Computer exercise C2a[*]

(<u>10 %</u> of the Final Grade)

April 2024

Deadline: see the webpage of the course.
(This is an individual assignment!)

PART A (60 points)

Consider the following <u>infinite</u> order PDE in one space dimension:

$$\frac{\partial u}{\partial t} = \alpha \sum_{k=0}^{\infty} \frac{\partial^k u}{\partial x^k}, \quad u(x,\cdot) \in \mathcal{C}^{\infty}(\mathbb{R}),\ t > 0,\ \alpha \in \mathbb{R}. \tag{1}$$

(a) Show that the solution $u(x,t)$ of PDE (1) must satisfy:

$$u_t = \alpha u + u_{xt}. \tag{2}$$

We choose $x \in [0,1]$, $t \in [0,20]$, the initial condition $u(x,0) = \sin(2\pi x)$ and we prescribe periodic boundary conditions.

(b) Apply the first step in the method-of-lines to PDE (2) and write it as a system of ODEs of the form:

$$\dot{\vec{u}} = \mathcal{M}\vec{u}. \tag{3}$$

Describe the matrix $\mathcal{M}$.

(c) In the second step of the method of lines, make use of the Implicit Euler method for the choice $\alpha = -1$. Plot the numerical solutions for different step sizes Δx and Δt. Check the numerical accuracy[1]. Also, plot the eigenvalues of the matrix $\mathcal{M}$.

(d) Secondly, the same question as in (c) for the Explicit Euler method. How small should you take Δt for numerical stability? (via numerical experiments)

(e) Finally, we treat the case $\alpha = 1$. Why doesn't it make sense to use Explicit, or even Implicit, Euler? Explain.
BONUS (<u>**5 points**</u>): What is the exact solution for $\alpha = 1$?

---

[*] Exercise C2**b**: 5 % of the Final Grade, deals with the topics discussed in the two guest lectures: check the webpage!

[1] The exact solution for $\alpha = -1$ reads: $\mathrm{e}^{-\beta t}[-\sin(2\beta\pi t)\cos(2\pi x) + \cos(2\beta\pi t)\sin(2\pi x)]$ with $\beta = \frac{1}{1+4\pi^2}$.

(f) Apply the BVM (boundary-value-method) as discussed in Lecture 9 for the case $\alpha = 1$. Choose appropriate step sizes Δx and Δt and solve the linear system. Plot the numerical solutions and check the numerical accuracy again.

PART B (40 points)

Consider the nonlinear second-order ODE[2] for the mathematical pendulum with mass $m = 1$ and length $l = \frac{(2\pi)^2}{g}$:

$$\begin{cases} \varphi''(t) + (2\pi)^2 \sin(\varphi(t)) = 0, \ t \in [0,1], \\ \varphi(0) = 1, \quad \varphi'(0) = 0. \end{cases} \tag{4}$$

For small angles φ we may assume the following linear approximation:

$$\begin{cases} \varphi''(t) + (2\pi)^2 \varphi(t) = 0, \ t \in [0,1], \\ \varphi(0) = 1, \quad \varphi'(0) = 0. \end{cases} \tag{5}$$

The exercise: apply the midpoint-based boundary-value method (BVM) from Lecture 9 to model (5) with <u>periodic conditions</u> and numerically solve the underlying linear system $\mathcal{A}\vec{\varphi} = \vec{b}$ for $t \in [0,1]$. It is known that the solution satisfies:

$$\begin{cases} \varphi(t) &= \cos(2\pi t), \\ \varphi'(t) &= -2\pi \sin(2\pi t), \end{cases} \tag{6}$$

and that the total energy of the system $\mathcal{E} = \varphi^2 + \frac{(\varphi')^2}{4\pi^2}$ should remain constant for all time $t \geq 0$. Furthermore, the solution in the phase plane (φ, φ') is represented by a circle.

Choose several (relevant) values for the time step size Δt to compare the numerical (BVM) solutions with the exact solution. Also check the accuracy of the "numerical energy "and the "numerical curve" in the phase plane.

Next, describe, implement and test a BVM for the nonlinear ODE (4). You could use, for example, *fsolve* in Matlab or its equivalent in Python.

<u>BONUS: 5 points:</u> As a final model, consider for $\trianglerighteq$(t) (predator) and $\circledcirc$(t) (prey), the ODE system:

$$\begin{cases} \trianglerighteq'(t) &= \alpha \trianglerighteq(t) - \beta \trianglerighteq(t)\circledcirc(t), \quad \trianglerighteq(0) = 0.9, \\ \circledcirc'(t) &= -\gamma \circledcirc(t) + \delta \trianglerighteq(t)\circledcirc(t), \quad \circledcirc(0) = 0.4, \end{cases} \tag{7}$$

$\Rightarrow$ P.T.O. $\rightsquigarrow$

---

[2]it is useful to write this ODE as a system of first order ODEs.

where $\alpha = 0.8$, $\beta = 1.2$, $\gamma = 0.4$, $\delta = 0.5$ and final time $T = 11.4$ (= the period of the solutions on the closed curve in the phase plane of this dynamical system). Work out and implement the BVM for model (7) with the given parameters. Plot the numerical BVM solutions for decreasing values of the time step Δt. Do the numerical solutions (☺(t), ⚐(t)) in the phase plane converge to the closed curve in the system?