Numerical Methods for Time-Dependent PDEs

Spring 2024

Solutions for Tutorial 5

Exercise 5.1

Consider the advection equation

$$u_t + cu_x = 0. \tag{1}$$

Show that for the CTCS-method (Leapfrog') the local truncation error is of the form

$$\tau = -1/6\Delta t^2 u_{ttt}|_i^n - \frac{c}{6}\Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{t and } \Delta\text{x}. \tag{2}$$

Solution:

Taylor expanding the solution around (x_i, t^n) we obtain

$$u_i^{n\pm1} = u_i^n \pm \Delta t u_t|_i^n + \frac{\Delta t^2}{2} u_{tt}|_i^n \pm \frac{\Delta t^3}{3!} u_{ttt}|_i^n + \text{H.O.T in } \Delta\text{t}, \tag{3}$$

$$u_{i\pm1}^n = u_i^n \pm \Delta x u_x|_i^n + \frac{\Delta x^2}{2} u_{xx}|_i^n \pm \frac{\Delta x^3}{3!} u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{x}, \tag{4}$$

By substituting the above expansions into the difference equation we have

$$\frac{2\Delta t u_t|_i^n + \frac{\Delta t^3}{3} u_{tt}|_i^n + \text{H.O.T in } \Delta\text{t}}{2\Delta t} + c\frac{2\Delta x u_x|_i^n + \frac{\Delta x^3}{3} u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{x}}{2\Delta x}, \tag{5}$$

Because u satisfies the PDE hence $(u_t + cu_x)|_i^n = 0$, simplifying the above equation gives the local truncation error

$$\tau = -1/6\Delta t^2 u_{ttt}|_i^n - \frac{c}{6}\Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{t and } \Delta\text{x}. \tag{6}$$

Exercise 5.2

Compute the local truncation error for the Lax-Friedrichs method when applied to advection equation.

Solution:

The Lax-Friedrichs for (1) reads

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0, \tag{7}$$

Taylor expanding the solution around (x_i, t^n) we obtain

$$u_i^{n\pm1} = u_i^n \pm \Delta t u_t|_i^n + \frac{\Delta t^2}{2} u_{tt}|_i^n + \text{H.O.T in } \Delta\text{t}, u_{i\pm1}^n = u_i^n \pm \Delta x u_x|_i^n + \frac{\Delta x^2}{2} u_{xx}|_i^n + \text{H.O.T in } \Delta\text{x}, \tag{8}$$

By substituting the above expansions into the difference equation we have

$$\frac{\Delta t u_t|_i^n + \frac{\Delta t^2}{2} u_{tt}|_i^n + \text{H.O.T in } \Delta\text{t} - [u_i^n + \frac{\Delta x^2}{2} u_{xx}|_i^n + \text{H.O.T in } \Delta\text{x}]}{\Delta t} + \frac{\Delta x u_x|_i^n + \frac{\Delta x^3}{3!} u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{x}}{\Delta x} \tag{9}$$

Because u satisfies the PDE hence $(u_t + cu_x)|_i^n = 0$, simplifying the above equation gives the local truncation error

$$\tau = -\Delta t u_{tt}|_i^n + \frac{\Delta x^2}{2\Delta t} u_{xx}|_i^n - \frac{c}{6} \Delta x^2 u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{t and } \Delta\text{x}. \tag{10}$$

Exercise 5.3

Use the Von Neumann stability analysis to discuss the (in)stability of the Leapfrog method for PDE (1)

Solution:

Let $r = \frac{a\Delta t}{\Delta x}$, the Leapfrog scheme can be rewritten as

$$u_j^{n+1} = u_j^{n-1} - r(u_{j+1}^n - u_{i-1}^n), \tag{11}$$

By substituting $u_j^n = g^n \exp(ij\Delta x\alpha)$ in to the above equation, we have

$$g^2 = 1 - rg(\exp(i\Delta x\alpha) - \exp(-i\Delta x\alpha)), \tag{12}$$

which implies that

$$g_\pm = -ir\sin(\Delta x\alpha) \pm \sqrt{1 - r^2\sin(\Delta x\alpha)}. \tag{13}$$

To determine the stability we consider two cases:

- $|r| > 1$: worst case for $\sin(\Delta x\alpha) = 1$, hence
  $|g_\pm| = -i(r \mp \sqrt{r^2 - 1})$, with $|g_-| > 1$. Hence, the scheme is unstable.
- $|r| \leq 1$: $|g_\pm| = (-r\sin(\Delta x\alpha))^2 + (1 - r^2\sin^2(\Delta x\alpha)) = 1, \quad \forall j$, hence the scheme is stable for all $|r| \leq 1$.

Exercise 5.4 Consider the advection equation

$$u_t + cu_x = 0. \tag{14}$$

Find a modified PDE for which the Lax-Wendroff method applied to PDE (14) gives an $\mathcal{O}(\Delta t^3)$ approximation.

