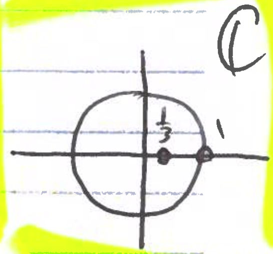
Exercise 3.1:

(a)- Implicit Gear method is zero stable:



$$u^{n+1} = \frac{1}{3}(4u^n - u^{n-1}) + \frac{2\Delta t}{3} f(u^{n+1})$$

Applying LMM, gives

$$3u^{n+1} - 4u^n + u^{n-1} = 0$$

Characteristic polynomial is;

$$P(\zeta) = 3\zeta^2 - 4\zeta + 1 = 0$$

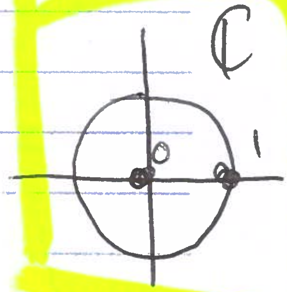
Roots:

$$\zeta_1 = \frac{1}{3}, \quad \zeta_2 = 1$$

Root Condition:

the roots ζ_1 and ζ_2 lie within and on the unit circle respectivel so root condition satisfied. Hence Implicit Gear method is zero stable.

(b)- Explicit 3-step Adams method is zero stable:



$$u^{n+3} = u^{n+2} + \frac{\Delta t}{12}\left[5f(u^n) - 16 f(u^{n+1}) + 23 f(u^{n+2})\right]$$

Applying LMM, we get

$$u^{n+3} - u^{n+2} = 0$$

Characteristic polynomial is;

$$P(\zeta) = \zeta^3 - \zeta^2 = 0$$

Roots:

$$\zeta^2(\zeta - 1) = 0 \quad ; \quad \zeta_1 = 0, \zeta_2 = 1$$

Roots ζ_1, ζ_2 lie on the circle. so root condition satisfied. Hence zero stable.

(C)- Linear multistep method is not zero Stable :

$$u^{n+2} - 3u^{n+1} + 2u^{n} = -\Delta t\, f(u^n)$$

Apply LMM, we have

$$u^{n+2} - 3u^{n+1} + 2u^{n} = 0$$

Characteristic polynomial is ;

$$P(\zeta) = \zeta^2 - 3\zeta + 2 = 0$$

Roots of the above equation is given by;

$$\zeta_1 = 1, \quad \zeta_2 = 2$$

As the root $\zeta_2 = 2$ does not lie within or on the unit circle. So root condition does not satisfy. Hence the given Linear multistep method is not zero stable.

Exercise 3.2 :

(a)- **Stability polynomial $\pi(\zeta, z)$ and its root for Trapezoidal method:**

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left[f(u^n) + f(u^{n+1})\right]$$

rewrite the above eq^n;

$$u^{n+1} - u^n = \frac{\Delta t}{2}\left[f(u^n) + f(u^{n+1})\right]$$

Apply LMM, we get

$$u^{n+1} - u^n = 0$$

Characteristic polynomial;

$$\rho(\zeta) = \zeta - 1$$

and

$$\sigma(\zeta) = \frac{1}{2}(1 + \zeta)$$

Stability polynomial:

$$\pi(\zeta, z) = \rho(\zeta) - z\,\sigma(\zeta)$$

$$= \zeta - 1 - \frac{z}{2}(1 + \zeta)$$

$$\pi(\zeta, z) = \zeta\left(1 - \frac{z}{2}\right) - \left(1 + \frac{z}{2}\right)$$

Roots: $\pi(\zeta, z) = 0$

$$\zeta_1 = \frac{1 + z/2}{1 - z/2}$$

(b)- for midpoint (leapfrog) method:

$$u^{n+1} - u^{n-1} = 2\Delta t\, f(u^n)$$

After applying LMM, we obtain

$$\rho(\zeta) = \zeta^2 - 1, \quad \sigma(\zeta) = 2\zeta$$

Stability polynomial:

$$\pi(\zeta, z) = \rho(\zeta) - z\sigma(\zeta)$$

$$\pi(\zeta, z) = \zeta^2 - 2z\zeta - 1$$

Roots:

$$\zeta_1 = z + \sqrt{z^2+1}, \quad \zeta_2 = z - \sqrt{z^2+1}$$

Exercise 3.3(a)

