v_{xy} \end{cases}$$

PDE: $u_{xx} + u_{yy} = -v_{xy} + v_{xy} = 0$ (Solved)

(c)- $$\begin{cases} u_x + v_y = 0 & \Rightarrow u_x = -v_y \Rightarrow u_{xx} = -v_{xy} \\ v_x + u = 0 & \Rightarrow u = -v_x \Rightarrow u_y = -v_{xy} \end{cases}$$

PDE

$u_{xx} - u_y = -v_{xy} + v_{xy} = 0$ (Solved).

Exercise 1·6 :-

(a). $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y - 2 = 0$

Classification:

Here $A = 4$, $B = 5$, $C = 1$

$B^2 - 4AC = 9 > 0$; Hyperbolic

Characteristics:

The differential equations for the family of characteristic curves are:

$$\frac{dy}{dx} = \frac{+B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{+5 \pm 3}{8}$$

$$\Rightarrow \quad \frac{dy}{dx} = +\frac{1}{4} \quad , \quad \frac{dy}{dx} = +1$$

The solutions are: $C_2 = y - \frac{1}{4}x$, $C_1 = y - x$ are the characteristic curves.

* Canonical form:

Consider the transformation in ξ and η independent variables i-e from (x, y) to (ξ, η) as

$$\xi = y - x \quad \text{and} \quad \eta = y - \frac{1}{4}x$$

$$\Rightarrow \quad \xi_x = -1 , \quad \xi_y = 1 , \quad \eta_x = -\frac{1}{4} , \quad \eta_y = 1$$

$$\xi_{xx} = 0 , \quad \xi_{yy} = 0 , \quad \xi_{xy} = 0 , \quad \eta_{xx} = 0 , \quad \eta_{yy} = 0 , \quad \eta_{xy} = 0$$

(10)

Using the 'Chain Rule', we notice that

$$u_x = u_\xi \xi_x + u_\eta \eta_x \quad , \quad u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}.$$

We have

$$u_x = -u_\xi - \frac{1}{4}u_\eta \quad , \quad u_y = u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{16}u_{\eta\eta}$$

$$u_{xy} = -u_{\xi\xi} - \frac{5}{4}u_{\xi\eta} - \frac{1}{4}u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

So given PDE transforms to :

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y - 2 = -\frac{9}{4}u_{\xi\eta} - \frac{3}{4} - 2 = 0$$

$$\Rightarrow \boxed{u_{\xi\eta} = \frac{1}{3}\left(u_\eta - \frac{8}{3}\right)}$$, is the required canonical form.

(b)-

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$$

Classification:

Here $A = y$, $B = (x+y)$, $C = x$

$$B^2 - 4AC = (x+y)^2 - 4(y)(x)$$

$$= x^2 + y^2 + 2xy - 4yx$$

$$= (x^2 + y^2 - 2xy) = (x-y)^2 > 0 \text{; Hyperbolic}$$

[If $x = y$, then parabolic]

Characterization:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{(x+y) \pm (x-y)}{2y}$$

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \frac{dy}{dx} = 1$$

Solution is:

$$y^2 - x^2 = c_1, \qquad y - x = c_2$$

* Canonical form :-

Choose

$$\xi = y^2 - x^2, \qquad \eta = y - x$$

The given PDE transform to in (ξ, η) coordinate as:

$$u_{\xi\eta} + \frac{1}{\eta} u_{\xi} = 0$$

(C)- $y u_{xx} - 2u_{xy} + e^{x} u_{yy} + x^2 u_x - u = 0$

Classification:-

Here $A = y$, $B = -2$, $C = e^x$

$$B^2 - 4AC = 4 - 4(y)(e^{-x}) = 4 - 4e^x y$$

$$B^2 - 4AC = 4 - 4e^x y : \begin{cases} \text{Elliptic} ; & y > e^{-x} \\ \text{parabolic} ; & y = e^{-x} \\ \text{hyperbolic} ; & y < e^{-x} \end{cases}$$

Characterization:

We discuss here the parabolic case :-

Characteristic polynomial is given by

$$\lambda = B/2A = -1/y$$

Characteristic eq^n:

$$\frac{dy}{dx} = -1/y$$

$$y^2 + x = C$$

new variables ξ, η are given by.

$$\xi = y^2 + x \quad , \quad \eta = x$$

Canonical form:

$$\sqrt{\xi - \eta}\,(u_{\eta\eta} - 3u_{\xi\xi} - 2u_{\xi\eta}) + e^{\eta}\big((4\xi - 4\eta)u_{\xi\xi} + 2u_{\xi}\big) + \eta^2(u_{\xi} + u_{\eta}) - u = c$$

(d)- $$u_{xx} + y\,u_{yy} = 0$$

Classification:

Here $A = 1$, $B = 0$, $C = y$

$$B^2 - 4AC = -4y \;:\; \begin{cases} \text{hyperbolic} & ; \quad y < 0 \\ \text{Elliptic} & ; \quad y > 0 \\ \text{parabolic} & ; \quad y = 0 \end{cases}$$

we discuss here the hyperbolic case:

Characteristics:

$$\frac{dy}{dx} = \frac{\pm\sqrt{-2y}}{2}$$

$$C_1 = x + 2\sqrt{-y} \;, \quad C_2 = x - 2\sqrt{-y} \quad \text{and} \quad y = -\frac{1}{4}(x-c)^2$$

*** Canonical form:-**

Choose $\quad \xi = x + 2\sqrt{-y} \;, \quad \eta = x - 2\sqrt{-y}$

Canonical form of PDE: $u_{xx} - y\,u_{yy} = 0$, in (ξ, η) coordinate:

$$u_{\xi\eta} = \frac{1}{2(\xi - \eta)}\left(u_\eta - u_\xi\right)$$

(e). $y^2 u_{xx} + x^2 u_{yy} = 0$

If $x=0$ or $y=0$, then parabolic

Classification:

Here $A = y^2$, $B=0$, $C = x^2$

$B^2 - 4AC = -4x^2y^2 < 0$; therefore the PDE is elliptic.

