"EXTRA"

Exercise 6.1:

Consider the *kinematic wave equation*

$$\rho_t + c(\rho)\,\rho_x = 0. \tag{5.2.10}$$

This equation often arises in nonlinear wave phenomena when the effects of dissipation, such as viscosity and diffusion, are neglected. We next investigate the development of shocks from the *initial-value problem* for $u(x,t)$

(a)-

$$u_t + c(u)\,u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{5.2.11}$$

$$u(x,0) = f(x), \quad -\infty < x < \infty. \tag{5.2.12}$$

where $c(u)$ and $f(x)$ are $C^1(R)$ functions of their arguments, that is, they are smooth functions.

The characteristic equations associated with (5.2.11) are

$$\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0}.$$

These equations give

$$\frac{du}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} = c(u)\,. \tag{5.2.13ab}$$

Clearly, the solution of (5.2.13b) represents characteristics of equation (5.2.11). Along these characteristics,

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = c(u)\, u_x + u_t = 0.$$

This means that u is constant on the characteristics which propagate with speed $c(u)$. The dependence of c on u produces a gradual nonlinear distortion of the wave profile as it propagates in the medium. It also follows that $c(u)$ is constant on the characteristics, and therefore, the characteristics must be straight lines in the (x,t)-plane with constant slope $1/c(u)$. Equations of these lines are given by

Here $u_0(\xi) = F(\xi) = c(f(\xi))$

$$x - t\,c(u) = \text{constant} = A, \tag{5.2.14}$$

where A is a constant, that is, $x = x(t) = t\,c(u) + A$.

If any one of these characteristics intersects that x-axis ($t = 0$) at $x(0) = \xi$, then, by the initial condition, $u(\xi, 0) = f(\xi)$. Thus, the equation of a typical characteristic line (see Figure 5.2) joining two points $(\xi, 0)$ and (x, t) is

$$x = \xi + t\,F(\xi)\,, \tag{5.2.15}$$

where $F(\xi) = c(f(\xi))$.

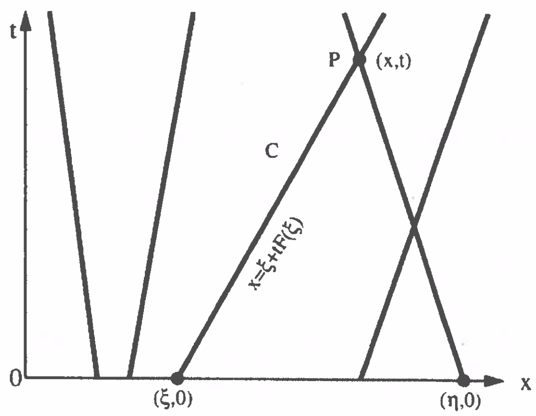


Figure 5.2 Characteristic lines of different slopes.

Since $u(x,t)$ is constant on the characteristics, it follows from (5.2.15) and the initial condition that

$$u(x,t) = u(\xi + t\,F(\xi), t) = u(\xi, 0) = f(\xi).$$

Thus, if a solution of the initial-value problem exists for $t > 0$, then the solution can be written in the parametric form

$$\left.\begin{aligned} u(x,t) &= f(\xi) \\ \xi &= x - t\,F(\xi) \end{aligned}\right\}, \tag{5.2.16ab}$$

where $F(\xi) = c(f(\xi))$.

If there are two points $(\xi, 0)$ and $(\eta, 0)$, with $\xi < \eta$ and

$$m_1 = \frac{1}{F(\xi)} < \frac{1}{F(\eta)} = m_2,$$

then the characteristics starting at $(\xi, 0)$ and $(\eta, 0)$ will intersect at the point $P(x,t)$ for $t > 0$. At the point of intersection $P(x,t)$, the solution $u(x,t)$ has two different values $f(\xi)$ and $f(\eta)$. This means that u is double valued, and hence, the solution i s *not unique* at the point of intersection of the characteristics. Thus, the solution *must* be *discontinuous* at the point of intersection. The conclusion is that if no two characteristic lines intersect in the half plane $t > 0$, there exists a solution of the initial-value problem (5.2.11), (5.2.12) as a differentiable function for all $t > 0$. This can happen only if the reciprocal of the slope is an increasing function of the intercept, that is,

$$F(\xi) \le F(\eta) \quad \text{for} \quad \xi \le \eta. \tag{5.2.17}$$

In other words, the family of characteristics spreads only for $t > 0$ and generates a solution of the problem that is at least as smooth as $f(x)$. Such a solution is called an *expansive* (or *refractive*) *wave.*

We now verify that (5.2.16ab) represents an analytical solution of the problem. Differentiating (5.2.16ab) with respect to x and t, we obtain $u_x = f'(\xi)\,\xi_x$, $u_t = f'(\xi)\,\xi_t$, $1 = \{1 + t\,F'(\xi)\}\,\xi_x$, $0 = F(\xi) + \{1 + t\,F'(\xi)\}\,\xi_t$.