We consider the method

$$\begin{cases} k_1 = f(u^n) \\ k_2 = f(u^n + k_1 \Delta t) \\ u^{n+1} = u^n + \frac{\Delta t}{2}(k_1 + k_2) \end{cases}$$

Substituting $f(u^n) = \lambda u^n$ and $z = \lambda \Delta t$ we find that $k_1 = \lambda u^n$ and for k_2 we find that

$$\begin{aligned} k_2 &= f(u^n + k_1 \Delta t) \\ &= f([1+z]\, u^n) \\ &= \lambda (1+z)\, u^n \end{aligned}$$

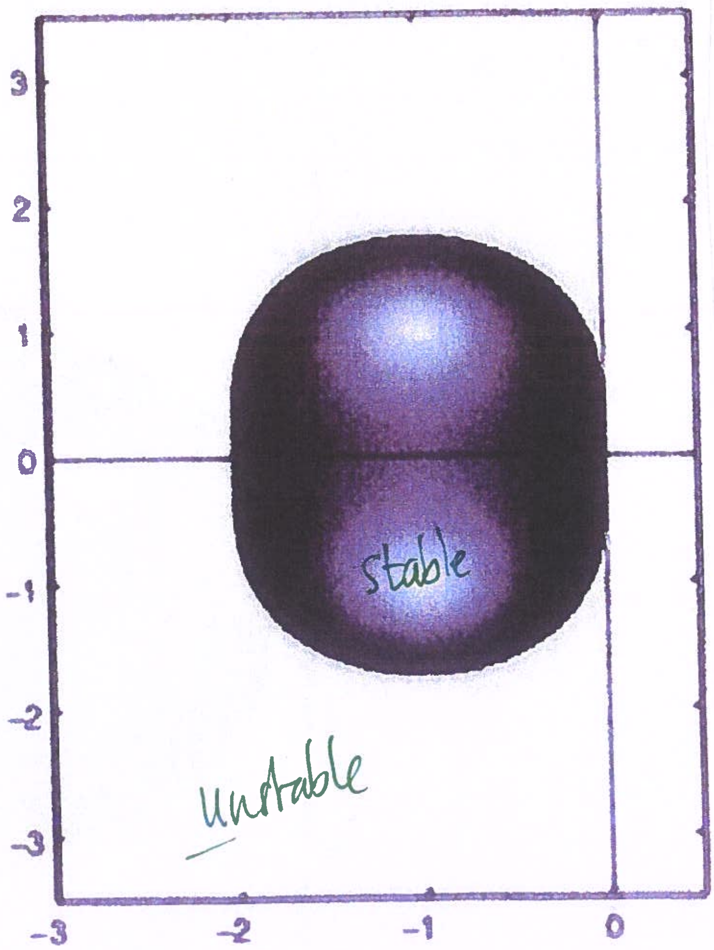
So it follows that

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{2}(k_1 + k_2) \\ &= u^n + \frac{\Delta t}{2}(\lambda u^n + \lambda(1+z)\, u^n) \\ &= u^n + \frac{1}{2}(z + z(1+z))\, u^n \\ &= \left(1 + z + \frac{z^2}{2}\right) u^n \end{aligned}$$

Hence we find that the stability polynomial is given by

$$R(z) = 1 + z + \frac{z^2}{2}$$

The stability region sketch is given below.



3.3(b)

We consider the method

$$\begin{cases} k_1 = f(u^n) \\ k_2 = f(u^n + k_1 \frac{\Delta t}{2}) \\ k_3 = f(u^n + k_2 \frac{\Delta t}{2}) \\ k_4 = f(u^n + k_3 \Delta t) \\ u^{n+1} = u^n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \end{cases}$$

Again substituting $f(u^n)$ and $z = \lambda \Delta t$ we find that $k_1 = \lambda u^n$, for k_2 we have that

$$\begin{aligned} k_2 &= f(u^n + k_1 \frac{\Delta t}{2}) \\ &= f\left(\left[1 + \frac{z}{2}\right] u^n\right) \\ &= \lambda\left(1 + \frac{z}{2}\right) u^n \end{aligned}$$