Characterization:

The roots of the characterization polynomial are given by

$$\lambda_1 = \frac{B - i\sqrt{4AC-B^2}}{2A}, \quad \lambda_2 = \frac{B + i\sqrt{4AC-B^2}}{2A}$$

$$\lambda_1 = -i\frac{2xy}{2y^2}, \quad \lambda_2 = i\, 2xy/2y^2$$

$$\lambda_1 = -i\frac{x}{y}, \quad \lambda_2 = i\, x/y$$

Characteristic eqⁿ's are:

$$\frac{dy}{dx} = -i\frac{x}{y}, \quad dy/dx = i\, x/y$$

Integrating the above two ODE, we obtain

$$y^2 + ix^2 = c_1, \quad y^2 - ix^2 = c_2$$

Canonical form:

for the transformation of elliptic eqⁿ, we have

$$\alpha = y^2 + ix^2, \quad \beta = y^2 - ix^2$$

the new coordinates (ξ, η) are given by

$$\xi = \frac{\alpha+\beta}{2} \quad , \quad \eta = \frac{\alpha-\beta}{2i}$$

$$\xi = y^2 \quad , \quad \eta = x^2$$

Hence the given PDE is transformed into new coordinate (ξ, η) as:

$$\boxed{2\xi\eta(u_{\xi\xi} + u_{\eta\eta}) + \eta u_\xi + \xi u_\eta = 0}$$

(f)-

$$x u_{xx} + u_{yy} = x^2$$

Classification:

Here $A = x$, $B = 0$, $C = 1$

$$B^2 - 4AC = -4x : \begin{cases} x > 0 & ; \text{ Elliptic} \\ x < 0 & ; \text{ hyperbolic} \\ x = 0 & ; \text{ parabolic} \end{cases}$$

Characterization:

for parabolic case; the characteristic polynomial of the given PDE has only one root i-e

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda(x,y)$$

$$\frac{dy}{dx} = \frac{0}{2x} = 0$$

$$y = c_1$$

the new coordinates are: $\xi = y$, $\eta = x$

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* The canonical form of $xu_{xx} + u_{yy} - x^2 = 0$ is;

$$u_{\xi\xi} + \eta u_{\eta\eta} = \eta^2$$

Exercise 1.7 :

(a).

The given nonlinear PDE is : $u^2 u_{xx} + 2u_x u_y u_{xy} - u^2 u_{yy} = 0$

Here $A = u^2$, $B = 2u_x u_y$, $C = -u^2$

$B^2 - 4AC = 4u_x^2 \cdot u_y^2 - 4u^2(-u^2) = (u_x^2 u_y^2 + u^4)4 > 0$;

Hence the given PDE is always hyperbolic..

(b).

The given PDE : $(1-u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1-u_y^2)u_{yy} = 0$

Here $A = (1-u_x^2)$, $B = -2u_x u_y$, $C = (1-u_y^2)$

$B^2 - 4AC = 4u_x^2 u_y^2 - 4(1-u_x^2)(1-u_y^2)$

$= -4 + 4(u_x^2 + u_y^2)$ $\quad (\because |\nabla u| = \sqrt{u_x^2 + u_y^2})$

$= -4 + |\nabla u|^2 \cdot 4$

$$B^2 - 4AC = -4 + 4|\nabla u|^2 = \begin{cases} \text{Elliptic} & ; \quad |\nabla u| < 1 \\ \text{parabolic} & ; \quad |\nabla u| = 1 \\ \text{hyperbolic} & ; \quad |\nabla u| > 1 \end{cases}$$

Exercise 1.8 :

(a) $2u_{xx} + 3u_{xy} + u_{yy} = 0$

Classification:

Here $A = 2, \quad B = 3, \quad C = 1$

$B^2 - 4AC = 9 - 4(2)(1) = 9 - 8 = 1 > 0$; Hyperbolic

Characterization equations:

$$2y - x = c_1 \quad , \quad y - x = c_2$$

Canonical form:

choose $\xi = 2y - x \quad , \quad \eta = y - x$

the canonical form of the given PDE is given by

$$2u_{xx} + 3u_{xy} + u_{yy} = 2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + 3(-2u_{\xi\xi} - 3u_{\xi\eta} - u_{\eta\eta})$$
$$+ (4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}) = 0$$

$$\Rightarrow \qquad u_{\xi\eta} = 0$$

After two times integration w.r.t ξ and η respectively, we obtain

$$u(\xi, \eta) = \eta f(\eta) + g(\xi)$$

Back into original coordinate;

$$u(x, y) = (y - x) f(y - x) + g(2y - x)$$

(b)- $u_{xx} + 4u_x u_y u_{xy} + 4u_{yy} = 0$

Classification: $A = 1$, $B = 4u_x u_y$, $C = 4$.

$B^2 - 4AC = 16u_x^2 u_y^2 - 4(1)(4)$

$= 16u_x^2 u_y^2 - 16$:
$$\begin{cases} \text{Hyperbolic} ; & u_x^2 u_y^2 > 1 \\ \text{parabolic} ; & u_x^2 u_y^2 = 1 \\ \text{Elliptic} ; & u_x^2 u_y^2 < 1 \end{cases}$$

Exercise 1.9:

(a)- $$u_t = u_{xx} - 12u$$

$$u(x,0) = u_0(x)$$

We apply FT $\mathcal{F}$ to the given PDE and use the properties of FT to reduce given PDE to an ODE.

Let $\mathcal{F}[u] = \hat{u}(\omega,t)$, then

$$\mathcal{F}\{u_t\} = \mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{d\hat{u}}{dt}, \quad \mathcal{F}[u_{xx}] = (+i\omega)^2 \hat{u} = -\omega^2 \hat{u}(\omega,t)$$

the given PDE is transformed to 1st order ODE

$$\frac{d\hat{u}}{dt} = -\omega^2 \hat{u} - 12\hat{u}$$

with

$$\hat{u}(\omega,0) = \hat{u}_0(\omega)$$

The solution of above ODE is :

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{-\omega^2 t - 12t}$$

Applying the inverse Fourier transformation to obtain $u(x,y)$

i.e

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{u}(\omega,t)\right]$$

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{u}_0(\omega)\, e^{-\omega^2 t - 12t}\right]$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{u}_0(\omega)\, e^{-\omega^2 t - 12t}\right] e^{i\omega x}\, d\omega .$$

with

$$\hat{u}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}\, u_0(x)\, dx$$

(b).

$$u_t = K u_{xx} + \Upsilon u_x$$

Apply the Fourier transform (FT) $\mathcal{F}$ to the given PDE:

Let $\mathcal{F}[u(x,t)] = \hat{u}(\omega,t)$, then $u(x,0) = u_0(x) = \hat{u}_0(\omega)$

the given PDE transform to 1st order ODE as:

$$\frac{d\hat{u}}{dt} = -\omega^2 K \hat{u} + i\omega\Upsilon \hat{u} \quad ; \quad \hat{u}(\omega,0) = \hat{u}_0(\omega)$$

