Exercise 2.1 (a)

$$u_x\big|_{x_i} \approx A u_{i+2} + B u_{i+1} + C u_i + D u_{i-1}$$

Taylor:

$$u_{i+2} = u_i + 2\Delta x\, u_x + \tfrac{1}{2}(2\Delta x)^2 u_{xx} + \tfrac{1}{6}(2\Delta x)^3 u_{xxx} + O(\Delta x^4)$$

$$u_{i+1} = u_i + \Delta x\, u_x + \tfrac{1}{2}(\Delta x)^2 u_{xx} + \tfrac{1}{6}(\Delta x)^3 u_{xxx} + O(\Delta x^4)$$

$$u_i = u_i$$

$$u_{i-1} = u_i - \Delta x\, u_x + \tfrac{1}{2}(\Delta x)^2 u_{xx} - \tfrac{1}{6}(\Delta x)^3 u_{xxx} + O(\Delta x^4)$$

$$\Leftrightarrow \begin{cases} A+B+C+D=0 \\ (2A+B-D)\Delta x = 1 \\ 4A+B+D=0 \\ 8A+B-D=0 \end{cases} \Rightarrow A = -\frac{1}{6\Delta x},\quad B = \frac{1}{\Delta x},\quad C = -\frac{1}{2\Delta x},\quad D = -\frac{1}{3\Delta x}$$

error:

$$E = \frac{16}{24}(\Delta x)^4 A + \frac{1}{24}(\Delta x)^4 B + \frac{1}{24}(\Delta x)^4 D + \text{H.O.T.}$$

$$= \text{constant} \cdot (\Delta x)^3 + \text{H.O.T.}$$

$$= O((\Delta x)^3)$$

(b)- $u_{xx}\Big|_{x_i}$

$$u_{xx}\Big|_{x_i} \approx \left[A u_{i-1} + B u_i + C u_{i+1} + D u_{i+2}\right] \Big/ \left(2(\Delta x)^2\right)$$

Taylor expansion:

$$u_{i-1} = u_i - \Delta x\, u_x + \frac{(\Delta x)^2}{2} u_{xx} - \frac{(\Delta x)^3}{6} u_{xxx} + O(\Delta x)^4$$

$$u_{i+1} = u_i + \Delta x\, u_x + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta x)^3}{6} u_{xxx} + O(\Delta x^4)$$

$$u_{i+2} = u_i + 2\Delta x\, u_x + \frac{4(\Delta x)^2}{2} u_{xx} + \frac{8(\Delta x)^3}{6} u_{xxx} + O(\Delta x^4)$$

$$\Rightarrow \quad A + B + C + D = 0$$

$$\Delta x(-A + C + 2D) = 0$$

$$\frac{(\Delta x)^2}{2}(A + C + 4D) = 1$$

$$\Rightarrow \quad A = \frac{1}{2(\Delta x)^2}, \quad B = -\frac{1}{2(\Delta x)^2}, \quad C = -\frac{1}{2(\Delta x)^2}$$

$$D = \frac{1}{2(\Delta x)^3}$$

$$u_{xx}\Big|_{x_i} \approx \frac{1}{2(\Delta x)^2}\left(u_{i-1} - u_i - u_{i+1} + u_{i+2}\right)$$

Since

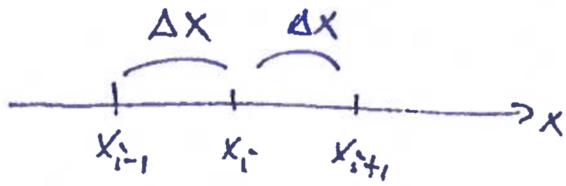
$$\frac{(\Delta x)^3}{6}(-A + C + 8D) \neq 0$$

$$u_{i-1} - u_i - u_{i+1} + u_{i+2} = 2(\Delta x)^2 u_{xx}\Big|_{x_i} + O((\Delta x)^3)$$

Thus $\quad u_{xx}\Big|_{x_i} = u_{i-1} - u_i - u_{i+1} + u_{i+2} \Big/ 2(\Delta x^2) + O(\Delta x)$

Exercise 2.2

$$u_{x,i} \approx A u_{i+1} + B u_i + C u_{i-1}$$



$$u_{i+1} = u_i + \Delta x\, u_{x,i} + \frac{(\Delta x)^2}{2} u_{xx,i} + \frac{(\Delta x)^3}{6} u_{xxx,i} + \dots$$

$$u_i = u_i$$

$$u_{i-1} = u_i - \Delta x\, u_{x,i} + \frac{(\Delta x)^2}{2} u_{xx,i} - \frac{(\Delta x)^3}{6} u_{xxx,i} \dots$$

$$u_{x,i} = A u_i + A \Delta x\, u_{x,i} + A \frac{(\Delta x)^2}{2} u_{xx,i} + \dots$$
$$+ B u_i$$
$$+ C u_i - C \Delta x\, u_{x,i} + C \frac{(\Delta x)^2}{2} u_{xx,i} \dots$$

$$\Rightarrow \begin{cases} A + B + C = 0 \\ (A - C)\Delta x = 1 \end{cases}$$

two equations with three unknowns

$\Rightarrow$ ∞-many solutions

define $B = \frac{\gamma}{\Delta x}$, $\gamma \in \mathbb{R}$

then $A = \frac{1-\gamma}{2\Delta x}$, $C = \frac{-\gamma - 1}{2\Delta x}$.