For k_3 we have that

$$\begin{aligned} k_3 &= f(u^n + k_2 \frac{\Delta t}{2}) \\ &= f\left(\left[1 + \frac{z}{2}\left(1 + \frac{z}{2}\right)\right] u^n\right) \\ &= \lambda\left(1 + \frac{z}{2} + \frac{z^2}{4}\right) u^n \end{aligned}$$

Lastly for k_4 we have that

$$\begin{aligned} k_4 &= f(u^n + k_3 \Delta t) \\ &= f\left(\left[1 + z\left(1 + \frac{z}{2} + \frac{z^2}{4}\right)\right] u^n\right) \\ &= \lambda\left(1 + z + \frac{z^2}{4} + \frac{z^3}{4}\right) u^n \end{aligned}$$

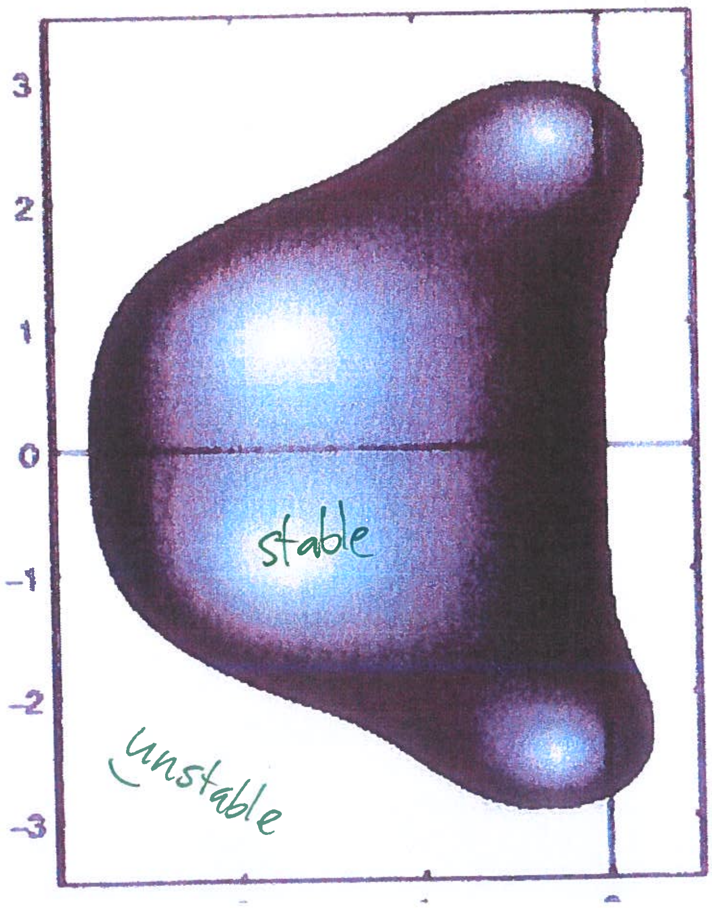
Combining these results, we find that

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ &= u^n + \frac{1}{6}\left(z + 2z + z^2 + \frac{z^3}{2} + z^2 + 2z + \frac{z^4}{4} + \frac{z^3}{2} + z^2 + z\right) \\ &= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right) u^n \end{aligned}$$

So we find the stability polynomial

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$$

The sketch of the stability region is given below



3.3(c)

