Exercise 4.1 :

As we have the heat equation

$$u_t = u_{xx} \longrightarrow (1)$$

We write the equation at the point $(x_i, t^{n+1/2})$, then the centered difference approximation for u_t can be written as:

$$u_t\Big|_i^{n+1/2} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Now we approximate the term u_{xx} at $(x_i, t^{n+1/2})$. We use the average of the centered differences for $u_{xx}\Big|_i^{n+1/2}$ i.e

$$u_{xx}\Big|_i^{n+1/2} \approx \frac{1}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

So (1) becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2(\Delta x)^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]. \quad \begin{array}{l} n = 0, 1, \ldots \\ i = 1, 2, \ldots, N-1 \end{array}$$

Exercise 4.2:

As we know the CTCS ("Leapfrog") scheme for the heat equation is;

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{1}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Stability analysis shows that this second order accurate for time derivative Scheme is unconditionally unstable. A small modification of this scheme where the term $2u_i^n$ is split into two time levels according to

$$2u_i^n = u_i^{n+1} + u_i^{n-1}$$

leads to the DuFort-Frankel scheme:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{1}{(\Delta x)^2}\left(u_{i+1}^n + u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right) \longrightarrow (*)$$

for error term: expanding the terms in the corresponding Taylor series i.e

$$u_i^{n\pm1} = u_i^n \pm \Delta t\, u_t\Big|_i^n + \frac{(\Delta t)^2}{2} u_{tt}\Big|_i^n \pm \frac{(\Delta t)^3}{3!} u_{ttt}\Big|_i^n + H.O.T$$

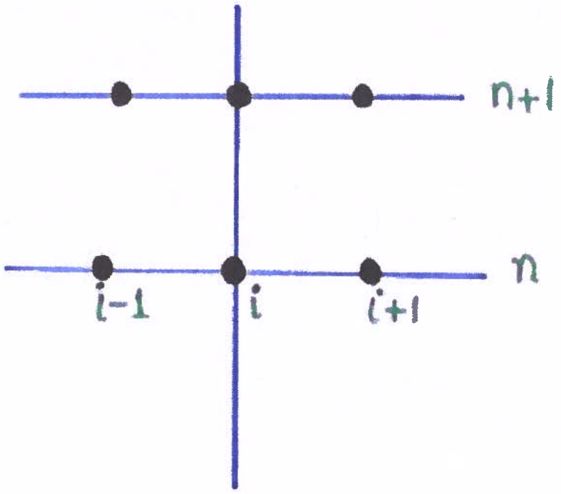
$$u_{i\pm1}^n = u_i^n \pm \Delta x\, u_x\Big|_i^n + \frac{(\Delta x)^2}{2} u_{xx}\Big|_i^n \pm \frac{(\Delta x)^3}{3!} u_{xxx}\Big|_i^n + H.O.T$$

Subsituiting in equation (*), we have

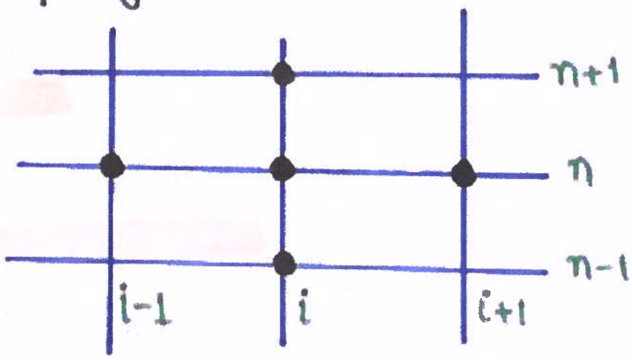
$$u_t\Big|_i^n - u_{xx}\Big|_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 u_{tt}\Big|_i^n + O(\Delta t^2, \Delta x^2) = 0$$

Exercise 4.3 :