$\gamma = 0 \Rightarrow$ --- (central approximation) $\rightarrow O((\Delta x)^2)$

$\gamma = \pm 1 \Rightarrow$ --- (forward/backward approximation) $\rightarrow O(\Delta x)$

Exercise 2.3

$$(C_5 = A,\ C_4 = B,\ C_3 = C,\ C_2 = D,\ C_1 = E)$$

$$u_{xx,i} \approx c_1 u_{i-2} + c_2 u_{i-1} + c_3 u_i + c_4 u_{i+1} + c_5 u_{i+2}$$

apply Taylor expansions:

$$u_{i-2} = u_i - 2\Delta x\, u_{x,i} + \frac{4(\Delta x)^2}{2} u_{xx,i} \cdots$$

$$u_{i-1} = u_i - \Delta x\, u_{x,i} + \frac{(\Delta x)^2}{2} u_{xx,i} \cdots$$

$$u_i = u_i$$

$$u_{i+1} = u_i + \Delta x\, u_{x,i} + \frac{(\Delta x)^2}{2} u_{xx,i} \cdots$$

$$u_{i+2} = u_i + 2\Delta x\, u_{x,i} + \frac{4(\Delta x)^2}{2} u_{xx,i} \cdots$$

collect / re-arrange:

$$c_1 + c_2 + c_3 + c_4 + c_5 = 0$$

$$-2c_1 - c_2 + c_4 + 2c_5 = 0$$

$$2c_1 + \frac{1}{2}c_2 + \frac{1}{2}c_4 + 2c_5 = \frac{1}{(\Delta x)^2}$$

$$-\frac{4}{3}c_1 - \frac{1}{6}c_2 + \frac{1}{6}c_4 + \frac{4}{3}c_5 = 0$$

$$\frac{2}{3}c_1 + \frac{1}{24}c_2 + \frac{1}{24}c_4 + \frac{2}{3}c_5 = 0$$

solve: $c_1 = -\frac{1}{12(\Delta x)^2}$, $c_2 = \frac{4}{3(\Delta x)^2}$, $c_3 = \frac{-5}{2(\Delta x)^2}$, $c_4 = -\frac{1}{12(\Delta x)^2}$,

$$c_5 = -\frac{1}{12(\Delta x)^2}$$

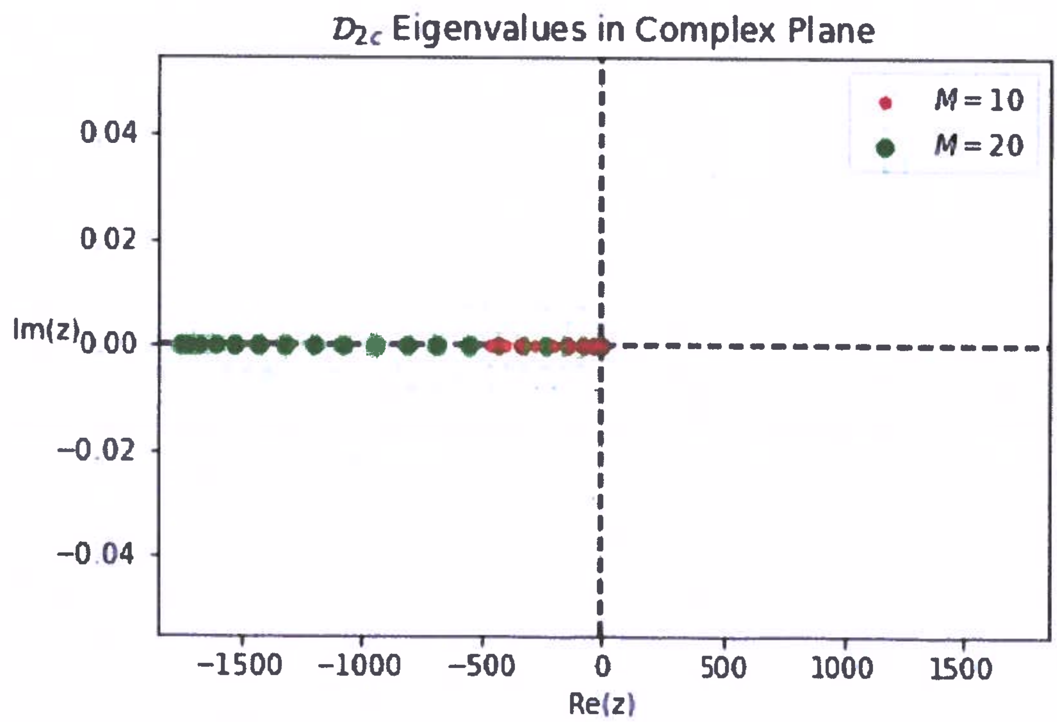
err (use the evalues $c_1, \ldots, c_5$ in the next term of Taylor):

$$O\left((\Delta x)^4\right)$$

exercise 2.4

(a) follow the "recipe" as mentioned in the lecture notes!

(b)



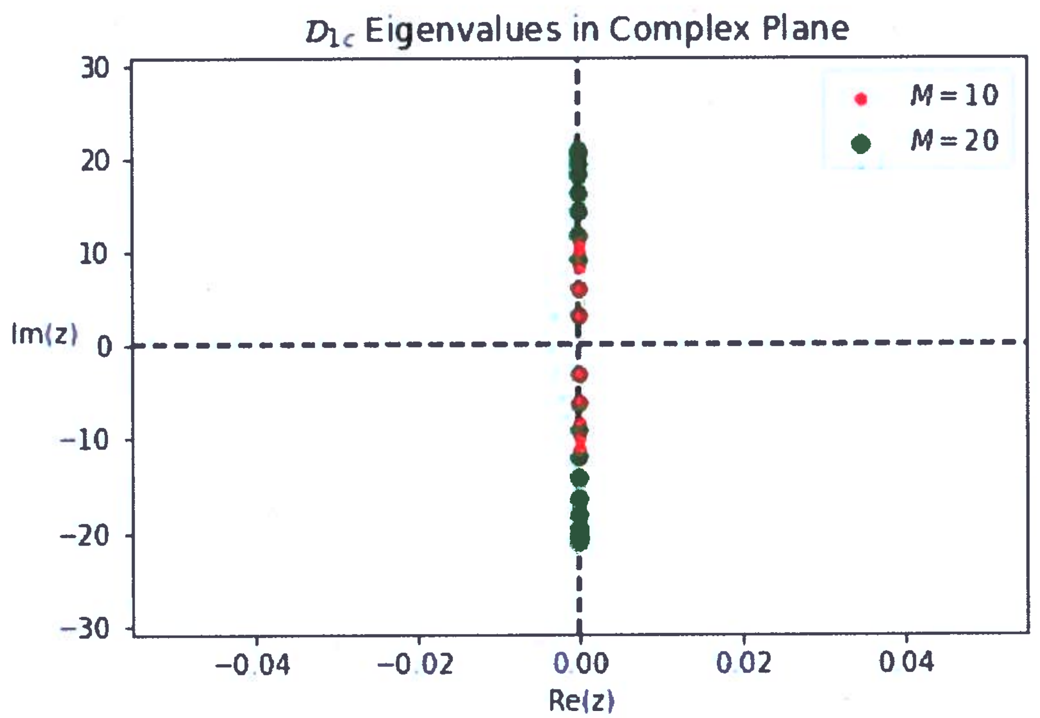
exercise 2.5

(a)