We now consider the method

$$u^{n+1} = u^n + \Delta t f(u^n) + \frac{1}{2}(\Delta t)^2 f'(u^n) f(u^n)$$

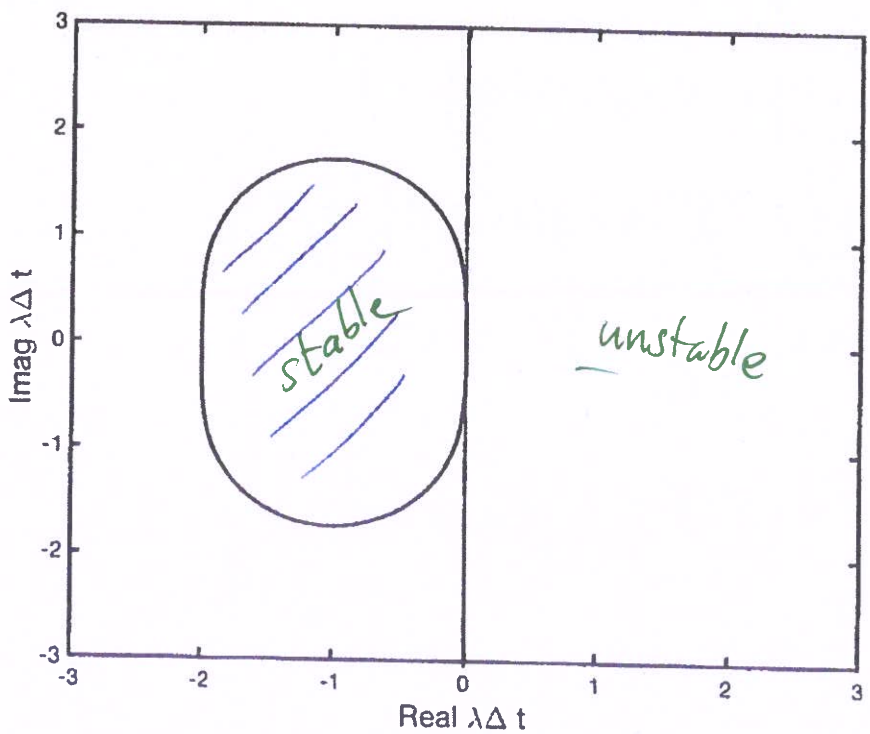
Substituting $f(u^n) = \lambda u^n$, $z = \Lambda \Delta t$ and using that $f'(u^n) = \Lambda$ we find that

$$\begin{aligned} u^{n+1} &= u^n + \Delta t f(u^n) + \frac{1}{2}(\Delta t)^2 f'(u^n) f(u^n) \\ &= u^n + z u^n + \frac{1}{2} z^2 u^n \\ &= \left(1 + z + \frac{z^2}{2}\right) u^n \end{aligned}$$

So we find that we have the stability polynomial

$$R(z) = 1 + z + \frac{z^2}{2}$$

which is the same as in part (a).



Exercise 3.4 :-

$$u^{n+1} = u^n + \Delta t\, f\left(u^n + \Delta t\, f(u^n)\right)$$

take $f = \lambda u$

$$\Rightarrow R(z) = 1 + z + z^2$$

$z = \lambda \Delta t$

for absolute stability;

$$|1 + z + z^2| = 1$$

Here

$$z = x + iy$$

$$|1 + x + iy + (x+iy)^2| = 1$$

$$|1 + x + iy + x^2 - y^2 + 2ixy| = 1$$

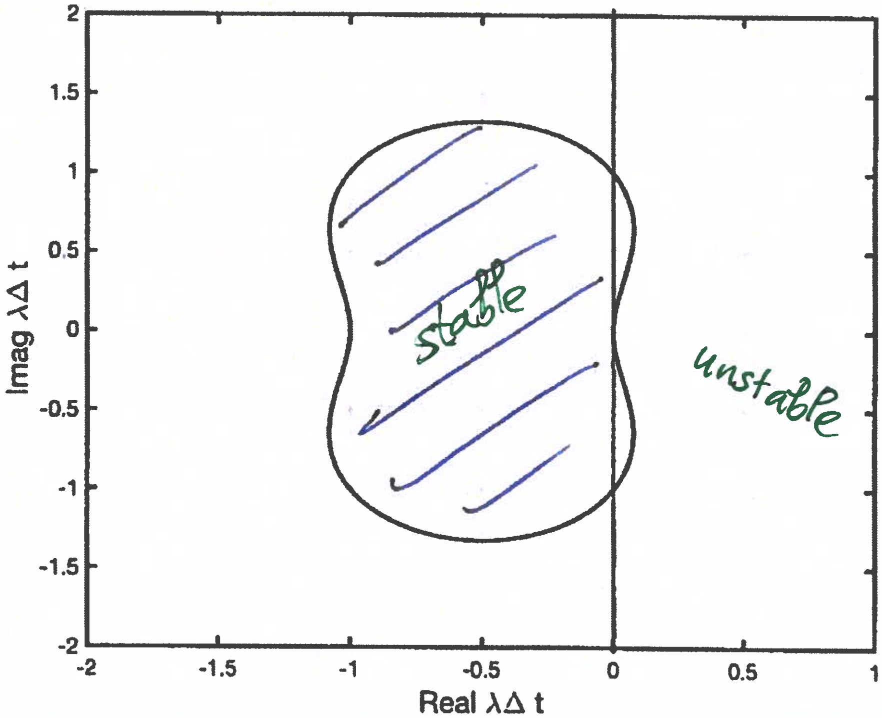
$$|(1 + x + x^2 - y^2) + i\, y(1 + 2x)| = 1$$