The solution of the ODE is:

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{(-\omega^2 K + i\omega\Upsilon)t}$$

Applying the inverse Fourier transformation to obtain $u(x,y)$. i-e

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{u}(\omega,t)\right] = \mathcal{F}^{-1}\left[\hat{u}_0(\omega)\, e^{(-\omega^2 K + i\omega\Upsilon)t}\right]$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, \hat{u}_0(\omega)\, e^{(-\omega^2 K + i\omega\Upsilon)t}\, d\omega$$

Exercise 1·10:

$$u_t = u_{xxx} \quad ; \quad u(x,0) = u_0(x)$$

Let $\mathcal{F}[u] = \hat{u}(\omega, t)$, then

$$\frac{d\hat{u}}{dt} = -i\omega^3 \hat{u}(\omega,t)$$; is the Fourier transformed form to an ODE.

with $\hat{u}(\omega, 0) = \hat{u}_0(\omega)$

The solution of the above ODE is :

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{-i\omega^3 t}$$

Applying the inverse transformation; we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, \hat{u}_0(\omega)\, e^{-i\omega^3 t}\, d\omega$$

or

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\omega)\, e^{i\omega(x-\omega^2 t)}\, d\omega.$$

Exercise 1.11 :

$$u_t = -u_{xxxx} \quad ; \quad u(x,0) = u_0(x)$$

After FT; the given PDE is transformed to an ODE as:

$$\frac{d\hat{u}}{dt} = -\xi^4 \hat{u} \quad \text{with} \quad \hat{u}(\xi,0) = \hat{u}_0(\xi)$$

the Solution of the above ODE is given by:

$$\hat{u}(\xi,t) = e^{-\xi^4 t}\,\hat{u}_0(\xi)$$

Solution to this equation behaves much like the solution of the heat equation but with even more damping of oscillatory data. for example; the PDE : $u_t = u_{xx}$ with $u(x,0) = u_0(x)$

we have

$$\hat{u}(\xi,t) = e^{-\xi^2 t}\,\hat{u}_0(\xi);$$

and for PDE :

$$u_t = u_{xxxx}$$

$$\hat{u}(\xi,t) = e^{+\xi^4 t}\,\hat{u}_0(\xi) \quad ;$$

Here $\hat{u}(\xi,t)$ does not decay in time exponential fast. (it grows very fast)

Note: For $u_t = -u_{xx}$, a similar conclusion holds : $e^{+\xi^2 t}\hat{u}_0(\xi)$

Exercise 1.12

(a) set $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \begin{cases} X'' + 4X' + k^2 X = 0 \\ \dot{T} + k^2 T = 0 \end{cases} \quad (-k^2: \text{separation constant})$$

assumption: $\alpha^2 = k^2 - 4 > 0$

$$\Rightarrow X(x) = e^{-2x}\cdot\left(A\cos(\alpha x) + B\sin(\alpha x)\right)$$

$$\xRightarrow{BC_1} A = 0 \; ; \; BC_2 \Rightarrow \underset{B\neq 0}{e^{-2}\sin(\alpha) = 0} \Rightarrow \alpha_n = n\pi, \; n = 1, 2, \ldots$$

$$\Rightarrow X_n(x) = B_n e^{-2x}\sin(n\pi x), \; n = 1, 2, 3, \ldots$$

$$\Rightarrow T_n(t) = C_n e^{-k_n^2 t}, \quad k_n^2 = \alpha_n^2 + 4$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k_n^2 t - 2x}\sin(\alpha_n x)$$

IC: $u(x,0) = \sum_{n=1}^{\infty} a_n e^{-2x}\sin(\alpha_n x) = u_0(x)$

$$\Rightarrow a_n = 2e^{2x}\int_0^1 u_0(x)\sin(n\pi x)\,dx$$

$$\Rightarrow u(x,t) = \ldots$$

1.12(b) extra IC needed!

set $u_t(x,0)=0$

$u(x,t)=X(x)T(t)$

$$\Rightarrow \begin{cases} c^2 X'' + (k^2-d^2)X = 0 \\ \ddot{T} + k^2 T = 0 \end{cases}$$

($-k^2$ separation constant)

$k^2>d^2$: $X(x)=A\cos(\mu x)+B\sin(\mu x)$

$$\mu = \frac{k^2-d^2}{c^2} > 0$$

$X(0)=0 \Rightarrow A=0$

$X(1)=0 \Rightarrow \sin(\mu x)=0$, $B\neq 0$ $\quad \mu_n = n\pi$, $n=1,2,\ldots$

$$\frac{k_n^2-d^2}{c^2} = n\pi$$

$k_n^2 = n\pi c^2 + d^2$ $\quad n=1,2\ldots$

$T(t)=C\cos(k_n t)+D\sin(k_n t)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x)\left[C_n\cos(k_n t)+D_n\sin(k_n t)\right]$$

$IC_2: \rightarrow u_t(x,t) = \sum_{n=1}^{\infty} \sin(\mu_n x)\left[-k_n\tilde{C}_n\sin(k_n t)+k_n\tilde{D}_n\cos(k_n t)\right]$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(\mu_n x)\left[0 + k_n\tilde{D}_n\right] \equiv 0$$

$\Rightarrow \tilde{D}_n = 0 \quad \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \tilde{C}_n \sin(\mu_n x)\cos(k_n t)$

$\tilde{C}_n$ follows from $u(x,0)=u_0(x)$ ------

Exercise 1.13

PDE:

$$\begin{cases} u_{tt} = -u_{xx} \\ u(x,0) = \sin(2\pi x) \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) \\ u_x(0,t) = u_x(1,t) \end{cases} \qquad (**)$$

(a) Show that we end up at the system (only for a_1 and b_1).

$$\boxed{\begin{cases} \ddot{a}_1(t) - 4\pi^2 a_1(t) = 0 \\ \ddot{b}_1(t) - 4\pi^2 b_1(t) = 0 \end{cases}}$$

Solution As we know from previous ex.

$$u_x = \frac{\partial u(x,t)}{\partial x} = \sum_{k=1}^{\infty} 2\pi k \cdot a_k(t)\cos(2\pi k x) - \sum_{k=0}^{\infty} 2\pi k \cdot b_k(t) \cdot \sin(2\pi k x)$$