It is allowed to check the eigenvalues and eigenvectors by substitution:

$$D_{1c}\vec{v} = \lambda\vec{v}.$$

(b)



Exercise 2.6: (you may use Matlab as well)

(a).

$$D_{1+} D_{1-} = \frac{1}{(\Delta x)^2} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & 0 \\ 1 & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{(\Delta x)^2} \begin{pmatrix} -1-1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -1-1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \\ 0 & \cdots & & 0 & 1 \;\; -1-1 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -1-1 \end{pmatrix}$$

$$= \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 \;\; 1 & -2 \end{pmatrix} = D_{2C}$$

(b).

$$D_{4C} = (D_{2C})^2$$

$$(D_{2C})^2 = D_{2C} \cdot D_{2C}$$

$$= \frac{1}{(\Delta x)^4} \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & & 1 & -2 & 1 \\ 1 & 0 & \cdots & & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & 0 \\ 0 & & & & 1 & -2 & 1 \\ 1 & 0 & \cdots & & 0 & 1 & -2 \end{pmatrix}$$

$$= \frac{1}{(\Delta x)^4}\begin{pmatrix}
4+1+1 & -2-2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -4 \\
-2-2 & 1+4+1 & -2-2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\
1 & -2-2 & 1+4+1 & -2-2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 1 & -2-2 & 1+4+1 & -2-2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \cdot \\
\cdot & \cdot & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\
\cdot & & & & & 1 & -2-2 & 1+4+1 & & -2-2 & 1 \\
0 & 0 & & & & & 1 & -2-2 & 1+4+1 & & -2-2 \\
-2-2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -2-2 & 1+4+1
\end{pmatrix}$$

$$= \frac{1}{(\Delta x)^4}\begin{pmatrix}
6 & -4 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -4 \\
-4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\
1 & -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\cdot & & & & & \ddots & \ddots & 1 & -4 & 6 & -4 \\
0 & & & & & & & & & & \\
-4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -4 & 6
\end{pmatrix}$$

$$= D_{4c}$$

Remaining part of (a):

$$\frac{1}{2}\left(D_{1+} + D_{1-}\right) = \frac{1}{2\Delta x}\left\{ \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & -1 & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & & -1 & 1 \end{pmatrix} \right.$$

$$= \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & \cdots & & -1 \\ -1 & 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{pmatrix}$$

$$\frac{1}{2}\left(D_{1+} + D_{1-}\right) = D_{1c}$$

(C)- for $U_{6x} := U_{xxxxxx}$:

$$D_{6c} = \underbrace{D_{2c} \cdot D_{2c}}_{D_{4c}} \cdot D_{2c}$$

$$D_{6c} = D_{4c} \cdot D_{2c}$$

from part (b); we have

$$D_{4c} \cdot D_{2c} = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 0 & 1 & -4 & 6 & -4 \\ -4 & 1 & 0 & \cdots & & 0 & 1 & -4 & 6 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & & \\ 0 & \cdots & & & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & & 0 & 1 & -2 \end{pmatrix}$$

$$
= \begin{pmatrix}
-12-4-4 & 6+8+1 & -4-2 & 1 & 0 & 0 & \cdots & 0 & 1 & -2-4 & 6+1+8 \\
8+6+1 & -4-12-4 & 6+8+1 & -4-2 & 1 & 0 & \cdots & 0 & 1 & -2-4 & \\
-2-4 & 1+8+6 & -4-12-4 & 6+8+1 & -4-2 & 1 & 0 & \cdots & 0 & 0 & 1 \\
1 & -2-4 & 1+8+6 & -4-12-4 & 6+8+1 & -4-2 & 1 & \cdots & & & 0 \\
0 & 1 & -2-4 & 1+8+6 & -4-12-4 & 1+6+8 & -4-2 & 1 & \cdots & & 0 \\
\vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \\
8+6+1 & -4-2 & 1 & 0 & \cdots & 0 & 1 & -2-4 & 1+8+6 & & -4-12-4
\end{pmatrix}
$$

$$D_{4c} \cdot D_{2c} = \begin{pmatrix} -20 & 15 & -6 & 1 & 0 & 0 & \cdots & 0 & 1 & -6 & 15 \\ 15 & -20 & 15 & -6 & 1 & 0 & \cdots & 0 & 0 & 1 & -6 \\ -6 & 15 & -20 & 15 & -6 & 1 & 0 & \cdots & & 0 & 1 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -6 & 15 & -20 & +15 & -6 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & & & & \\ 15 & -6 & 1 & 0 & \cdots & & & 0 & 1 & -6 & 15 & -20 \end{pmatrix}$$

$$= D_{6c}$$

Exercise 2.7:

(a)- Consider the one-dimensional stationary convection-diffusion models

(1) $\longrightarrow \epsilon u''(x) - u'(x) = 0$, $x \in [0,1]$, $0 < \epsilon < 1$

$u(0) = 0$, $u(1) = 1$.

Analytical Solution:

The roots of the given DEq:

$$m = 0, \quad m = \frac{1}{\epsilon}$$

$$u(x) = C_1 e^0 + C_2 e^{x/\epsilon}$$

$$u(x) = C_1 + C_2 e^{x/\epsilon}$$

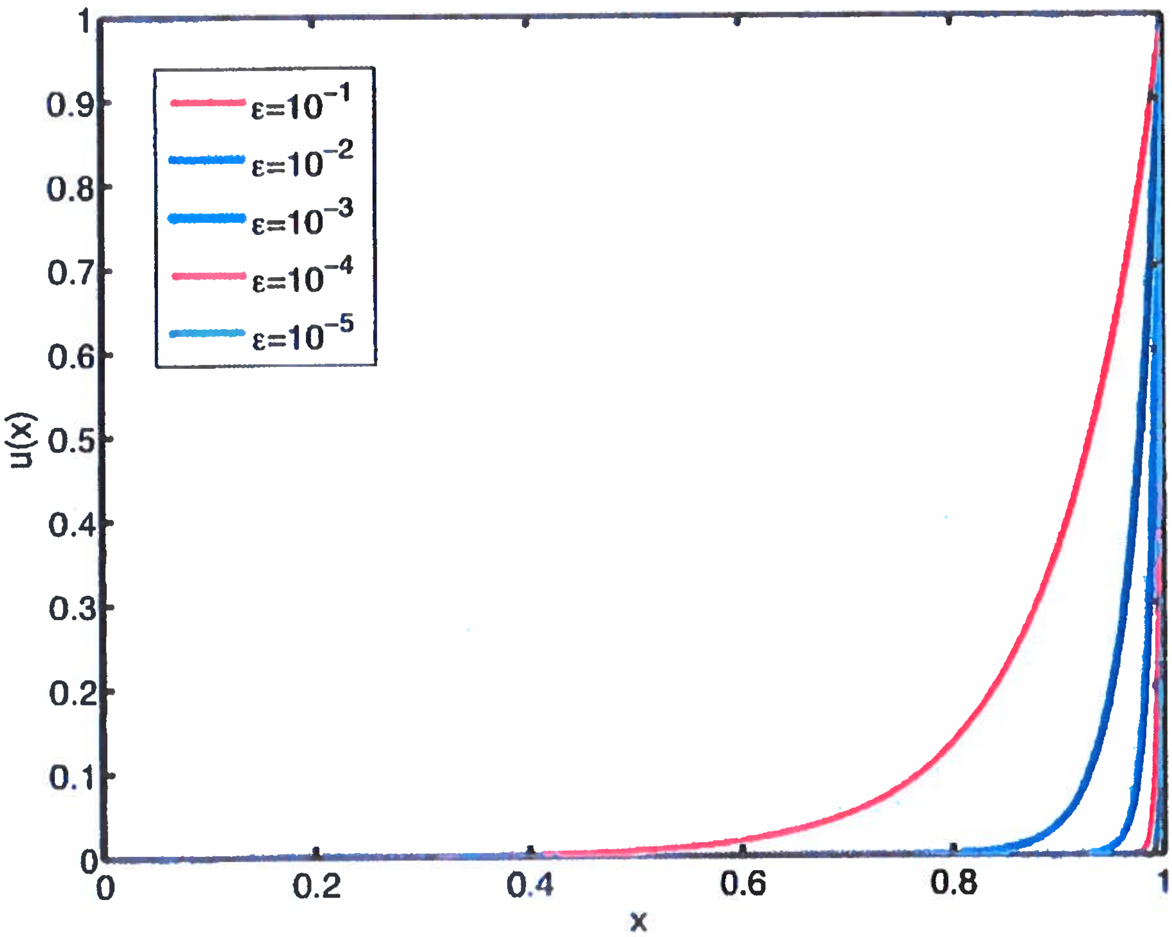
$u(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$

$u(1) = 1 \Rightarrow C_2 = -1/1 - e^{1/\epsilon}$

$$u(x) = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$, is the required solution.

(a)-

Solutions for decreasing values of ϵ



(b):-

As the exact solution of (1) is:

$$u_i = \frac{\left(\frac{1+P_e}{1-P_e}\right)^i - 1}{\left(\frac{1+P_e}{1-P_e}\right)^M - 1} \;, \quad i = 1, \dots, M-1 \;; \quad P_e = \frac{\Delta x}{2\epsilon} \longrightarrow (*)$$

Use central differences for $u_{x,i}$ on three-point stencils

$$u_{x,i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

We approximate the given equation[(1)] by:

$$\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0 \longrightarrow (2)$$

Here,

$$u_{i+1} = \left(\frac{1+P_e}{1-P_e}\right)^{i+1} - 1 \Big/ \left(\frac{1+P_e}{1-P_e}\right)^M - 1 \longrightarrow (**)$$

$$u_{i-1} = \left(\frac{1+P_e}{1-P_e}\right)^{i-1} - 1 \Big/ \left(\frac{1+P_e}{1-P_e}\right)^M - 1 \longrightarrow (***)$$

So equation (2) by using (*), (**) & (***) expressions, we obtain

$$0 = \frac{2\epsilon}{\Delta x}\left[\left(\frac{1+P_e}{1-P_e}\right) - 2 + \left(\frac{1-P_e}{1+P_e}\right)\right] - \left[\left(\frac{1+P_e}{1-P_e}\right) - \left(\frac{1-P_e}{1+P_e}\right)\right]$$

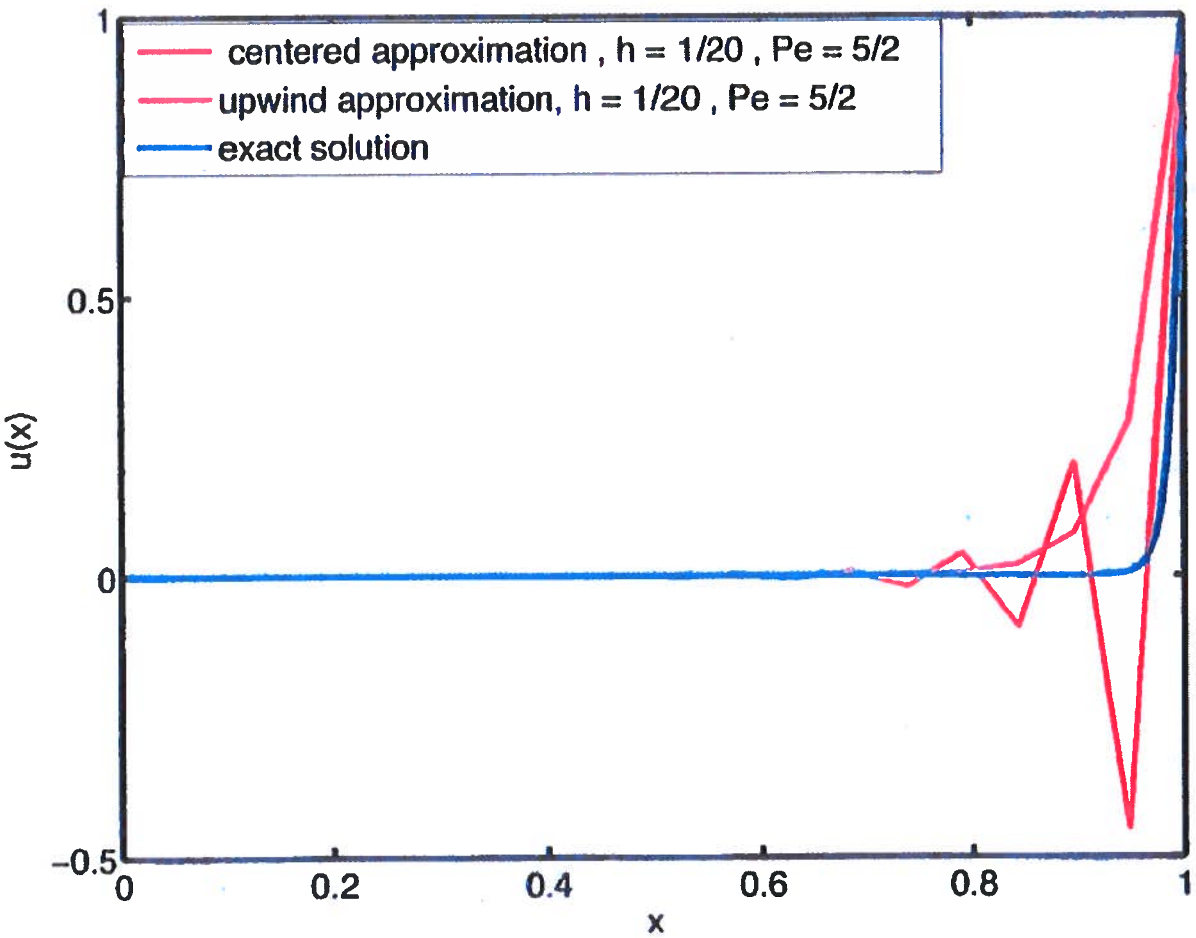
$$0 = \frac{1}{P_e}\left[\frac{(1+P_e)^2 - 2(1+P_e)(1-P_e) + (1-P_e)^2}{(1+P_e)(1-P_e)}\right] - \left[\frac{(1+P_e)^2 - (1-P_e)^2}{(1-P_e)(1+P_e)}\right]$$