Eliminating ξ_x and ξ_t gives

$$u_x = \frac{f'(\xi)}{1 + t\,F'(\xi)}, \qquad u_t = -\frac{F(\xi)\,f'(\xi)}{1 + t\,F'(\xi)}, \tag{5.2.18ab}$$

Substituting u_x and u_t, equation (5.2.11) is satisfied provided $\{1 + t\,F'(\xi)\} \neq 0$. The solution (5.2.16ab) also satisfies the initial condition at $t = 0$ since $\xi = x$, and hence, it is unique.

Suppose that $u(x,t)$ and $v(x,t)$ are two solutions. Then, on $x = \xi + t\,F(\xi)$,

$$u(x,t) = u(\xi, 0) = f(\xi) = v(x,t).$$

Thus, we proved the following.

Theorem 5.2.1 The nonlinear initial-value problem given by (5.2.11), (5.2.12) has a unique solution provided that $\{1 + t\,F'(\xi)\} \neq 0$ and f and c are $C^1(R)$ functions where $F(\xi) = c(f(\xi))$. The solution is given by the parametric form (5.2.16ab).

Extra

Remark. When $c(u) = \text{constant} = c > 0$, we obtain the linear initial-value problem

$$u_t + c\,u_x = 0, \quad x \in R, \quad t > 0, \tag{5.2.19}$$

$$u(x,0) = f(x), \quad x \in R. \tag{5.2.20}$$

This linear problem has a unique solution given by

$$u(x,t) = f(x - ct). \tag{5.2.21}$$

This solution represents a traveling wave moving with constant velocity c in the positive direction of the x-axis *without* any change of shape.

In this case, the characteristics $x = \xi + c\,t$ form a family of parallel straight lines in the (x,t)-plane as shown in Figure 5.3.

(b)-

Physical Significance of Solution (5.2.16ab).

For the general nonlinear problem, the dependence of the wave speed c on u produces a gradual nonlinear distortion of the wave as it propagates in the medium. This means that some parts of the wave travel faster than others. When $c'(u) > 0$, $c(u)$ is an increasing function of u. In this case, higher values of u propagate faster than lower ones. On the other hand, when $c'(u) < 0$, $c(u)$ is a decreasing function of u and higher values of u travel slower than the lower ones. This means that the wave profile progressively distorts itself, leading to a multiple-valued solution with a vertical slope, and hence, it breaks. In the linear case, c is constant, there is no such distortion of the wave, and hence, it propagates *without* any change of shape. Thus, there is a striking difference between the linear and nonlinear solutions.

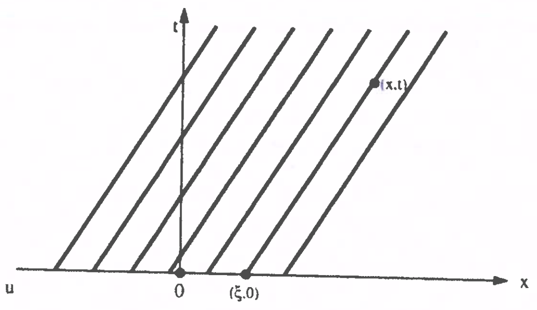


Figure 5.3 Parallel characteristic lines.

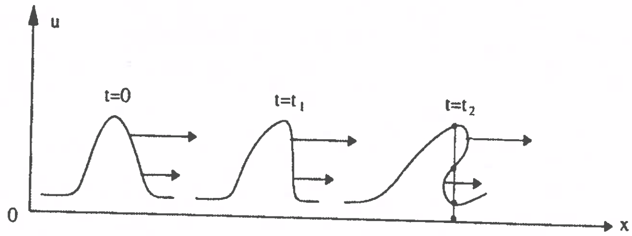


Figure 5.4 Distortion of wave profile with increasing time $t_2 \gg t_1 > 0$.