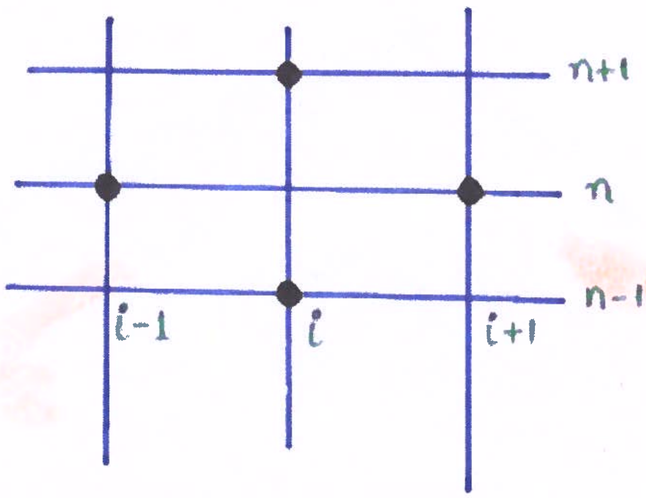
- Stencil for the Crank - Nicolson method :



- Stencil for Leapfrog Scheme :



- Stencil for the DuFort Frankel Scheme :



Exercise 4.4:

$\boxed{*}$

Du Fort Frankel for heat: $\dfrac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \dfrac{u_{i+1}^n - u_i^n - u_i^{n-1} + u_{i-1}^n}{(\Delta x)^2}$

Taylor: $u_i^{n\pm1} = u_i^n \pm \Delta t\, u_t\big|_i^n + \dfrac{(\Delta t)^2}{2} u_{tt}\big|_i^n \pm \dfrac{(\Delta t)^3}{6} u_{ttt}\big|_i^n + \dots$

$u_{i\pm1}^n = u_i^n \pm \Delta x\, u_x\big|_i^n + \dfrac{(\Delta x)^2}{2} u_{xx}\big|_i^n \pm \dfrac{(\Delta x)^3}{6} u_{xxx}\big|_i^n + \dots$

substituting $\Longrightarrow$ in $\boxed{*}$

$$\frac{2\Delta t\, u_t + O((\Delta t)^3)}{2\Delta t} = \Big\{ (\Delta x)^2 u_{xx}\big|_i^n - (\Delta t)^2 u_{tt}\big|_i^n + \frac{1}{12}(\Delta x)^4 u_{xxxx}\big|_i^n - \frac{1}{12}(\Delta t)^4 u_{tttt}\big|_i^n + 2\cdot\frac{1}{6!}(\Delta x)^6 u_{xxxxxx}\big|_i^n + O((\Delta t)^6) + O((\Delta x)^8) \Big/ (\Delta x)^2$$

$u_t + O((\Delta t)^2)$

using $u_t = u_{xx} \Rightarrow u_{tt} = u_{xxxx}$

and $\Delta t = \dfrac{(\Delta x)^2}{\sqrt{12}} \Rightarrow \begin{cases} (\Delta t)^2 = \dfrac{(\Delta x)^4}{12} \\ (\Delta t)^4 = O((\Delta x)^8) \end{cases}$

$= u_{xx} + O((\Delta x)^6) \left(+ O(\Delta t^{\text{"higher"}}) \right)$

two terms cancel; see

Exercise 4.5 :

The Crank-Nicolson method for the heat equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{2(\Delta x)^2}$$

Taylor expanding the solution around $(x_i, t^{n+1/2})$, we obtain

$$u_i^{n+1} = u_i^{n+1/2} \pm \frac{(\Delta t)}{2} u_t\Big|_i^{n+1/2} + \frac{(\Delta t/2)^2}{2!} u_{tt}\Big|_i^{n+1/2} \pm \frac{(\Delta t/2)^3}{3!} u_{ttt}\Big|_i^{n+1/2} + H.O.T$$

$$u_{i\pm 1}^{n+1} = u_i^{n+1/2} \pm (\Delta x)\, u_x\Big|_i^{n+1/2} + \frac{(\Delta x)^2}{2!} u_{xx}\Big|_i^{n+1/2} \pm \frac{(\Delta x)^3}{3!} u_{xxx}\Big|_i^{n+1/2} + H.O.T$$