After simplification, we obtain

$$= \frac{4P_e - 4P_e}{(P_e+1)(1-P_e)} = 0$$

(b)-

For $P_e > 1$, $M = 20$, $\epsilon = 0.01$

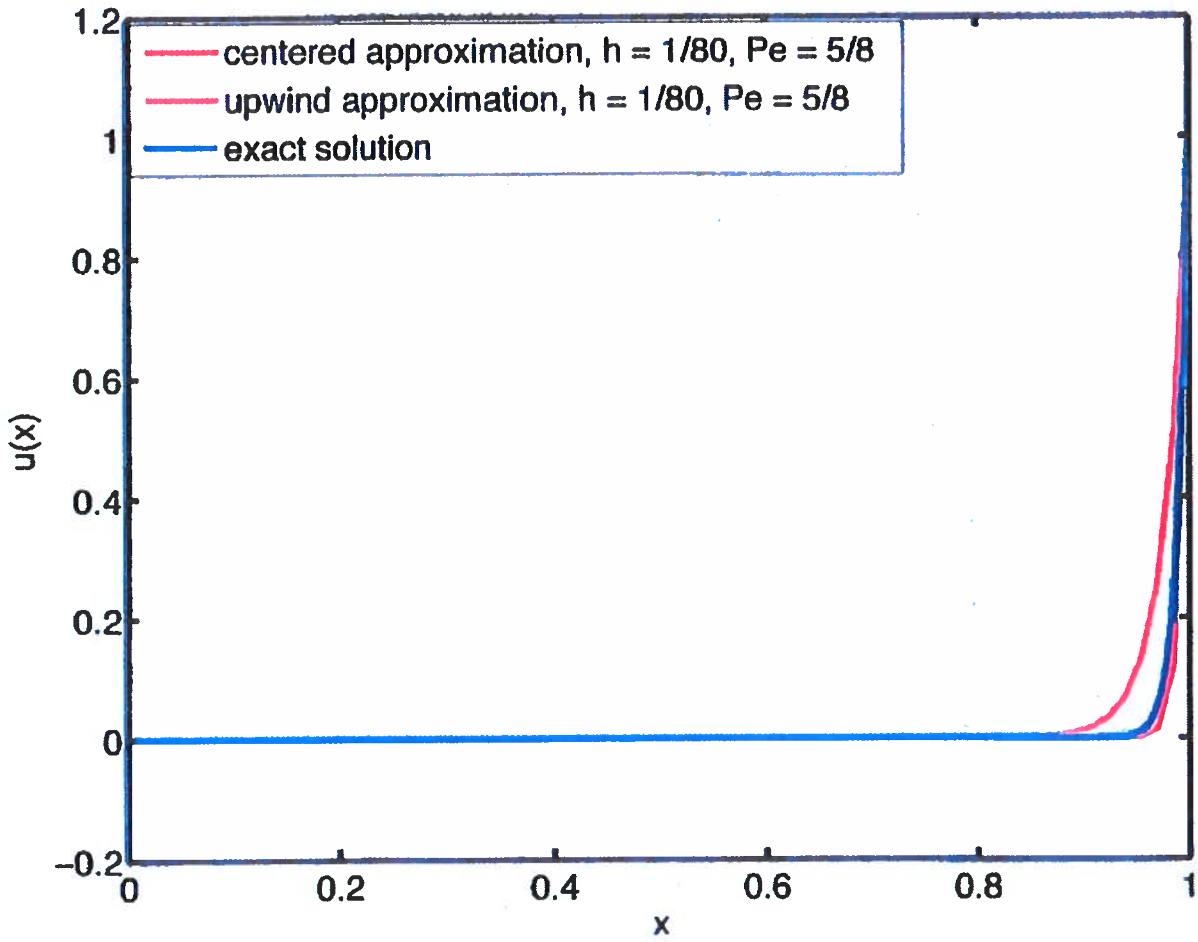


(C) :

$\boxed{\text{Please check handin-exercise C1}}$

(C) -

for $P_e < 1$ & $M = 80$, $\epsilon = 0.01$



When we choose $P_e < 1$ and increase the M value, then the oscillation in solution is removed. The solution becomes smooth.