At any *compressive* part of the wave, the wave speed is a decreasing function of x, as shown in Figure 5.4. The wave profile distorts progressively to produce a triple-valued solution for $u\,(x,t)$, and hence it ultimately breaks.

It follows from Theorem 5.2.1 that the solution of the nonlinear initial-value problem exists provided that $1 + t\,F'\,(\xi) \neq 0$. This condition is always satisfied for a sufficiently small time t. It also follows from (5.2.18ab) that both u_x and u_t tend to infinity as $1 + t\,F'\,(\xi) \to 0$. This means that the solution develops a *discontinuity* (*singularity*) when $1 + t\,F'\,(\xi) = 0$. Thus, on any characteristic for which $F'\,(\xi) < 0$, a discontinuity occurs at time t given by

$$t = -\frac{1}{F'\,(\xi)}, \tag{5.2.22}$$

which is positive because $F'\,(\xi) = c'\,(f)\,f'\,(\xi) < 0$. If we assume $c'\,(f) > 0$, this inequality implies that $f'\,(\xi) < 0$. Hence, the solution (5.2.16ab) ceases to exist for all time if the initial data is such that $f'\,(\xi) < 0$ for some value of ξ. Suppose that $t = \tau$ is the time when the solution first develops a discontinuity (singularity) for some value of ξ. Then,

(C)-

$$\tau = -\frac{1}{\min_{-\infty<\xi<\infty}\{c'\,(f)\,f'\,(\xi)\}} > 0. \tag{5.2.23}$$

Thus, the shape of the initial curve for $u\,(x,t)$ changes continuously with increasing values of t, and the solution becomes multiple valued with a vertical slope for $t \geq \tau$. Therefore, the solution breaks down when $F'\,(\xi) < 0$ for some ξ, and such breaking is a strikingly nonlinear phenomenon. Indeed, Whitham (1974) emphasized that: "This breaking phenomenon is one of the most intriguing long-standing problems of water wave theory." In the linear theory, such breaking will *never* occur.

More precisely, the development of a discontinuity in the solution for $t \geq \tau$ can also be seen in the (x,t)-plane. If $f'\,(\xi) < 0$, then we can find two values of $\xi = \xi_1, \xi_2\,(\xi_1 < \xi_2)$ on the initial line $(t = 0)$, such that the characteristics through them have different slopes $1/c\,(u_1)$ and $1/c\,(u_2)$, where $u_1 = f\,(\xi_1)$, $u_2 = f\,(\xi_2)$, and $c\,(u_2) > c\,(u_1)$. These two characteristics will intersect at a point in the (x,t)-plane for some $t > 0$. Since the characteristics carry constant values of u, the solution ceases to be single valued at the point of intersection. As Figure 5.4 shows, the solution $u(x,t)$ progressively distorts itself, and, at any instant of time, there exists an interval on the x-axis where u becomes triple valued for a given x. The end result is the development of multivalued solutions, and hence, it leads to breaking. This is exactly the situation always observed on beaches when water waves break. Finally, we conclude the above discussion by stating the remarkable fact that both the distortion of the wave profile and the development of a discontinuity or a shock are typical nonlinear phenomena.

Therefore, when $1+t\,F'(\xi)=0$, the solution develops a discontinuity known as a *shock*. The analysis of shock involves an extension of a solution to allow for discontinuities. It is also necessary to impose certain restrictions on the solution to be satisfied across its discontinuity.

(d)-

Example 5.2.1 Solve the nonlinear initial-value problem

$$u_t + u u_x = 0, \quad x \in R, \quad t > 0, \tag{5.2.24}$$

Here we take:
$a = 1$

$$u(x,0) = \begin{Bmatrix} (a^2 - x^2), & |x| \le a \\ 0, & |x| \ge a \end{Bmatrix}. \tag{5.2.25}$$

and in (5.2.24):
$c(u) = u$

In this case, $c(u) = u$, and the solution follows from (5.2.16ab) as

$$u(x,t) = \begin{Bmatrix} (a^2 - \xi^2), & |x| \le a \\ 0, & |x| \ge a \end{Bmatrix},$$

where

$$\xi = x - t\left(a^2 - \xi^2\right).$$

This is a quadratic equation in ξ giving

$$\xi = \frac{1}{2t}\left[1 \pm \left\{1 - 4t\left(x - ta^2\right)\right\}^{\frac{1}{2}}\right], \quad t \neq 0.$$