$$\sqrt{(1 + x + x^2 - y^2)^2 + y^2(1+2x)^2} = 1$$

or

$$(1 + x + x^2 - y^2)^2 + y^2(1+2x)^2 = 1$$

Plot of the curve: $[1+x+x^2-y^2]^2 + y^2[1+2x]^2 = 1$



Exercise 3.5:

(a)- Hints:

As in the exercises 2.4 and 2.5, the eigenvalues of D_{2c} and D_{1c} are given.

- So for D_{3c}; we can find the eigenvalues by using the eigenvalues of D_{1c} and D_{2c} as:

$$\lambda(D_{3c}) = \lambda(D_{2c}\, D_{1c}) \quad \text{or} \quad [\lambda(D_{1c})]^3$$

Similarly; for D_{4c}:

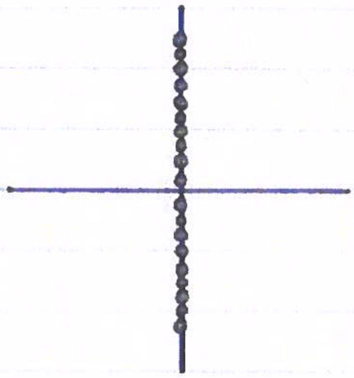
$$\lambda(D_{4c}) = \lambda(D_{2c} \cdot D_{2c}) = [\lambda(D_{2c})]^2$$

and

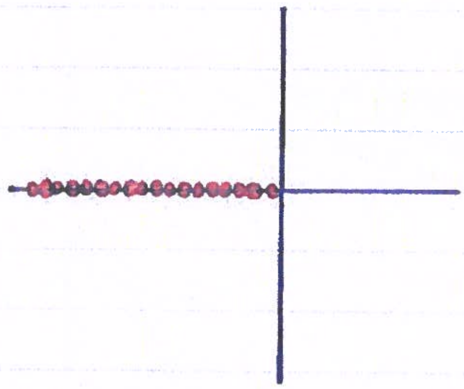
$$\lambda\, D_{6c} = \lambda(D_{4c} \cdot D_{2c}) = [\lambda(D_{2c})]^3$$

(b)- Eigenvalues in the Complex plane :

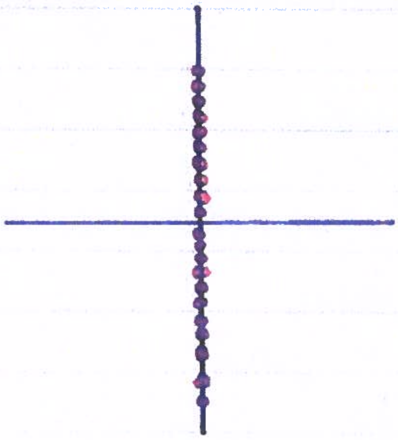
- for D_{1c} :



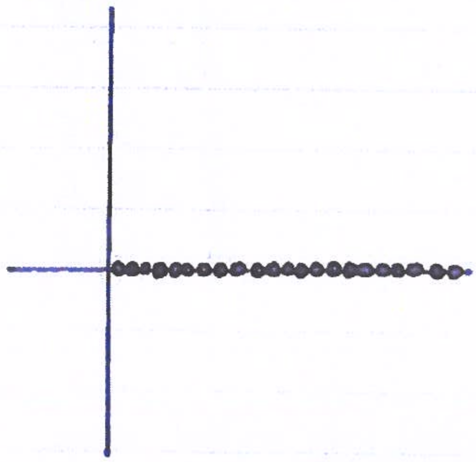
- for D_{2c} :



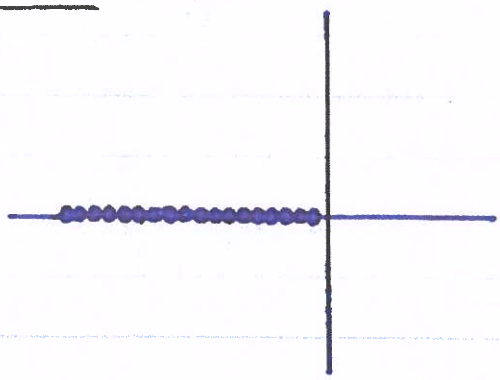
- for D_{3c} :