Then

$$u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2} = -\sum_{k=1}^{\infty} 4\pi^2 k^2 a_k(t)\sin(2\pi k x) - \sum_{k=0}^{\infty} 4\pi^2 k^2 \cdot b_k(t)\cdot\cos(2\pi k x)$$

Substituting partial derivatives in $(**)$ for general case we receive:

$$\sum_{k=1}^{\infty} \ddot{a}_k(t)\sin(2\pi k x) + \sum_{k=0}^{\infty} \ddot{b}_k(t)\cos(2\pi k x) =$$

$$= + \sum_{k=1}^{\infty} 4\pi^2 k^2 a_k(t)\sin(2\pi k x) + \sum_{k=0}^{\infty} 4\pi^2 k^2 b_k(t)\cos(2\pi k x)$$

For only a_1 and b_1, we receive the following!

$$\ddot{a}_1(t)\sin(2\pi x)+\ddot{b}_1(t)\cos(2\pi x)-4\pi^2 a_1(t)\sin(2\pi x)$$
$$-4\pi^2 b_1(t)\cos(2\pi x)=0$$

Grouping:

$$\sin(2\pi x)\left(\ddot{a}_1(t)-4\pi^2 a_1(t)\right)+$$
$$+\cos(2\pi x)\left(\ddot{b}_1(t)-4\pi^2 b_1(t)=0\right.$$

Finally we end up with the following system

$$\boxed{\begin{cases}\ddot{a}_1(t)-4\pi^2 a_1(t)=0\\ \ddot{b}_1(t)-4\pi^2 b_1(t)=0\end{cases}}$$

1.13 (b) again by letting:

$$u_1=a_1$$
$$u_2=b_1$$
$$u_3=\dot{a}_1$$
$$u_4=\dot{b}_1$$

$$\dot{u}_1=\dot{a}_1=u_3$$
$$\dot{u}_2=\dot{b}_1=u_4$$
$$\dot{u}_3=\ddot{a}_1=4\pi^2 a_1=4\pi^2 u_1$$
$$\dot{u}_4=\ddot{b}_1=4\pi^2 b_1=4\pi^2 u_2$$

$$\begin{pmatrix}\dot{a}_1\\ \dot{b}_1\\ \ddot{a}_1\\ \ddot{b}_1\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\4\pi^2&0&0&0\\0&4\pi^2&0&0\end{pmatrix}\begin{pmatrix}a_1\\ b_1\\ \dot{a}_1\\ \dot{b}_1\end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In order to find the exact solution of second PDE (**), we need to solve the following:

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = e^{t\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4\pi^2 & 0 & 0 & 0 \\ 0 & 4\pi^2 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Following the procedure of finding matrix exponential, we find:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4\pi^2 & 0 & 0 & 0 \\ 0 & 4\pi^2 & 0 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) =$$

$$= 1 \begin{vmatrix} -\lambda & 0 & 1 \\ 4\pi^2 & -\lambda & 0 \\ 0 & 4\pi^2 & -\lambda \end{vmatrix} = -\lambda + 16\pi^4 = 0$$

$\Rightarrow$ The eigenvalues are

$$\lambda_1 = -2\pi$$
$$\lambda_2 = -2\pi$$
$$\lambda_3 = 2\pi$$
$$\lambda_4 = 2\pi$$

.....

Corresponding eigenvectors are:

$\lambda_1 = -2\pi$

$$V_1 = \begin{pmatrix} 0 \\ -\frac{1}{2\pi} \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2 = -2\pi$

$$V_2 = \begin{pmatrix} -\frac{1}{2\pi} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_3 = 2\pi$

$$V_3 = \begin{pmatrix} 0 \\ \frac{1}{2\pi} \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_4 = 2\pi$

$$V_4 = \begin{pmatrix} \frac{1}{2\pi} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\exp(At) = \exp(Z \Lambda Z^{-1} t) = Z \exp(\Lambda t) Z^{-1}$$

$$Z = \begin{pmatrix} 0 & -\frac{1}{2\pi} & 0 & 1/2\pi \\ -\frac{1}{2\pi} & 0 & 1/2\pi & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\Lambda t = \begin{pmatrix} -2\pi & 0 & 0 & 0 \\ 0 & -2\pi & 0 & 0 \\ 0 & 0 & 2\pi & 0 \\ 0 & 0 & 0 & 2\pi \end{pmatrix}$$

$$Z^{-1} = \begin{pmatrix} 0 & -\pi & 0 & \frac{1}{2} \\ -\pi & 0 & 1/2 & 0 \\ 0 & \pi & 0 & 1/2 \\ \pi & 0 & 1/2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp(At) = \begin{pmatrix} 0 & -\frac{1}{2\pi} & 0 & 1/2\pi \\ -\frac{1}{2\pi} & 0 & 1/2\pi & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} e^{-2\pi} & 0 & 0 & 0 \\ 0 & e^{-2\pi} & 0 & 0 \\ 0 & 0 & e^{2\pi} & 0 \\ 0 & 0 & 0 & e^{2\pi} \end{pmatrix} \times$$

$$\times \begin{pmatrix} 0 & -\pi & 0 & 1/2 \\ -\pi & 0 & 1/2 & 0 \\ 0 & \pi & 0 & 1/2 \\ \pi & 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} e^{-2\pi}(1 + e^{4\pi}) \\ 0 \\ 2e^{-2\pi}(-\pi + e^{4\pi}\pi) \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = \begin{pmatrix} \cosh(2\pi t) \\ 0 \\ 2\pi \sinh(2\pi) \\ 0 \end{pmatrix} \Rightarrow a_1 = \cosh(2\pi)$$

Substituting a_1 in general solution, we obtain:

$$\boxed{u(x,t) = \cosh(2\pi t)\sin 2\pi x}$$

This is the exact solution to PDE (**)

Now we check if this solution satisfies PDE 2 + initial conditions

$$u(x,t) = \cosh(2\pi t)\sin(2\pi x)$$

$$u_t = \frac{\partial u(x,t)}{\partial t} = 2\pi \sinh(2\pi t)\sin(2\pi x)$$

$$u_{tt} = \frac{\partial^2 u(x,t)}{\partial t^2} = 4\pi^2 \cos(2\pi t)\sin(2\pi x)$$

$$u_x = \frac{\partial u(x,t)}{\partial x} = 2\pi \cosh(2\pi t)\cos(2\pi x)$$

$$u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2} = -4\pi^2 \cosh(2\pi t)\sin(2\pi x)$$

Substituting partial derivatives into PDE 2 + IC.

$$\begin{cases} 4\pi^2 \cosh(2\pi t)\sin(2\pi x) = 4\pi^2 \cosh(2\pi t)\sin(2\pi x) \\ u(x,0) = \underbrace{\cosh(0)}_{=1} \sin 2\pi x = \sin 2\pi x \\ u_t(x,0) = 2\pi \sinh(0)\sin(2\pi x) = 0 \end{cases}$$

Hence our solution satisfies PDE 2 + ICs.