The solution of (5.2.24) becomes

$$\begin{aligned} u(x,t) &= \left(a^2 - \xi^2\right) \quad \text{for } |\xi| \le a, \\ &= \frac{1}{2t^2}\left[2xt - 1 \pm \left\{1 - 4xt + 4a^2t^2\right\}^{\frac{1}{2}}\right], \quad t \neq 0 \end{aligned} \tag{5.2.26}$$

and

$$u(x,t) = 0 \quad \text{for } |\xi| \ge a.$$

For small values of $t\,(t \to 0)$, only the positive sign before the radical in (5.2.26) is acceptable so that the initial condition is satisfied. On the other hand, when $t > T$, both signs are admissible for $x > a$.

We next draw the characteristics in the (x,t)-plane for $a = 1$ in Figure 5.5.

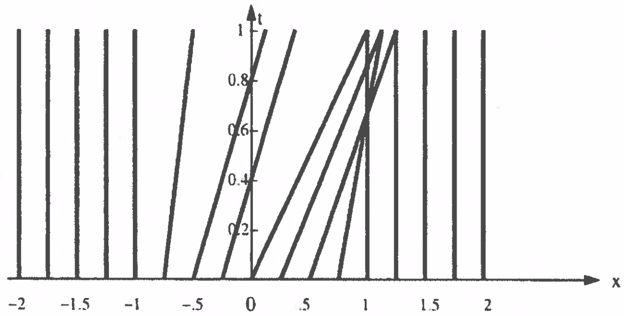


Figure 5.5 Characteristics of (5.2.24) with the condition for $a = 1$.

As stated before, characteristics are straight lines with speed u. Several characteristic lines intersect. At the point where two characteristics intersect, the solution becomes double valued and the slope in the (x, t)-plane becomes infinite. From this point onward, the solution is a discontinuous function of position, and corresponds to the onset of a shock wave. Moreover, it follows that all characteristics, originating from $(x, 0)$ where $x > -1$, intersect the characteristics starting from the point x, where $x \geq 1$, at some point or other. In particular, the two characteristics initially at $x = 0$ and $x = 1$ intersect at the point $(x, t) = (1, 1)$. The solution would be double valued at $(1, 1)$. Figure 5.6 represents the propagation of the initial parabolic pulse with $a = 1$.

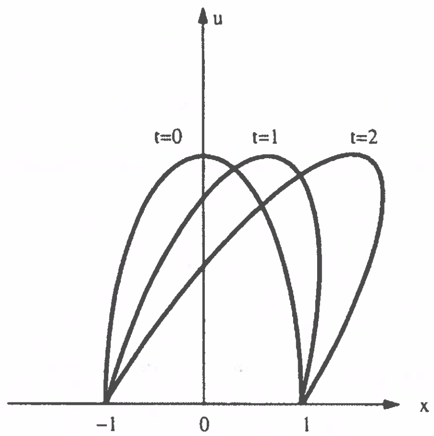


Figure 5.6 Propagation of a parabolic pulse for $a = 1$.

As t increases, the initial pulse distorts progressively. This progressive change in the initial wave pulse is the result of the nonlinear term in the equation. In the linear case ($u_t + c\,u_x = 0, \quad c = \text{constant}$) with the same initial data (5.2.25) for $a = 1$, the initial parabolic pulse propagates with constant velocity c in the positive direction of the x-axis *without change of shape.*

Exercise 6.2

Show that for the nonlinear hyperbolic PDE

$$u_t + [F(u)]_x = 0, \qquad \boxed{*}$$

the following property holds:

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx, \forall t \geq 0.$$

Answer:
Integrate the Eq. $\boxed{*}$ over $[-\infty, \infty] \times [0, t]$, we obtain

$$\int_0^t \int_{-\infty}^{\infty} (u_t + [F(u)]_x)dxdt = \int_{-\infty}^{\infty} [u(x,t) - u(x,0)]dx + \int_0^t [F(u(\infty,t)) - F(u(-\infty,t))]dt = 0$$