Exercise 2.8:

(a)-

Consider the one-dimensional non-linear Gelfand-Bratu model:

$$(1) \longrightarrow \begin{cases} u''(x) + \lambda e^{u(x)} = 0, & x \in [0,1] \\ u(0) = 0,\ u(1) = 0, & \lambda \in \mathbb{R} \end{cases}$$

We need to check the exact solution of (1) is:

$$2) \longrightarrow \begin{cases} u(x) = -2 \ln\left[\dfrac{\cosh\left((x-\frac{1}{2})\theta/2\right)}{\cosh(\theta/4)}\right] \\ \theta = \sqrt{2\lambda}\cosh(\theta/4) \end{cases}$$

Check:

$$u'(x) = -\theta \tanh\left((x-\tfrac{1}{2})\theta/2\right)$$

$$u''(x) = -\frac{\theta^2}{2}\operatorname{sech}^2\left((x-\tfrac{1}{2})\theta/2\right)$$

or

$$u''(x) = -\frac{2\lambda\cosh^2(\theta/4)}{2}\cdot\frac{1}{\cosh^2\left((x-\frac{1}{2})\theta/2\right)}$$

$$u''(x) = -\lambda\left(\frac{\cosh(\theta/4)}{\cosh\left((x-\frac{1}{2})\theta/2\right)}\right)^{2}$$

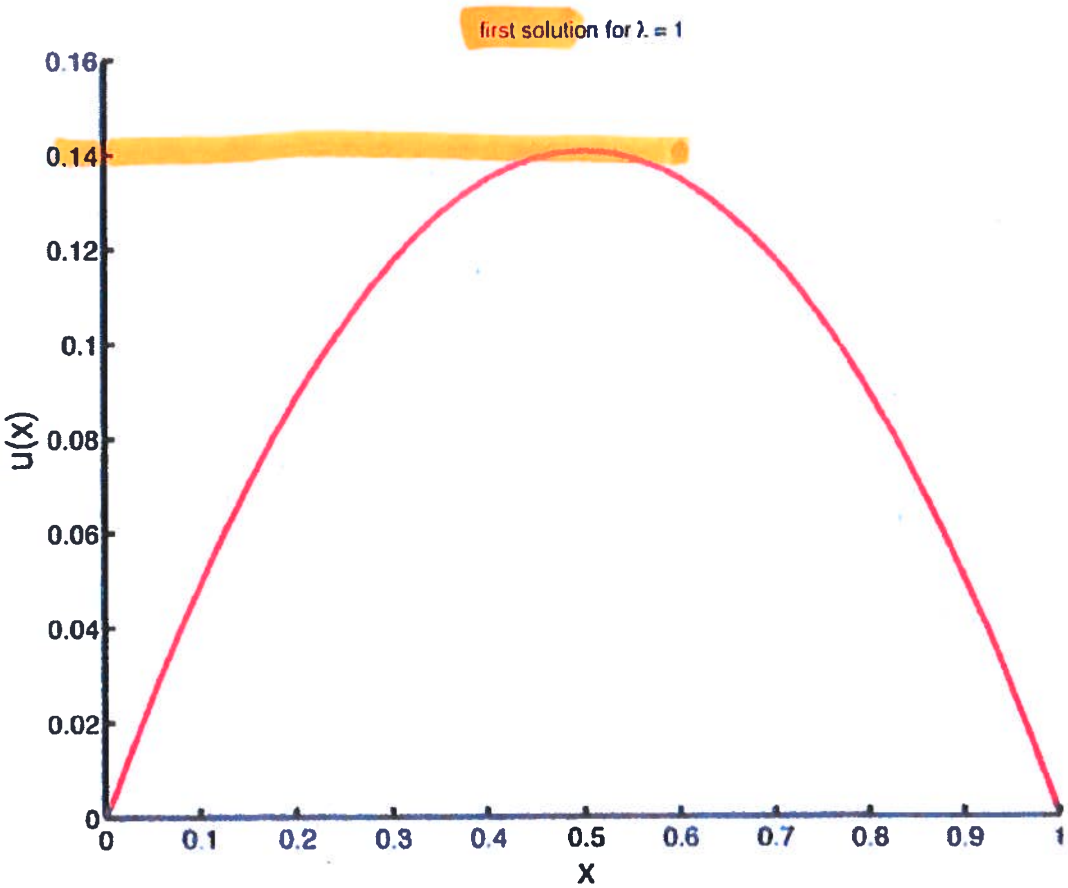
so (1) $\Rightarrow$

$$u''(x) + \lambda e^{u(x)} = -\lambda\left(\frac{\cosh(\theta/4)}{\cosh\left((x-\frac{1}{2})\theta/2\right)}\right)^2 + \lambda e^{-2\ln\left[\frac{\cosh\left((x-\frac{1}{2})\theta/2\right)}{\cosh(\theta/4)}\right]}$$

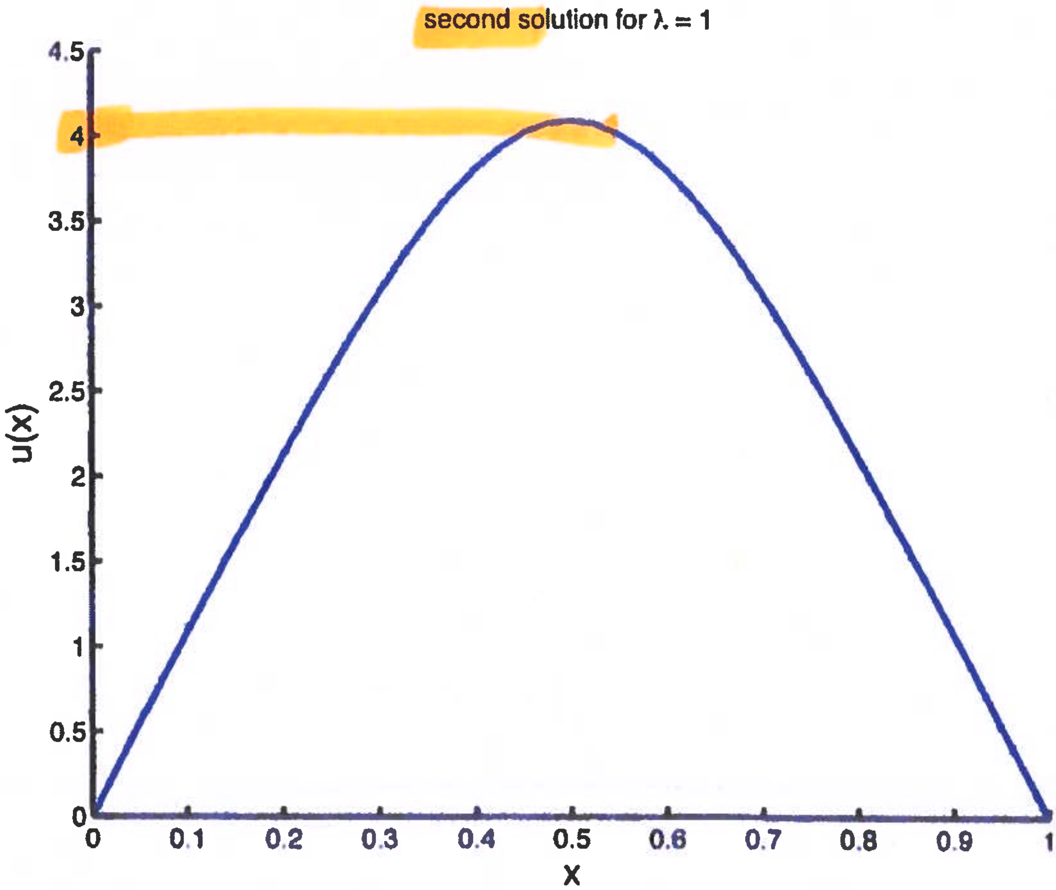
$$= -\lambda\left(\frac{\cosh(\theta/4)}{\cosh\left((x-\frac{1}{2})\theta/2\right)}\right)^2 + \lambda\left(\frac{\cosh(\theta/4)}{\cosh\left((x-\frac{1}{2})\theta/2\right)}\right)^2$$