By subsituting the above expansions into the difference equation, we have

$$\frac{\Delta t\, u_t\big|_i^{n+1/2} + 2\frac{(\Delta t/2)^3}{3!} u_{ttt}\big|_i^{n+1/2} + O(\Delta t)^5}{\Delta t} = \frac{1}{2(\Delta x)^2}\Bigg[(\Delta x)^2\left(u_{xx}\Big|_i^{n+1/2} + \frac{\Delta t}{2} u_{xxt}\Big|_i^{n+1/2} + \frac{(\Delta t)^2}{2!} u_{xxtt}\Big|_i^{n+1/2} + O(\Delta t)^3\right)$$

$$+ 2\frac{(\Delta x)^4}{4!}\left(u_{xxxx}\Big|_i^{n+1/2} + \frac{\Delta t}{2} u_{xxxxt}\Big|_i^{n+1/2} + O(\Delta t)^2\right)$$

$$+ (\Delta x)^2\left(u_{xx}\Big|_i^{n+1/2} - \frac{\Delta t}{2} u_{xxt}\Big|_i^{n+1/2} + \frac{(\Delta t)^2}{2!} u_{xxtt}\Big|_i^{n+1/2} + O(\Delta t)^3\right)$$

$$+ 2\frac{(\Delta x)^4}{4}\left(u_{xxxx}\Big|_i^{n+1/2} + \frac{\Delta t}{2} u_{xxxxt}\Big|_i^{n+1/2} + O(\Delta t)^2\right)$$

$$+ O(\Delta x)^6\Bigg].$$

As

$$u_t\Big|_i^{n+1/2} = u_{xx}\Big|_i^{n+1/2}$$

The dominant term in τ is:

$$\tau = -\frac{(\Delta t)^2}{24} u_{ttt}\Big|_i^{n+1/2} + \frac{(\Delta x)^2}{12} u_{xxxx}\Big|_i^{n+1/2} + O(\Delta t)^3 + O(\Delta x)^4$$

Exercise 4.6

index

$$\vec{u}^{\,n+1} = B\vec{u}^{\,n} + \vec{b}^{\,n}_{n}$$

$$\vec{u}_*^{\,n+1} = B\vec{u}_*^{\,n} + \vec{b}^{\,n} + \vec{\tau}^{\,n}\cdot\Delta t$$

$$\vec{E}^{\,n} \overset{d}{=} \vec{u}_*^{\,n} - \vec{u}^{\,n} \quad : \quad \vec{E}^{\,n+1} = B\vec{E}^{\,n} - \Delta t\,\vec{\tau}^{\,n}$$

index

$$= B(B\vec{E}^{\,n-1} - \Delta t\,\vec{\tau}^{\,n-1}) - \Delta t\,\vec{\tau}^{\,n}$$

$$= B(B(B\vec{E}^{\,n-2} - \Delta t\,\vec{\tau}^{\,n-2}) - \Delta t\,\vec{\tau}^{\,n-1}) - \Delta t\,\vec{\tau}^{\,n}$$

$$= \ldots$$

power

"power"

$$= B^{n+1}\vec{E}^{\,0} - \Delta t \sum_{m=1}^{n+1} B^{n-m}\,\vec{\tau}^{\,m-1}$$

index

$$\Rightarrow \|\vec{E}^{\,n+1}\| \le \|B^{n+1}\|\cdot\|\vec{E}^{\,0}\| + \Delta t \sum_{m=1}^{n+1} \|B^{n-m}\|\cdot\|\vec{\tau}^{\,m-1}\|$$

method converges if
- it is consistent: $\vec{\tau}^{\,m-1} \to 0$ in each step
- it is stable: $\|B^{n+1}\|$ is bounded for all Δt and n with $\underbrace{n\Delta t}_{=t^n} \le T_{end}$