Solution: The derivation of a modified PDE is similar to computing the local truncation error. Now we insert $v(x,t)$ into the FD equation to derive a PDE that $v(x,t)$ satisfies exactly, thus

$$v(x,t+\Delta t) = v(x,t) - \frac{c\Delta t}{2\Delta x}(v(x+\Delta x,t) - v(x-\Delta x,t)) + \frac{\Delta t^2}{2\Delta x^2}c^2(v(x+\Delta x,t) - 2v(x,t) + v(x-\Delta x,t)) \tag{15}$$

Expanding the terms in Taylor series about (x,t) and simplifying yields

$$v_t + \frac{\Delta t}{2}v_{tt} + \frac{\Delta t^2}{6}v_{ttt} + \mathcal{O}(\Delta t^3) = -cv_x - \frac{c\Delta x^2}{6}v_{xxx} + \frac{\Delta t c^2}{2}v_{xx} + \mathcal{O}(\Delta t\Delta x^2), \tag{16}$$

Then we have

$$v_{tt} = -cv_{xt} + \mathcal{O}(\Delta t, \Delta x) = c^2 v_{xx} + \mathcal{O}(\Delta t, \Delta x) \tag{17}$$

$$v_{ttt} = -c^3 v_{xxx} + \mathcal{O}(\Delta t, \Delta x) \tag{18}$$

so the leading modified PDE is

$$v_t + cv_x = -\frac{c\Delta x^2}{6}(1 - (\frac{c\Delta t}{\Delta x})^2)v_{xxx} \tag{19}$$

Exercise 5.5

Compute the local truncation error for the Beam-Warming method when applied to advection equation:

Solution:

Taylor expanding the solution around (x_i, t^n) we obtain

$$u_i^{n\pm1} = u_i^n \pm \Delta t u_t|_i^n + \frac{\Delta t^2}{2}u_{tt}|_i^n \pm \frac{\Delta t^3}{3!}u_{ttt}|_i^n + \text{H.O.T in } \Delta\text{t}, \tag{20}$$

$$u_{i\pm1}^n = u_i^n \pm \Delta x u_x|_i^n + \frac{\Delta x^2}{2}u_{xx}|_i^n \pm \frac{\Delta x^3}{3!}u_{xxx}|_i^n + \text{H.O.T in } \Delta\text{x}, \tag{21}$$

By substituting the above expansions into the difference equation we have

$$\Delta t u_t|_i^n + \frac{\Delta t^2}{2}u_{tt}|_i^n + \frac{\Delta t^3}{6}u_{ttt} + \mathrm{O}(\Delta t^4) = -\frac{c\Delta t}{2\Delta x}(2\Delta x u_x|_i^n - \frac{2\Delta x^3}{3}u_{xxx}|_i^n + \mathcal{O}(\Delta x^4)) \tag{22}$$

$$+ \frac{c^2\Delta t^2}{2\Delta x^2}(\Delta x^2 u_{xx} - \Delta x^3 u_{xxx}|_i^n + \frac{7\Delta x^4}{12}u_{xxxx}|_i^n + \mathcal{O}(\Delta x^5)), \tag{23}$$

Because u satisfies the PDE hence $(u_t + cu_x)|_i^n = 0$, and

$$u_{tt} = (-cu_x)_t = c^2 u_{xx}, \tag{24}$$

thus we have $u_{tt}|_i^n = c^2 u_{xx}|_i^n$, simplifying the above equation gives the local truncation error

$$\tau = \frac{c\Delta x^2}{3} u_{xxx}|_i^n - \frac{c^2 \Delta t \Delta x}{2} u_{xxx}|_i^n - \frac{\Delta t^2}{6} u_{ttt} + \text{H.O.T in } \Delta\text{t and } \Delta\text{x}. \quad (25)$$

Exercise 5.6

Show that the Beam-Warming method is stable for $0 \leq c\frac{\Delta t}{\Delta x} \leq 2$, if we assume that $c > 0$.

Solution:

Let $\lambda = \frac{c\Delta t}{\Delta x}$, by substituting $u_j^n = g^n \exp(ij\xi)$ into the FD equation, we have obtained an isolate expression for the amplification factor g:

$$g = 1 - \frac{\lambda}{2}(3 - 4\exp(-i\xi) + \exp(-2i\xi)) + \frac{\lambda^2}{2}(1 - 2\exp(-i\xi) + \exp(-2i\xi)), \quad (26)$$

since

$$\exp(i\xi) = \cos(\xi) + i\sin(\xi), \quad (27)$$

$$\cos(2\xi) = 2\cos^2(\xi) - 1, \qquad \sin(2\xi) = 2\sin(\xi)cos(\xi), \quad (28)$$

we obtain

$$g = i\lambda^2 \sin(\xi)\cos(\xi) - i\lambda^2 \sin(\xi) + \lambda^2 \cos^2(\xi) + i\lambda \sin(\xi)\cos(\xi) - \lambda^2 \cos(\xi)$$
$$-2i\lambda \sin(\xi) + \lambda \cos^2(\xi) - 2\lambda\cos(\xi) + \lambda + 1,$$