$$u''(x) + \lambda e^{u(x)} = 0$$

(a) -

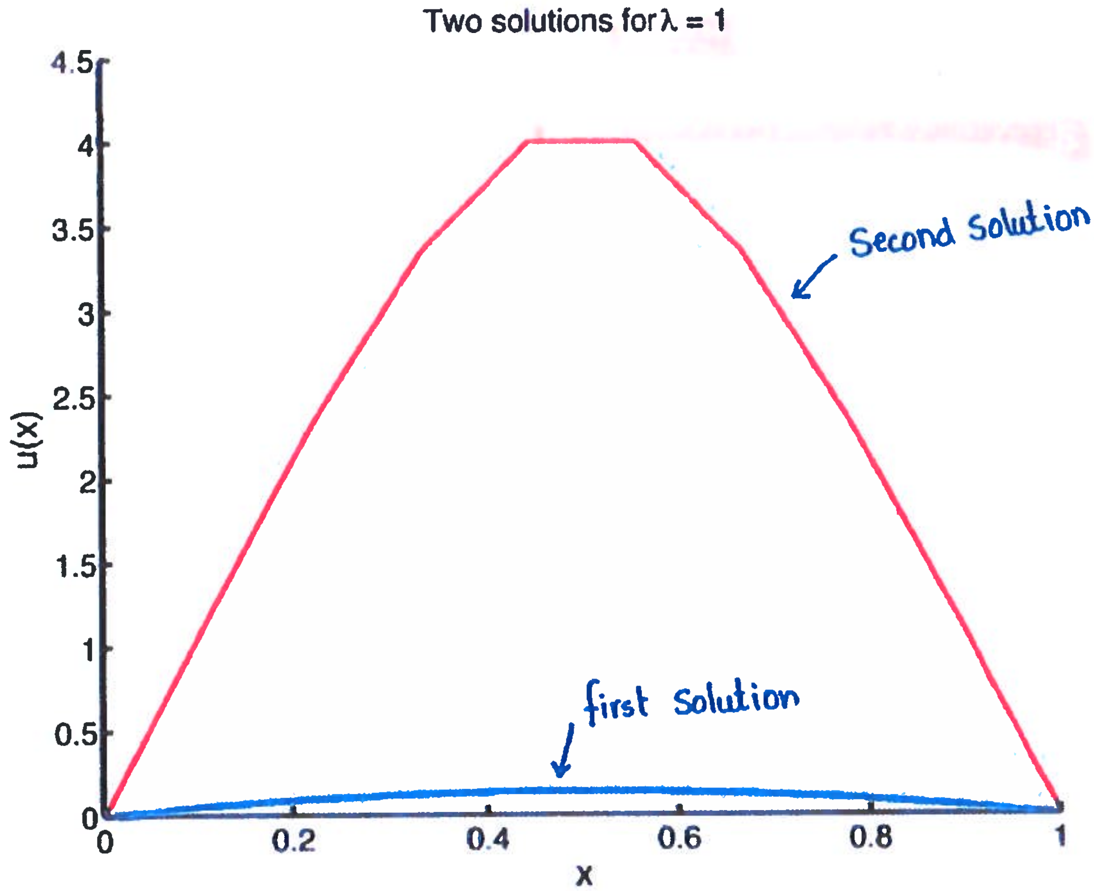

first solution for λ = 1
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
u(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

(a)


second solution for λ = 1
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
u(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Two Solutions for $\lambda = 1$ numerically; using fsolve.m

Here $M = 10$



Exercise 2.9: (see also the lecture in Part II of the Course, on adaptive grids!)

- for $u_{x,j}$

Non-uniform discretization of the first derivative is given by

$$u'_j \approx \frac{u_{j+1} - u_{j-1}}{2\,\Delta x_j} \quad ; \quad j = 1, 2, \dots, J-1. \longrightarrow (1)$$

where $\Delta x_j = \Delta x_{j+1} + \Delta x_{j-1}/2$, $\quad \Delta x_{j+1} = x_{j+1} - x_j$, $\Delta x_{j-1} = x_j - x_{j-1}$

Taylor expansion of terms appeared in (1), we have

$$\frac{u_{j+1} - u_{j-1}}{\Delta x_{j+1} - \Delta x_{j-1}} = u' + \frac{(\Delta x_{j+1} - \Delta x_{j-1})}{2!} u'' + \frac{\Delta x_{j+1}^2 - \Delta x_{j+1}\Delta x_{j-1} + \Delta x_{j-1}^2}{3!} u''' + \text{H.O.T} \longrightarrow (2)$$

and also we present the Taylor expansion of the terms Δx_{j+1}, Δx_{j-1} as:

$$\Delta x_{j+1} = x_{j+1} - x_j = x(\xi_{j+1}) - x(\xi_j)$$

$$\Delta x_{j+1} = \Delta\xi\, x_\xi + \frac{(\Delta\xi)^2}{2!} x_{\xi\xi} + \frac{(\Delta\xi)^3}{3!} x_{\xi\xi\xi} + \text{H.O.T} \quad \left(\because \Delta\xi = H = \frac{1}{M}\right) \longrightarrow (*)$$