- for D_{4c} :

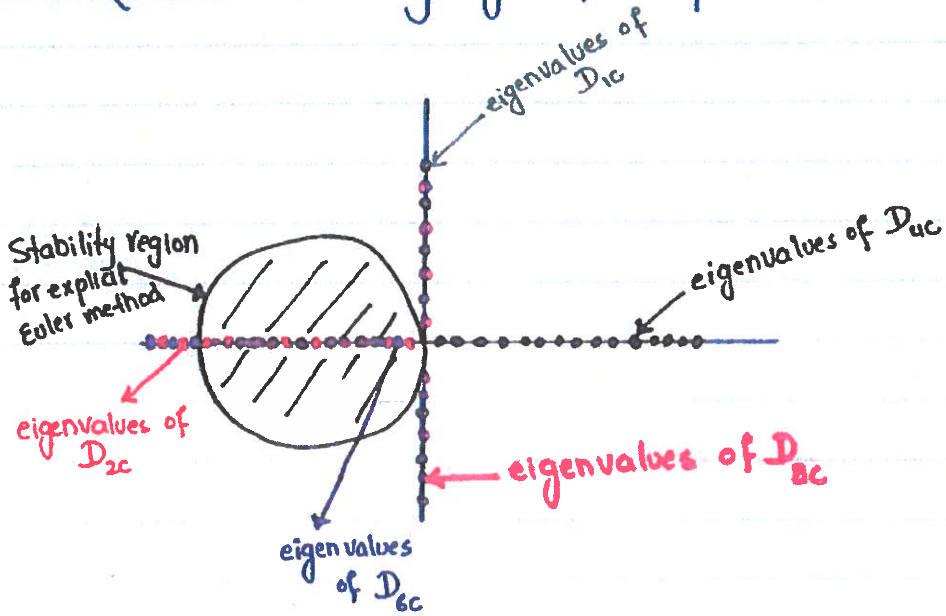


- for D_{6c} :



(C)- Comment:

As we know the stability region for explicit Euler method;



So the explicit Euler method when applied to semi-discrete ODE systems:

$\dot{\vec{u}} = D_{1c}\vec{u}$; unstable

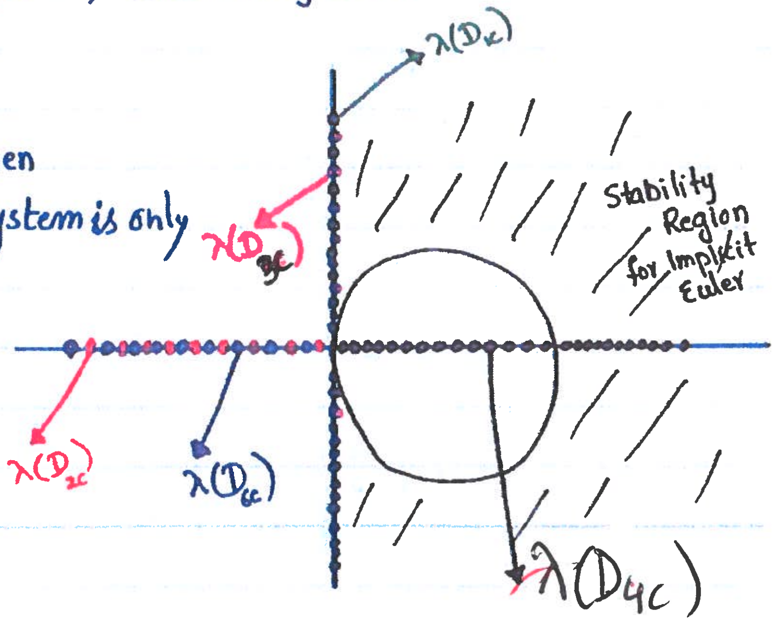
$\dot{\vec{u}} = D_{2c}\vec{u}$; Conditionally stable

$\dot{\vec{u}} = D_{3c}\vec{u}$; unstable

$\dot{\vec{u}} = D_{4c}\vec{u}$; unstable

$\dot{\vec{u}} = D_{6c}\vec{u}$; Conditionally stable

So the Implicit Euler method when applied to Semi-discrete ODE system is only unstable for the D_{4c} case.