$\boxed{1.13\ (c)}$

PDE :
(*)
$$\begin{cases} u_{tt} = u_{xx} & x \in [0,1) \\ u(x,0) = \sin(2\pi x) & t > 0 \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) \end{cases}$$

i) Write the solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin(2\pi k x) + \sum_{k=0}^{\infty} b_k(t) \cos(2\pi k x)$$

Show that

$$\begin{cases} a_k(0) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \\ b_k(0) = 0 \quad \forall k \\ \dot{a}_k(0) = \dot{b}_k(0) = 0 \quad \forall k \end{cases}$$

Substituting the initial conditions in the solution.

$$\underline{u(x,0)} = \sum_{k=1}^{\infty} a_k(0) \sin 2\pi k x + \sum_{k=0}^{\infty} b_k(0) \cos(2\pi k x) = \underline{\sin(2\pi x)}$$

$$\Rightarrow \underline{a_1(0) = 1 \overset{\text{if } k=1}{\vee} b_1(0) = 0 \ , \ a_k(0) = 0 \text{ otherwise}}$$

$$u_t = \frac{\partial u(x,t)}{\partial t} = \sum_{k=1}^{\infty} \dot{a}_k(t) \sin(2\pi k x) + \sum_{k=0}^{\infty} \dot{b}_k(t) \cos(2\pi k x)$$

$$u_t(x,0) = \sum_{k=1}^{\infty} \dot{a}_k(0) \sin(2\pi k x) + \sum_{k=0}^{\infty} \dot{b}_k(0) \cos(2\pi k x)$$

$$= 0 \Rightarrow \underline{\dot{a}_k(0) = 0 \ , \ \dot{b}_k(0) = 0}$$

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ii) Solve using

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = e^{t\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\pi & 0 & 0 \\ 2\pi & 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

<u>Solution</u>

Calculate e^{At} for matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\pi & 0 & 0 \\ 2\pi & 0 & 0 & 0 \end{pmatrix}$$

First we find eigenvalues and eigenvectors of A

$$0 = \det(A - \lambda I) =$$

$$= \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & -2\pi & -\lambda & 0 \\ 2\pi & 0 & 0 & \lambda \end{vmatrix} =$$

$$= -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ -2\pi & -\lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & -2\pi & 0 \\ 2\pi & 0 & -\lambda \end{vmatrix} =$$

$$= -\lambda\left(-\lambda^3\right) + 4\pi^2 = 0$$

$$\lambda_1 = -\sqrt[4]{-1}\,\sqrt{2\pi}$$

$$\lambda_2 = \sqrt[4]{-1}\,\sqrt{2\pi}$$

$$\lambda_3 = (-1)^{3/4}\sqrt{2\pi}$$

$$\lambda_4 = -(-1)^{3/4}\sqrt{2\pi}$$

$$\lambda_1 = -\sqrt[4]{-1}\,\sqrt{2\pi}$$

$$\begin{bmatrix} \sqrt[4]{-1}\sqrt{2\pi} & 0 & 1 & 0 \\ 0 & \sqrt[4]{-1}\sqrt{2\pi} & 0 & 1 \\ 0 & -2\pi & \sqrt[4]{-1}\sqrt{2\pi} & 0 \\ 2\pi & 0 & 0 & \sqrt[4]{-1}\sqrt{2\pi} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{pmatrix} -\dfrac{\frac{1}{2} - \frac{i}{2}}{\sqrt{\pi}} \\ -\dfrac{\frac{1}{2} + \frac{i}{2}}{\sqrt{\pi}} \\ -i \\ 1 \end{pmatrix}$$

$$\lambda_2 = \sqrt[4]{-1}\,\sqrt{2\pi}$$

$$\begin{bmatrix} -\sqrt[4]{-1}\sqrt{2\pi} & 0 & 1 & 0 \\ 0 & -\sqrt[4]{-1}\sqrt{2\pi} & 0 & 1 \\ 0 & -2\pi & -\sqrt[4]{-1}\sqrt{2\pi} & 0 \\ 2\pi & 0 & 0 & -\sqrt[4]{-1}\sqrt{2\pi} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{pmatrix} -\dfrac{\frac{1}{2} + \frac{i}{2}}{\sqrt{\pi}} \\ -\dfrac{\frac{1}{2} - \frac{i}{2}}{\sqrt{\pi}} \\ i \\ 1 \end{pmatrix}$$

$$\lambda_3 = (-1)^{3/4}\sqrt{2\pi}$$

$$\begin{bmatrix} -(-1)^{3/4}\sqrt{2\pi} & 0 & 1 & 0 \\ 0 & -(-1)^{3/4}\sqrt{2\pi} & 0 & 1 \\ 0 & -2\pi & -(-1)^{3/4}\sqrt{2\pi} & 0 \\ 2\pi & 0 & 0 & -(-1)^{3/4}\sqrt{2\pi} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_3 = \begin{pmatrix} \frac{\frac{1}{2} + \frac{i}{2}}{\sqrt{\pi}} \\ \frac{\frac{1}{2} - \frac{i}{2}}{\sqrt{\pi}} \\ i \\ 1 \end{pmatrix}$$

$$\lambda_4 = -(-1)^{3/4} \sqrt{2\pi}$$

$$\begin{bmatrix} (-1)^{3/4}\sqrt{2\pi} & 0 & 1 & 0 \\ 0 & (-1)^{3/4}\sqrt{2\pi} & 0 & 1 \\ 0 & -2\pi & (-1)^{3/4}\sqrt{2\pi} & 0 \\ 2\pi & 0 & 0 & (-1)^{3/4}\sqrt{2\pi} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_4 = \begin{pmatrix} \frac{\frac{1}{2} - \frac{i}{2}}{\sqrt{\pi}} \\ \frac{\frac{1}{2} + \frac{i}{2}}{\sqrt{\pi}} \\ -i \\ 1 \end{pmatrix}$$

In order to find e^{At} we need to calculate

$$\exp(At) = \exp(Z \Lambda Z^{-1} t) = Z \exp(\Lambda t) Z^{-1}$$

where
- Z - matrix whose columns are eigenvectors of A
- Z^{-1} - inverse of Z
- $\Lambda(t)$ - diagonal matrix with eigenvalues of A on the diagonal.