Since

$$\lim_{x \to \pm\infty} F(u(x,t)) = 0, \forall t \geq 0$$

thus we have

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx, \forall t \geq 0.$$

Since for all I and $J(> I)$ the flux-differencing form gives

$$u_I^{n+1} = u_I^n - \frac{\Delta t}{\Delta x}(F_{I+1}^n - F_I^n),$$
$$u_{I+1}^{n+1} = u_{I+1}^n - \frac{\Delta t}{\Delta x}(F_{I+2}^n - F_{I+1}^n),$$
$$\vdots$$
$$u_J^{n+1} = u_J^n - \frac{\Delta t}{\Delta x}(F_{J+1}^n - F_J^n),$$

Summing the above equations gives

$$\Delta x \sum_{i=I}^{J} u_i^{n+1} = \Delta x \sum_{i=I}^{J} u_i^n - \Delta t(F_{J+1}^n - F_I^n)$$

Exercise 6.3

It can be easily checked (Taylor expansions ...)

that $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$ (**)

"$= u_i^n$" (in Taylor)

is a consistent discretization

of the PDE $u_t = -u u_x$ (*)

But $u_t + \left(\frac{u^2}{2}\right)_x = u_t + u u_x = 0$

and $(u^2)_t + \left(\frac{2u^3}{3}\right)_x = 2 \cdot u u_t + \frac{6}{3} u^2 u_x$

$= 2u \underbrace{(u_t + u u_x)}_{=0} = 0$

are "equivalent" exact

formulations of PDE (*)

(**) $\Rightarrow$ also consistent

Exercise 6.4

Wave equation in 1D

for $c = 1$

$$\begin{cases} u_{tt} = (u_t)_t = (u_x)_t = (u_t)_x = (u_x)_x = u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_x(x,0) = u_0'(x) \end{cases}$$

exact solution using d'Alembert formulation

$\Longrightarrow$

$$u(x,t) = \frac{1}{2}\left(u_0(x-t) + u_0(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_0'(s)\,ds$$

$$= \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t) + \frac{1}{2}\left[u_0(s)\right]_{s=x-t}^{s=x+t}$$

$$= \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t) + \frac{1}{2}u_0(x+t) - \frac{1}{2}u_0(x-t)$$

$= u_0(x+t) \equiv$ the solution of the "original" advection equation (before the doubling)

⊛ EXTRA OBSERVATION:

M.o.L.

$$u_{tt} = c^2 u_{xx}$$

$$\begin{cases} u_t = v \\ v_t = c^2 u_{xx} \end{cases} \qquad \text{or} \qquad \begin{cases} u_t = c v_x \\ v_t = c u_x \end{cases}$$

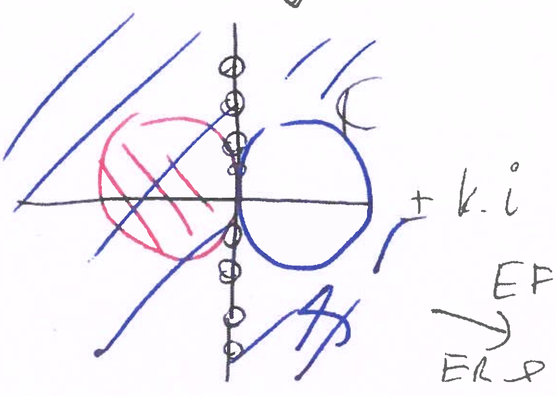
$v_t = u_{tt}$

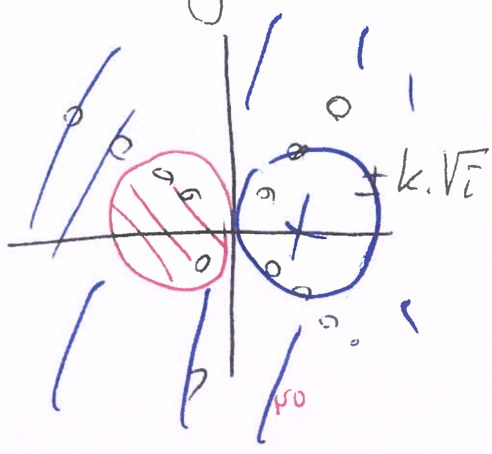
check: $u_{tt} = c v_{xt} = c v_{tx} = c (v_t)_x = c^2 (u_x)_x = c^2 u_{xx}$ ✓