Exercise 3.6:

(a)- $u_t = d u_{xx} + (u^2)_x - \mu u_{xxt}$

Discretized form of the given PDE for $u_i(t)$

$$\dot{u}_i = d\,\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \frac{u_{i+1}^2 - u_{i-1}^2}{2\Delta x} - \mu\,\frac{\dot{u}_{i+1} - 2\dot{u}_i + \dot{u}_{i-1}}{(\Delta x)^2}$$

System of (M-1) coupled ODEs:

$$M\,\dot{\vec{u}}(t) = A\,\vec{u}(t) + \vec{F}\,\vec{u}(t)$$

where

$$M = \begin{pmatrix} \left(1 - \frac{2\mu}{(\Delta x)^2}\right) & \mu/(\Delta x)^2 & & \Theta \\ \frac{\mu}{(\Delta x)^2} & \left(1 - \frac{2\mu}{(\Delta x)^2}\right) & \frac{\mu}{(\Delta x)^2} & \\ & \ddots & \ddots & \ddots \\ \Theta & & \ddots & \ddots \end{pmatrix}$$

$$A = \frac{d}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \Theta \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & \ddots & \ddots & \ddots \\ \Theta & & \ddots & \ddots \end{pmatrix}, \quad \vec{F} = \frac{1}{2\Delta x}\begin{pmatrix} \vdots \\ u_{i+1}^2 - u_{i-1}^2 \\ \vdots \end{pmatrix}$$

(b)- $U_t = U_{xxx} + 6UU_x$

$$\dot{u}_i = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2(\Delta x)^3} + 6u_i\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right)$$

System of ODEs;

$$\dot{\vec{u}}(t) = A\vec{u}(t) + F\vec{u}(t)$$

where

$$A = \begin{pmatrix} 0 & -2 & 1 & & & & \\ 2 & 0 & -2 & 1 & & \Theta & \\ -1 & 2 & 0 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \Theta & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \frac{1}{2(\Delta x)^3}$$

D_{3c}

and

$$\vec{F} = \frac{3}{\Delta x}\begin{pmatrix} \vdots \\ u_i u_{i+1} - u_i u_{i-1} \\ \vdots \end{pmatrix}$$

(C) -

$$U_{tt} = U_{xxx}$$

Two PDE's are;

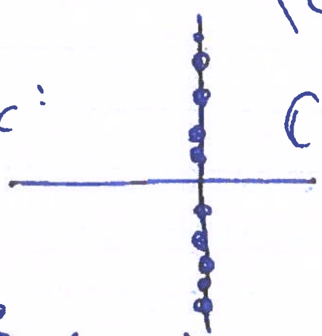
$$\begin{cases} U_t = V \\ V_t = U_{xxx} \end{cases} \Rightarrow \quad \dot{\vec{u}} = \vec{V}, \qquad \dot{V}_i = \frac{U_{i+2} - 2U_{i+1} + 2U_{i-1} - U_{i-2}}{2(\Delta x)^3}$$

$$\dot{\vec{V}}(t) = A\vec{U}(t)$$

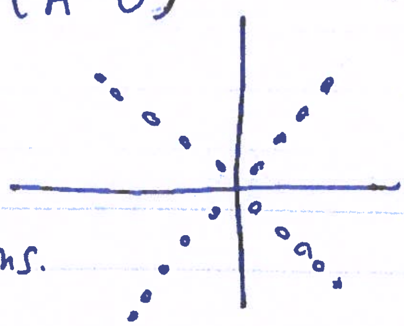
where

$$A = \frac{1}{2(\Delta x)^3}\begin{pmatrix} 0 & -2 & 1 & & & \Theta \\ 2 & 0 & -2 & 1 & & \\ -1 & 2 & 0 & -2 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ \Theta & & & & & \end{pmatrix}$$

eigenvalues of $A = D_{3C}$:



eigenvalues of M in $\dot{\vec{\eta}} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\vec{\eta}$ with $\vec{\eta} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$:



Explicit and Implicit Euler: eigenvalues of M are outside both stability regions.

(this holds, in fact, for all RK and LMM, explicit/implicit!!)