We compute the norm of g by summing the squares of the real and imaginary part. The scheme is stable if $|g|^2 \leq 1, \quad \forall \xi$. Preceding conditions gives the following form

$$\lambda(\lambda - 2)(\lambda + 1)^2(\cos(\xi) - 1)^2 \leq 0, \quad (29)$$

thus it implies the Beam-Warming scheme is stable if

$$\lambda = \frac{c\Delta t}{\Delta x} \leq 2. \quad (30)$$

Exercise 5.7

Show the amplification factor of the upwind method applied to advection equation.

Solution:

By substituting $u_j^n = g^n \exp(ij\Delta x \xi_m)$ in to the above equation, we have

$$g - 1 + \frac{c\Delta t}{\Delta x}(1 - \exp(-i\Delta x \xi_m)) = 0 \quad (31)$$

Thus

$$|g| = |1 - \frac{c\Delta t}{\Delta x} + \frac{c\Delta t}{\Delta x}\exp(-i\xi_m \Delta x) \tag{32}$$

$$= |(1-\lambda) + \lambda\exp(-i\xi_m\Delta x)| \tag{33}$$

$$= |1 - \lambda(1-\cos(\Delta x\xi_m)) - i\lambda\sin(\Delta x\xi_m)| \tag{34}$$

with magnitude

$$|g|^2 = 1 - 2(1-\lambda)\lambda(1-\cos(\Delta x\xi_m)), \tag{35}$$

since $c > 0$, so if $1 - \lambda \geq 0$ we have $|g| \leq 1$, then this method is stable.

Exercise 5.8

Apply FTCS scheme to the linear advection equation. Illustrate this i a figure with stencils in the $x - t$ domain and with characteristics of the PDE.

Solution:

The FTCS scheme for linear advection equation reads

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0, \tag{36}$$

when $|c\frac{\Delta t}{\Delta x}| \leq 1$ the scheme satisfies the CFL condition. However, substituting $u_j^n = g^n\exp(ij\Delta x\alpha)$ in to the scheme, we have

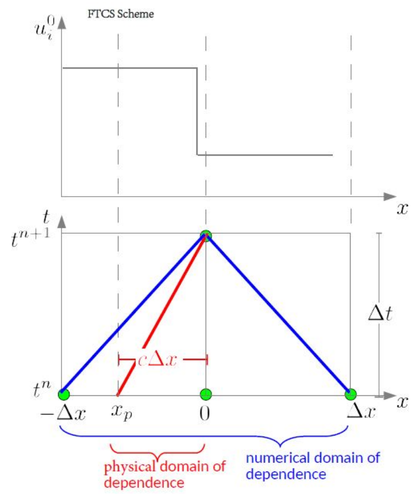
$$g = (1 - i\frac{c\Delta t}{\Delta x}\sin(\Delta x\alpha)) \tag{37}$$

since

$$|g| = \sqrt{1 + (\frac{c\Delta t}{\Delta x})^2\sin^2(\Delta x\alpha)} \geq 1 \tag{38}$$

doesn't satisfy von Neumann condition, thus it is always unstable.

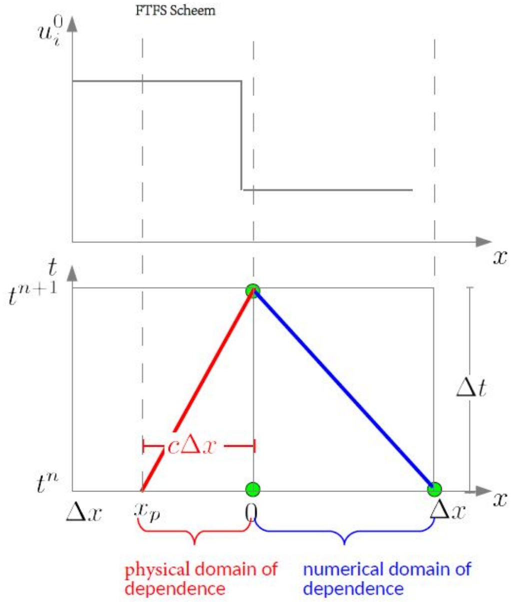
Below we illustrate this in a figure with stencils in the $x-t$ domain and with characteristics of the PDE



Exercise 5.9 Draw the domains of numerical and physical dependence for the FTBS and FTFS scheme applied to the linear advection equation.

Solution:

FTFS Scheem



FTBS Scheme