Exercise 4.7 : (using eigenvalues)

Crank-Nicolson Scheme reads for heat equation,

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{2}\left[\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}+u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right]+f_i \qquad \begin{array}{l} i=1,2,\dots,N-1 \\ n=0,1,2,\dots \end{array}$$

Let $r=\frac{\Delta t}{(\Delta x)^2}$, the above equation can be written as

$$-\frac{r}{2}u_{i+1}^{n+1}+(1+r)u_i^{n+1}-\frac{r}{2}u_{i-1}^{n+1}=\frac{r}{2}u_{i+1}^n+(1-r)u_i^n+\frac{r}{2}u_{i-1}^n+\Delta t f_i$$

The matrix form of the above equation is;

$$Au^{n+1}=Bu^n+\Delta t F$$

where

$u^n=(u_1^n,u_2^n,\dots,u_{N-1}^n)$ and A, B are matrices in $\mathbb{R}^{N-1\times N-1}$:

$$A=(1+r)I-\frac{r}{2}S$$

$$B=(1-r)I+\frac{r}{2}S$$

Since A is reversible, we have

$$u^{n+1}=A^{-1}Bu^n+A^{-1}\Delta t F$$

$$\Rightarrow \quad u^{n+1}=Cu^n+A^{-1}\Delta t F; \quad (\because C=A^{-1}B)$$

The eigenvalues for 'C' are

$$\lambda_i=\frac{1-2r\sin^2\left(\frac{i\pi\Delta x}{2}\right)}{1+2r\sin^2\left(\frac{i\pi\Delta x}{2}\right)}, \quad i=1,2,\dots,N-1$$

The denominator of λ_i is always positive, while numerator may be positive or negative depending on 'r' and 'i'. In any case $|\lambda_i|\leq 1$. So 'C-N' scheme is unconditionally stable.

here: $k=1$

4.7 (using "Von Neumann")

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{k}{2}\,\frac{\left\{u_{j-1}^n - 2u_j^n + u_{j+1}^n + u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right\}}{(\Delta x)^2}$$

$$LTE = O((\Delta t)^2) + O((\Delta x)^2)$$

stable?

insert $u^n = e^{at} \cdot e^{i\xi_m x}$ and check $|g|$

$$\Rightarrow e^{a\Delta t} = 1 + \frac{k\,\Delta t}{(\Delta x)^2}\left[e^{-i\xi_m \Delta x} - 2 + e^{+i\xi_m \Delta x}\right] * \left[1 + e^{a\Delta t}\right]$$

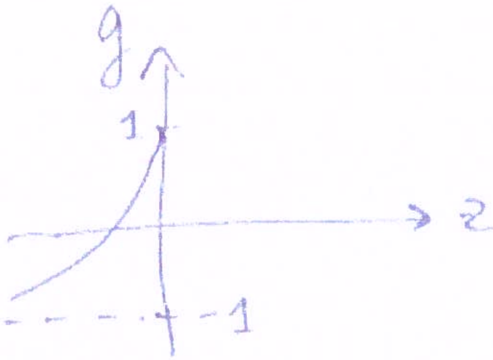
$$\Rightarrow g = e^{a\Delta t} = \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$$

where $z = \frac{k\,\Delta t}{(\Delta x)^2} \cdot \left\{e^{-i\xi_m \Delta x} - 2 + e^{+i\xi_m \Delta x}\right\}$

$= \frac{k\,\Delta t}{(\Delta x)^2} \cdot \left\{\cos(\xi_m \Delta x) - 1\right\}$

≤ 0

$\Rightarrow |g(z)| \leq 1$



CN is unconditionally stable

Exercise 4.8 :

Using finite difference approximation, the update equation writes as

$$u_j^{n+1} = \frac{1-\alpha}{1+\alpha}\, u_j^{n-1} + \frac{\alpha}{1+\alpha}\,(u_{j-1}^n + u_{j+1}^n), \qquad (*)$$

with $\alpha = 2b\frac{\Delta t}{\Delta x^2}$. (b = 1 here)