...

$$t = \begin{pmatrix} -\dfrac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} & \dfrac{-\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} & \dfrac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} & \dfrac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} \\ -\dfrac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} & -\dfrac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} & \dfrac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} & \dfrac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} \\ -i & i & i & -i \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\det(t) =$$

$$= \left(\frac{i}{2}-\frac{1}{2}\right)\frac{1}{\sqrt{\pi}} \begin{vmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1 \end{vmatrix} + \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} \begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1 \end{vmatrix}$$

$$+ 0 - i$$

$$\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \end{vmatrix} -$$

$$- 1 \begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1i \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1i \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i \end{vmatrix}.$$

$$\left(\frac{i}{2}-\frac{1}{2}\right)\frac{1}{\sqrt{\pi}} \begin{vmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} \; 0 + & i & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1 \end{vmatrix} +$$

$$+\left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}}\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1\end{vmatrix}+$$

$$+0-i\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\end{vmatrix}$$

$$-1\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i\end{vmatrix}$$

$$-\frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}}\begin{vmatrix}\left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1\end{vmatrix}+$$

$$=\frac{-\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}}\cdot\frac{-2-2i}{\sqrt{\pi}}=\frac{2}{\pi}\ ;)$$

$$\frac{2}{\pi}+\left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}}\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1\end{vmatrix}+0-$$

$$-i\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1\end{vmatrix}-$$

$$-1\begin{vmatrix}\left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i\\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\pi} & i\\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\pi} & -i\end{vmatrix}$$

$$\frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}}\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i & 1 \end{vmatrix} =$$

$$= \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} \cdot \frac{2-2i}{\sqrt{\pi}} = \frac{2}{\pi}$$

$$\frac{2}{\pi}+\frac{2}{\pi}+0-1\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -1 \end{vmatrix}$$

$$-1\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i \end{vmatrix}$$

$$-i\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & 1 \end{vmatrix} =$$

$$= -i \times \frac{2i}{\pi} = \frac{2}{\pi}$$

$$-1\begin{vmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(-\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i \\ \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & i \\ \left(\frac{1}{2}-\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & \left(\frac{1}{2}+\frac{i}{2}\right)\frac{1}{\sqrt{\pi}} & -i \end{vmatrix} =$$

$$= -\frac{-2}{\pi} = \frac{2}{\pi}$$

$$\frac{2}{\pi}+\frac{2}{\pi}+\frac{2}{\pi}+\frac{2}{\pi} = \frac{8}{\pi}$$

So $\det(Z) = \frac{8}{\pi}$

$$Z^{-1} = \begin{pmatrix} \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \left(-\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & -\frac{i}{4} & \frac{1}{4} \\ \left(-\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \frac{i}{4} & \frac{1}{4} \\ \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & \frac{i}{4} & \frac{1}{4} \\ \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix}$$

Finally: $\exp(At)$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\pi & 0 & 0 \\ 2\pi & 0 & 0 & 0 \end{pmatrix}$$

is equal to

$$\exp(Z\Lambda Z^{-1}t) = Z\exp(\Lambda t)Z^{-1} =$$

$$= \begin{pmatrix} \frac{-\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} & -\frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} \\ \frac{-\frac{1}{2}+\frac{i}{2}}{\sqrt{\pi}} & \frac{-\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} & \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{\pi}} \\ -i & i & i \\ 1 & 1 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} e^{-\sqrt{\pi}}\cos(\sqrt{\pi}) - i e^{-\sqrt{\pi}}\sin(\sqrt{\pi}) & 0 & 0 & 0 \\ 0 & e^{-\sqrt{\pi}}\cos(\sqrt{\pi}) + i e^{-\sqrt{\pi}}(\sin\sqrt{\pi}) & 0 & 0 \\ 0 & 0 & e^{\sqrt{\pi}}\cos(\sqrt{\pi}) - i e^{\sqrt{\pi}}\sin(\sqrt{\pi}) & 0 \\ 0 & 0 & 0 & e^{\sqrt{\pi}}\cos(\sqrt{\pi}) + i e^{\sqrt{\pi}}\sin(\sqrt{\pi}) \end{pmatrix}$$

$$\begin{pmatrix} \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \left(-\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & -\frac{i}{4} & 1/4 \\ \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \left(-\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \frac{i}{4} & 1/4 \\ \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & \frac{i}{4} & 1/4 \\ \left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{\pi} & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{\pi} & -\frac{i}{4} & 1/4 \end{pmatrix} =$$

see next page

$$\frac{1}{2} e^{-\sqrt{\pi t}} \quad \begin{matrix} e^{2\sqrt{\pi t}} \cos(\sqrt{\pi t}) + \cos(\sqrt{\pi t}) \\ e^{2\sqrt{\pi t}} \sin(\sqrt{\pi t}) - \sin(\sqrt{\pi t}) \end{matrix} \qquad \begin{matrix} \sin(\pi t) - e^{2\sqrt{\pi t}} \sin(\sqrt{\pi t}) \\ e^{2\sqrt{\pi t}} \cos(\sqrt{\pi t}) + \cos(\sqrt{\pi t}) \end{matrix}$$

$$e^{2\sqrt{\pi t}} \sqrt{\pi t} \cos(\sqrt{\pi t}) - \sqrt{\pi t} \cos(\sqrt{\pi t}) - $$
$$- e^{2\sqrt{\pi t}} \sqrt{\pi t} \sin(\sqrt{\pi t}) - \sqrt{\pi t} \sin(\sqrt{\pi t})$$

$$-e^{2\sqrt{\pi t}} \sqrt{\pi t} \cos(\sqrt{\pi t}) + \sqrt{\pi t} \cos(\sqrt{\pi t}) -$$
$$- e^{2\sqrt{\pi t}} \sqrt{\pi t} \sin(\sqrt{\pi t} - \sqrt{\pi t} \sin \sqrt{\pi t}$$

$$e^{2\sqrt{\pi t}} \sqrt{\pi t} \cos(\sqrt{\pi t}) - \sqrt{\pi t} \cos(\sqrt{\pi t}) +$$
$$+ e^{2\sqrt{\pi t}} \sqrt{\pi t} \sin(\sqrt{\pi t} - \sqrt{\pi t} \sin(\sqrt{\pi t})$$

$$e^{2\sqrt{\pi t}} \sqrt{\pi t} \cos(\sqrt{\pi t} - \sqrt{\pi t} \cos(\sqrt{\pi t}) - e^{2\sqrt{\pi t}} \times$$
$$\times \sqrt{\pi t} \sin(\sqrt{\pi t}) + \sqrt{\pi t} \sin(\sqrt{\pi t})$$