$$\begin{cases} \dot{\vec{u}} = \vec{v} \\ \dot{\vec{v}} = \underbrace{\frac{c^2}{\Delta x}\begin{pmatrix} 0 & 1 & -2 & 1 & 0 \end{pmatrix}}_{c^2 D_{2c}} \vec{u} \end{cases} \qquad \begin{cases} \dot{\vec{u}} = c\, D_{1c} \vec{v} \\ \dot{\vec{v}} = c\, D_{1c} \vec{u} \end{cases}$$

$$\dot{\vec{\eta}} = \begin{pmatrix} 0 & I \\ c^2 D_{2c} & 0 \end{pmatrix} \vec{\eta} \qquad \dot{\vec{\eta}} = \begin{pmatrix} 0 & c D_{1c} \\ c D_{1c} & 0 \end{pmatrix} \vec{\eta}$$

eigenvalues ? eigenvalues





check notation and indices

$k \to \Delta t$
$h \to \Delta x$
$j \to n$
$u_{i,j+1} \to u_i^{n+1}$
etcetera

Exercise 6.5

Two Exact Rows Given

The accuracy of the numerical approximations produced by the equations in (7) depends on the truncation errors in the formulas used to convert the partial differential equation into a difference equation. Although it is unlikely to know values of the exact solution for the second row of the grid, if such knowledge were available, using the increment $k = ch$ along the t-axis will generate an exact solution at all the other points throughout the grid.

u_i^0 $\quad$ u_i^1

Assume that the two rows of values $u_{i,1} = u(x_i, 0)$ and $u_{i,2} = u(x_i, k)$, for $i = 1, 2, \ldots, n$, are the exact solutions to the wave equation (1). If the step size $k = h/c$ is chosen along the t-axis, then $r = 1$ and formula (7) becomes

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}. \tag{17}$$

Furthermore, the finite-difference solutions produced by (17) throughout the grid are exact solution values to the differential equation (neglecting computer round-off error).

Proof. Use d'Alembert's solution and the relation $ck = h$. The calculation $x_i - ct_j = (i-1)h - c(j-1)k = (i-1)h - (j-1)h = (i-j)h$ and a similar one producing $x_i + ct_j = (i+j-2)h$ are used in equation (14) to produce the following special form of $u_{i,j}$:

$$u_{i,j} = F((i-j)h) + G((i+j-2)h) \tag{18}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Applying this formula to the term

De Alembert princ:
$U(x,t) = F(x+ct) + G(x-ct)$

$u_{i+1,j}$, $u_{i-1,j}$, and $u_{i,j-1}$ on the right side of (17) yields

$$
\begin{aligned}
u_{i+1,j} + u_{i-1,j} - u_{i,j-1}
&= F((i+1-j)h) + F((i-1-j)h) \\
&\quad - F((i-(j-1))h) + G((i+1+j-2)h) \\
&\quad + G((i-1+j-2)h) - G((i+j-1-2)h) \\
&= F((i-(j+1))h) + G((i+j+1-2)h) = u_{i,j+1},
\end{aligned}
$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Warning. Theorem . does not guarantee that the numerical solutions are exact when numerical calculations based on (9) and (13) are used to construct approximations $u_{i,2}$ in the second row. Indeed, truncation error will be introduced if $u_{i,2} \neq u(x_i, k)$ for some i, where $1 \le i \le n$. This is why we endeavor to obtain the best possible values for the second row by using the second-order Taylor approximations in equation (13).

see lecture notes

Exercise 6.6

Work out the Von Neumann stability analysis for the wave equation with the CTCS scheme.

Answer:

The wave equation reads

$$u_{tt} - c^2 u_{xx} = 0$$

The CTCS scheme gives

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0.$$

By substituting $u_j^n = g^n \exp(ij\Delta x\xi)$ into the FD equation,and letting $p = \frac{c^2 \Delta t^2}{\Delta x^2}$ we have

$$\begin{aligned} g^2 &= 2g - 1 + p^2 g(\exp(i\Delta x\xi) - 2 + \exp(-i\Delta x\xi)) \\ &= 2g[1 - 2p^2 \sin^2(\Delta\xi/2)] - 1 \end{aligned}$$

The roots are

$$g_\pm = 1 - 2p^2 \sin^2(\Delta x\xi/2) \pm \sqrt{4p^2 \sin^2(\Delta x\xi/2)[p^2 \sin^2(\Delta\xi/2) - 1]}.$$

Now we discuss the discriminant $\sqrt{4p^2 \sin^2(\Delta x\xi/2)[p^2 \sin^2(\Delta x\xi/2) - 1]}$