Similarly

$$\Delta x_{j-1} = \Delta\xi\, x_\xi - \frac{(\Delta\xi)^2}{2!} x_{\xi\xi} + \frac{(\Delta\xi)^3}{3!} x_{\xi\xi\xi} + \text{H.O.T} \longrightarrow (**)$$

So

Eq^n (2) becomes:

$$u' = \underbrace{\frac{u_{j+1} - u_{j-1}}{\Delta x_{j+1} - \Delta x_{j-1}}}_{\tilde{u}'} + \frac{(\Delta\xi)^2}{6} \underbrace{\left(3x_{\xi\xi} u'' + x_\xi^2 u'''\right)}_{\tau_1} + \text{H.O.T}$$

So the truncation error of the first derivative on the non-uniform grids:

$$\tau_1 = +\frac{1}{6}\left(3x_{\xi\xi} u'' + x_\xi^2 u'''\right)$$

- for $u_{xx,j}$:

Second derivative on non-uniform grid can be approximated by

$$u''_j \approx \frac{\dfrac{u_{j+1}-u_j}{\Delta x_{j+1}} - \dfrac{u_j - u_{j-1}}{\Delta x_{j-1}}}{\Delta x_j}$$

from Taylor expansion of the terms in the above expression, we have

$$\frac{u_{j+1}-u_j}{\Delta x_{j+1}} = u' + \frac{\Delta x_{j+1}}{2!} u'' + \frac{(\Delta x_{j+1})^2}{3!} u''' + \frac{(\Delta x_{j+1})^3}{4!} u'''' + H.O.T$$

$$\frac{u_j - u_{j-1}}{\Delta x_{j-1}} = u' + \frac{\Delta x_{j-1}}{2!} u'' + \frac{(\Delta x_{j-1})^2}{3!} u''' + \frac{(\Delta x_{j-1})^3}{4!} u'''' + H.O.T$$

By using the expressions (*), (**) from part (a), the expression for the non-uniform second derivative approximation becomes:

$$\frac{1}{\Delta x_j}\left[\frac{u_{j+1}-u_j}{\Delta x_{j+1}} - \frac{u_j - u_{j-1}}{\Delta x_{j-1}}\right] = u'' + \frac{\Delta x_{j+1} - \Delta x_{j-1}}{3} u''' + \frac{\left(\Delta x_{j+1}^2 - \Delta x_{j+1}\Delta x_{j-1} + \Delta x_{j-1}^2\right)}{60} u'''' + H.O.T$$

$$\tilde{u}'' = u'' + \frac{(\Delta\xi)^2}{3} x_{\xi\xi} u''' + \frac{(\Delta\xi)^2}{12} x_{\xi}^2 u'''' + H.O.T$$

Which can be written as

$$u'' = \tilde{u}'' \underbrace{- \frac{(\Delta\xi)^2}{12} \left(4x_{\xi\xi} u''' + x_{\xi}^2 u''''\right)}_{\tau_2} + H.O.T$$

So the truncation error of the second derivative approximation is:

$$\tau_2 = -\frac{1}{12} \left(4x_{\xi\xi} u''' + x_{\xi}^2 u''''\right)$$

Exercise 2.10:

Laplacian Δu can be approximated at the gridpoint $(x,y)=(x_{i,j}, y_{i,j})$ on a nine-point stencil:

$$\Delta u_{i,j} \approx \frac{1}{6h^2}\Big[4(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}) + u_{i-1,j-1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1} - 20u_{i,j}\Big] \longrightarrow (*)$$

first we work out Taylor expansion for the terms appeared in (*).

$$u_{i-1,j} = u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} + O(h^5)$$

$$u_{i+1,j} = u + hu_x + \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + \frac{h^4}{24}u_{xxxx} + O(h^5)$$

$$u_{i,j-1} = u - hu_y + \frac{h^2}{2}u_{yy} - \frac{h^3}{6}u_{yyy} + \frac{h^4}{24}u_{yyyy} + O(h^5)$$

$$u_{i,j+1} = u + hu_y + \frac{h^2}{2}u_{yy} + \frac{h^3}{6}u_{yyy} + \frac{h^4}{24}u_{yyyy} + O(h^5)$$

$$u_{i-1,j-1} = u - h(u_x+u_y) + \frac{h^2}{2}(u_{xx} + 2u_{xy} + u_{yy}) + \text{H.O.T}$$
$$- \frac{h^3}{6}(u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}) +$$
$$\frac{h^4}{24}(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \text{H.O.T}$$

$$u_{i-1,j+1} = u - h(u_x - u_y) + \frac{h^2}{2}(u_{xx} - 2u_{xy} + u_{yy}) -$$
$$\frac{h^3}{6}(u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy}) +$$
$$\frac{h^4}{24}(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \text{H.O.T}$$

$$U_{i+1,j-1} = U + h(U_x - U_y) + \frac{h^2}{2}(U_{xx} - 2U_{xy} + U_{yy}) +$$
$$\frac{h^3}{6}(U_{xxx} - 3U_{xxy} + 3U_{xyy} - U_{yyy}) +$$
$$\frac{h^4}{24}(U_{xxxx} - 4U_{xxxy} + 6U_{xxyy} - 4U_{xyyy} + U_{yyyy}) + H.O.T$$

$$U_{i+1,j+1} = U + h(U_x + U_h) + \frac{h^2}{2}(U_{xx} + 2U_{xy} + U_{yy}) +$$
$$\frac{h^3}{6}(U_{xxx} + 3U_{xxy} + 3U_{xyy} + U_{yyy}) +$$
$$\frac{h^4}{24}(U_{xxxx} + 4U_{xxxy} + 6U_{xxyy} + 4U_{xyyy} + U_{yyyy}) + H.O.T$$

Subsitute all the Taylor expansion of terms appeared in (*), we get the second order approximation.

$$\Delta U_{i,j} \approx (U_{xx} + U_{yy}) + O(h^4) \longrightarrow (**)$$

for fourth order:

$$\Delta U = 0$$

It is clear from the expression in (**), we get fourth order approximations.