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Substituing in

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = e^{t\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\Pi & 0 & 0 \\ 2\Pi & 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we finally receive

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = \frac{e^{-\sqrt{\pi t}}}{2} \begin{pmatrix} e^{2\sqrt{\pi t}} \cos(\sqrt{\pi t}) + \cos(\sqrt{\pi t}) \\ e^{2\sqrt{\pi t}} \sin(\sqrt{\pi t}) - \sin(\sqrt{\pi t}) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}(t) = \begin{pmatrix} \left(\frac{e^{\sqrt{\pi t}}}{2} + \frac{e^{-\sqrt{\pi t}}}{2}\right) \cdot \cos\sqrt{\pi t} \\ \left(\frac{e^{\sqrt{\pi t}}}{2} - \frac{e^{-\sqrt{\pi t}}}{2}\right) \sin\sqrt{\pi t} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} a_1 &= \cosh(\sqrt{\pi t}) \cdot \cos\sqrt{\pi t} \\ b_1 &= \sinh(\sqrt{\pi t}) \cdot \sin\sqrt{\pi t} \\ \dot{a}_1 &= 0 \\ \dot{b}_1 &= 0 \end{aligned}$$

Substituing a_1 and b_1 in solution

in the form

$$u(x,t) = \sum_{K=1}^{\infty} a_k(t) \sin(2\pi k x) + $$
$$+ \sum_{K=0}^{\infty} b_k(t) \cos(2\pi k x)$$

we receive:

$$\boxed{u(x,t)} = \underbrace{\overbrace{\cosh(\sqrt{\pi t}) \cos(\sqrt{\pi t})}^{a_1}} \cdot \sin(2\pi x) + $$
$$+ \underbrace{\sinh(\sqrt{\pi t}) \sin(\sqrt{\pi t})}_{b_1} \cdot \cos(2\pi x)$$

This is exact solution of PDE (1)

(*) Show that we end up at the system (only for a_1 and b_1)

$$* \begin{cases} \ddot{a}_1(t) + 2\pi b_1(t) = 0 \\ \ddot{b}_1(t) - 2\pi a_1(t) = 0 \end{cases}$$

<u>Solution</u> We can prove that we end up at (*) by sustituting solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin(2\pi k x) + \sum_{K=0}^{\infty} b_k(t) \cos(2\pi k x)$$ into

PDE 1 and initial conditions:

First find.

$$\boxed{u_x} = \frac{\partial u(x,t)}{\partial x} = \sum_{K=1}^{\infty} 2\pi k \cdot a_k(t) \cos(2\pi k x) -$$
$$- \sum_{k=0}^{\infty} 2\pi k \, b_k(t) \sin(2\pi k x)$$

$$\boxed{u_t} = \frac{\partial u(x,t)}{\partial t} = \sum_{k=1}^{\infty} \dot{a}_k(t) \sin(2\pi kx) +$$

$$+ \sum_{k=0}^{\infty} \dot{b}_k(t) \cos(2\pi kx)$$

$$\boxed{u_{tt}} = \frac{\partial^2 u(x,t)}{\partial t^2} = \sum_{k=1}^{\infty} \ddot{a}_k(t) \sin(2\pi kx) +$$

$$+ \sum_{k=0}^{\infty} \ddot{b}_k(t) \cos(2\pi kx)$$

Substituting partial derivatives in PDE 1:

$$\sum_{k=1}^{\infty} \ddot{a}_k(t) \sin(2\pi kx) + \sum_{k=0}^{\infty} \ddot{b}_k(t) \cos(2\pi kx) =$$

$$= \sum_{k=1}^{\infty} 2\pi k \cdot a_k(t) \cos(2\pi kx) - \sum 2\pi k \cdot b_k(t) \cdot \sin(2\pi kx)$$

As we know from the previous ex.

$$\begin{cases} a_k(0) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \\ b_k(0) = 0 \end{cases}$$

Hence we get:

$$\ddot{a}_1 \sin(2\pi x) + \ddot{b}_1(t) \cos(2\pi x) = 2\pi a_1(t) \cos(2\pi x) -$$

$$- 2\pi b_1(t) \sin(2\pi x)$$

$$\sin(2\pi x)(\ddot{a}_1 + 2\pi b_1(t)) - \cos 2(\pi x)(\ddot{b}_1 - 2\pi a_1(t)) = 0$$

$$\begin{cases} \ddot{a}_1 + 2\pi b_1(t) = 0 \\ \ddot{b}_1 - 2\pi a_1(t) = 0 \end{cases}$$

By letting

$$\beta_1 = a_1$$
$$\beta_2 = b_1$$
$$\beta_3 = \dot{a}_1$$
$$\beta_4 = \dot{b}_1$$

we obtain.

$$\dot{\beta}_1 = \dot{a}_1 = \beta_3$$
$$\dot{\beta}_2 = \dot{b}_1 = \beta_4$$
$$\dot{\beta}_3 = \ddot{a}_1 = -2\pi\, b_1 = -2\pi\, \beta_2$$
$$\dot{\beta}_4 = \ddot{b}_1 = 2\pi\, a_1 = 2\pi\, \beta_1$$

This corresponds to.

$$\begin{cases} \begin{pmatrix} \dot{a}_1 \\ \dot{b}_1 \\ \ddot{a}_1 \\ \ddot{b}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\pi & 0 & 0 \\ 2\pi & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix} \\ \begin{pmatrix} a_1 \\ b_1 \\ \dot{a}_1 \\ \dot{b}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{cases}$$