- Case 1 $p^2 < 1$: $|g_\pm|^2 = 1$,
- Case 2 $p^2 = 1$: gives $|g_\pm| = 1$, Hence the scheme is stable for $p^2 \leq 1$.
- Case 3 $p^2 > 1$: Consider the best scenario, i.e. $\Delta x\xi = \pi$,

$$g_\pm = 1 - 2p^2 \pm 2p\sqrt{p^2 - 1}$$

So $g_- < 1 - 2p^2 < -1, \quad \forall p^2 > 1$ thus $|g_-| > 1$ at $\Delta x\xi = \pi$,

Conclusion: The CTCS scheme is von Neumann stable for $p^2 \leq 1$.

Exercise 6.7

$u_{tt} = c^2 u_{xx}$

Higher Order Accurate Schemes for Second-Order Equations

$m \to i$
$k \to \Delta t$
$h \to \Delta x$
$a \to c$
$\lambda = \frac{\Delta t}{\Delta x}$

We now present two (2, 4) accurate schemes for the wave equation . The first is

$$\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} = a^2\left(\frac{-v_{m+2}^n + 16v_{m+1}^n - 30v_m^n + 16v_{m-1}^n - v_{m-2}^n}{12h^2}\right)$$
$$= a^2\left(1 - \frac{h^2}{12}\delta^2\right)\delta^2 v_m^n. \tag{8.2.4}$$

The equation for the amplification factors is

$$g - 2 + g^{-1} = a^2\lambda^2\left(\frac{-2\cos 2\theta + 32\cos\theta - 30}{12}\right)$$
$$= -\frac{4}{3}a^2\lambda^2\sin^2\frac{1}{2}\theta\left(3 + \sin^2\frac{1}{2}\theta\right)$$

or

$$g^{1/2} - g^{-1/2} = \pm 2i\left[\frac{a^2\lambda^2\sin^2\frac{1}{2}\theta\left(3 + \sin^2\frac{1}{2}\theta\right)}{3}\right]^{1/2}.$$

As in the previous analyses, the scheme is stable, i.e., $|g_\pm| \le 1$, if and only if

$$\frac{a^2\lambda^2\sin^2\frac{1}{2}\theta\left(3 + \sin^2\frac{1}{2}\theta\right)}{3} \le 1.$$

Obviously the maximum of the left-hand side of this inequality occurs when $\sin^2\frac{1}{2}\theta$ is 1, and so we obtain the stability condition

$$a\lambda \le \frac{\sqrt{3}}{2} \approx 0.8660$$

for the (2, 4) scheme (8.2.4).

EXTRA INFO

An implicit (2, 4) scheme for the wave equation (~~8.1.1~~) $u_{tt} = c^2 u_{xx}$ is given by

$$\delta_t^2 v_m^n = a^2\left(1 + \frac{h^2}{12}\delta_x^2\right)^{-1}\delta_x^2 v_m^n$$

or

$$v_{m+1}^{n+1} + 10v_m^{n+1} + v_{m-1}^{n+1} - 2\left(v_{m+1}^n + 10v_m^n + v_{m-1}^n\right) + v_{m+1}^{n-1} + 10v_m^{n-1} + v_{m-1}^{n-1}$$
$$= 12a^2\lambda^2\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right).$$

Exercise 6.8

$u_{tt} = -b^2 u_{xxxx}$

For the Euler–Bernoulli equation ( ), the simplest scheme is the second-order accurate scheme

$$\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} = -b^2 \frac{v_{m+2}^n - 4v_{m+1}^n + 6v_m^n - 4v_{m-1}^n + v_{m-2}^n}{h^4} = -b^2 \delta^4 v_m^n. \tag{8.2.3}$$

The equation for the amplification factors is

$$g - 2 + g^{-1} = -16b^2\mu^2 \sin^4 \tfrac{1}{2}\theta,$$

where $\mu = k/h^2$. The stability analysis is almost exactly like that of scheme (8.2.2) for the wave equation, and it is easy to see that scheme (8.2.3) is stable if and only if

$$2b\mu \sin^2 \tfrac{1}{2}\theta \le 1,$$

which requires that

$$b\mu \le \tfrac{1}{2}. \square$$

check notation and indices!

$m \to i$

$k \to \Delta t$

$h \to \Delta x$