Introducing any error mode

$$\begin{aligned}\epsilon(x,t) &= \hat{\epsilon}(k,t)\ \exp(ikx),\\ &= \hat{\epsilon}_n\ [\exp(ik\Delta x)]^j,\end{aligned}$$

and replace this expression in Eq (*)
We obtain the recurrence relation

$$(1+\alpha)\ \hat{\epsilon}_{n+1} + (\alpha-1)\ \hat{\epsilon}_{n-1} - 2\alpha\cos(k\Delta x)\ \hat{\epsilon}_n = 0. \qquad (**)$$

The characteristic polynomial of this recurrence relation

$$(1+\alpha)\ \xi^2 + (\alpha-1) - 2\alpha\cos(k\Delta x)\ \xi = 0,$$

admits two roots

$$\xi_{\frac{1}{2}} = \frac{\alpha\cos(k\Delta x) \pm \sqrt{1-\alpha^2\sin^2(k\Delta x)}}{1+\alpha}.$$

Provided both roots are distinct (*i.e* $\xi_1 \neq \xi_2$)), the solution of the recurrence relation in Eq.(**) has the following form

$$\hat{\epsilon}_n = A\xi_1^n + B\xi_2^n,$$ ⚠ n is an exponent $\boxed{\bullet}$

where A and B are both constants that depend on the initial error.
Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

$$|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \qquad (***)$$

Two different situations will be considered as the modulus of $\xi_{\frac{1}{2}}$ will depend on the sign of ρ.

Case 1: $\rho < 0$
If ρ is negative, the first term of $\xi_{\frac{1}{2}}$ is real while the second one is imaginary. Then, the solutions ξ_1 and ξ_2 have the same modulus which is equal to

$$\begin{aligned}\left|\xi_{\frac{1}{2}}\right| &= \sqrt{\frac{\alpha^2\left[\cos^2(k\Delta x)+\sin^2(k\Delta x)\right]-1}{(1+\alpha)^2}}\\ &= \sqrt{\frac{\alpha^2-1}{(1+\alpha)^2}}\end{aligned}$$

As $\alpha > 0$, then

$$\left|\xi_{\frac{1}{2}}\right|^2 = \frac{\alpha^2-1}{(1+\alpha)^2} \leq 1$$

In this case, the solution is always stable.

Case 2: $\rho > 0$

If ρ is positive, both terms are real. Then to be stable, if the most restrictive cases are considered, one has to verify:

$$\begin{aligned} &\xi_1 = \frac{\alpha \cos(k\Delta x) + \sqrt{1 - \alpha^2 \sin^2(k\Delta x)}}{1+\alpha} \leq 1, \\ \Rightarrow \quad &\xi_1 \leq \frac{\alpha + 1}{1+\alpha} = 1 \leq 1, \end{aligned}$$

and,

$$\begin{aligned} &\xi_2 = \frac{\alpha \cos(k\Delta x) - \sqrt{1 - \alpha^2 \sin^2(k\Delta x)}}{1+\alpha} \geq -1, \\ \Rightarrow \quad &\xi_2 \geq \frac{-\alpha - 1}{1+\alpha} = -1 \geq -1, \end{aligned}$$

The conditions on ξ_1 and ξ_2 in Eq. are valid and all the modes for which $\rho > 0$ are not divergent.

Case 3: $\rho = 0$

If $\rho = 0$ multiplicity of the root is two, the solution of Eq. is then given by

$$\hat{\epsilon}_n = (An + B)\,(\xi_{1,2})^n.$$

with

$$|\xi_{1,2}| = \left| \frac{\alpha \cos(k\Delta x)}{1+\alpha} \right| < 1$$

which implies that the solution is also stable when $\rho = 0$.

Conclusion In all the cases, the error does not diverge. It means that this method is unconditionally stable.