Also we check the correctness of initial condition

$$u(x,0) = \underbrace{a_1(0)}_{=1} \sin 2\pi x + 0 + \dots + 0 + \underbrace{b_0(t)\cos(2\pi k x)}_{0}$$

$$= \sin 2\pi x$$

$$u_t(x,0) = \sum_{k=1}^{\infty} \underbrace{\dot{a}_k(0)}_{=0} \sin 2\pi x + \overbrace{\sum_{k=0}^{\infty} \dot{b}_k(0) \underbrace{\cos 2\pi 0 x}_{1}}^{=0}$$

$$= 0$$

Exercise 1.14 :

$$u_t = K u_{xx} \quad , \; K \in \mathbb{R} \quad ; \; x \in [0,1] .$$

with

I.C : $u(x, 0) = \sin(\pi x)$

BCs : $u(0, t) = u(1, t) = 0$

Let a solution of the form $u(x,t) = X(x)\,T(t)$

the given heat equation becomes

$$\frac{\dot{T}(t)}{K.T(t)} = \frac{X''(x)}{X(x)} = C \;(\text{constant})$$

$$\Rightarrow \quad \begin{cases} \dot{T}(t) - K.C\,T(t) = 0 \\ X''(x) - C\,X(x) = 0 \end{cases}$$

The solution of the first ODE is ;

$$T_n(t) = b_n\, e^{-K(n\pi)^2 t} \qquad \text{with } C = -\lambda_n^2 = (n\pi)^2$$

from second ODE : we have three cases :

- $C > 0$: let $C = \lambda^2 \Rightarrow X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

  Applying BCs, we have

  $c_1 = c_2 = 0$

  $\Rightarrow \quad X(x) = 0$ (trivial Solution)

- $C = 0$: $\quad X(x) = c_3 + c_4 x \xrightarrow{BCs} c_3 = c_4 = 0$

  Hence $X(x) = 0$ (trivial Solution)

- $c < 0$: Let $c = -\lambda^2$, we have

$$X(x) = c_5 \cos(\lambda x) + c_6 \sin(\lambda x)$$

BCs: $\longrightarrow$

$$X_n(x) = a_n \sin(n\pi x) \quad ; \quad n = 1, 2, 3, \dots$$

with

$$\lambda_n = n\pi$$

Hence the Solution is:

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$= a_n \sin(n\pi x)\, b_n e^{-K.(n\pi)^2 t}$$

or

$$u_n(x,t) = d_n e^{-K(n\pi)^2 t} \sin(n\pi x) \qquad (\because d_n = a_n b_n)$$

Applying Superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-K(n\pi)^2 t} \sin(n\pi x)$$

with

$$c_n = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\, dx \qquad (\because u(x,0) = u_0(x) = \sin(\pi x))$$

Initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} c_n \cdot e^{0} \cdot \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} c_n \sin(n\pi x) \underset{\text{given}}{=\!=} \sin(\pi x)$$

$\Rightarrow$ $n=1$ only, remaining terms not present

$$\Rightarrow c_1 = 2\int_0^1 \sin^2(\pi x)\, dx = 1$$

$$\Rightarrow u(x,t) = e^{-K\pi^2 t} \cdot \sin(\pi x)$$

$\begin{cases} K > 0 & \begin{cases} \text{well-posed } t > 0 \\ \text{ill-posed } t < 0 \end{cases} \\ K < 0 & \begin{cases} \text{ill-posed } t > 0 \\ \text{well-posed } t < 0 \end{cases} \end{cases}$

Exercise 1.15:

Consider the hyperbolic PDE:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu = f(x,t) \quad ; \quad a \,\&\, b \text{ are constants}$$

Coordinate change: $(x,t) \longrightarrow (\xi, \tau)$

with $\tau = t$, $\xi = x - at$

inverse mapping: $t = \tau$, $x = \xi + a\tau$

Define

$$\hat{u}(\xi,\tau) = u(x,t)$$

the given PDE transforms to new coordinate as:

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial t}{\partial \tau}\cdot\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \tau}$$

$$= \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}$$

$$\frac{\partial \hat{u}}{\partial t} = -b\hat{u} + f(\xi + a\tau, \tau)$$

$\left(\because \text{ from given PDE: } \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -bu + f(x,t) \right)$

The solution of this ODE:

$$\hat{u}(\xi,\tau) = u_0(\xi)\, e^{-b\tau} + \int_0^{\tau} f(\xi + a\sigma, \sigma)\, e^{-b(\tau-\sigma)}\, d\sigma$$

Back to the original coordinate $(x,t) \Rightarrow$

$$u(x,t) = u_0(x-at)\, e^{-bt} + \int_0^{t} f(x - a(t-s), s)\, e^{-b(t-s)}\, ds.$$

Exercise 1.16:

$$u_t + \frac{1}{1+\frac{1}{2}\cos(x)} u_x = 0$$

We change the variables to τ and ξ ; where $\tau = t$.
We have

$$\frac{\partial \tilde{u}}{\partial \tau} = \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x$$

$$= u_t + \frac{\partial x}{\partial \tau} u_x$$

where the given PDE is transformed to ODE:

$$\frac{dx}{d\tau} = \frac{1}{1+\frac{1}{2}\cos(x)} \quad ; \quad x(0) = \xi \longrightarrow (*)$$

$\frac{d\tilde{u}}{d\tau} = 0$, $\tilde{u}(0,\xi) = u_0(\xi)$; from this ODE; u is constant along each characteristic curve.

The general solution of $(*)$ is given by

$$x(\tau) + \frac{1}{2}\sin(x) - \tau = C$$

and the initial condition $x(0) = \xi$ implies that

$$\xi + \frac{1}{2}\sin\xi - 0 = C$$

Hence the unique solution is in (x,t) coordinate:

$$x + \frac{1}{2}\sin x - t = \xi + \frac{1}{2}\sin\xi$$

Exercise 1.17

$$u_t + x u_x = 0$$

with

$$u(x,0) = u_0(x) = \begin{cases} 1 & ; \quad 0 \le x \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The given PDE corresponds to the system of the equations

$$\frac{d\tilde{u}}{d\tau} = 0 \quad ; \quad \tilde{u}(0, \xi) = u_0(\xi)$$

$$\frac{dx}{d\tau} = x \quad ; \quad x(0) = \xi$$

From the first equation; u is constant along each characteristic curve

i.e $\quad \tilde{u}(\xi, \tau) = u_0(\xi) \quad (\because c = u_0(\xi))$

the general solution of second equation $x(\tau)$ is

$$x(\tau) = c e^{\tau}$$

with $x(0) = \xi$ implies that

$$x(\tau) = \xi e^{\tau} \quad \text{or} \quad \xi = x e^{-t}$$

Thus

$$u(t,x) = \tilde{u}(\tau, \xi) = u_0(\xi) = u_0(x e^{-t})$$

So we have; for $t > 0$

$$u(t,x) = \begin{cases} 1 & ; \quad \text{if} \quad 0 \le x \le e^{t} \\ 0 & ; \quad \text{otherwise} \end{cases}$$