MECCANO MATH II*

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*These notes are the sequel of "Meccano Math I" by the same author.

1. Crossings

As we laid down in *Meccano Math I*, the rules of meccano math are that two-dimensional constructions are to be made using strips with holes in them, to be joined with bolts through the holes. The holes are placed at integer positions, as in the simplest strips of Meccano. The strips can be of any length, and any finite number of them can be used, although constructions using fewer strips are preferred. The constructions must be mechanically stable, while the bolts will always allow strips to rotate in them.

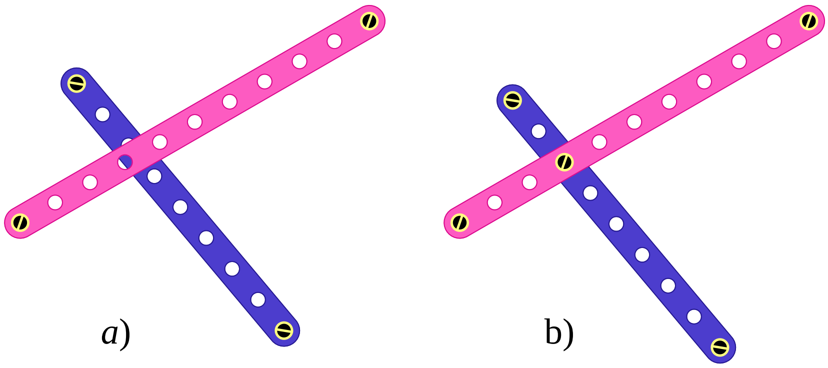

Figure 10: a) Loose crossing; b) bolted crossing.

We found that, in practice, many solutions to our problems require strips to cross one another. This is allowed. Sometimes, however, we wish to avoid this. Usually, it means that we continue to search for simpler solutions. In particular, we wish to avoid *loose crossings*, while *bolted crossings* are unavoidable, as they serve good purposes. This is illustrated in Fig. 10. In most of our problems, some attention to crossings will be given.

2. Bisectors

Let us, once again, focus on the problem of bisecting an arbitrary angle. In *Meccano Math I*, the suspicion was uttered that there should be better ways to bisect an angle than the method derived there. Indeed, there is. To discover how to do this, let us first concentrate on a much simpler problem:

> Given two strips bolted together at an arbitrary angle ψ . How do we construct a strip that fixes upon the bisectrix *approximately*?

We require $\psi = \alpha + \beta$, with α and β approximately equal. Of course, a strip with a slit instead of holes does not belong in our primary set.

This is basically just a problem of mechanics, and can be easily solved. The simplest construction is the one given in Figure 11. We call this the *false bisector*. For small angles are approximately divided in two, but not exactly: $\alpha \neq \beta$, as one can easily check.

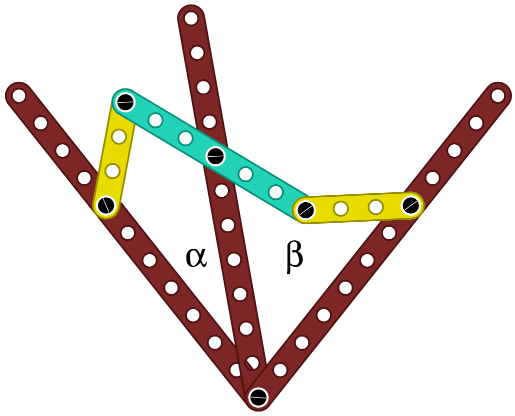

Figure 11: The false bisector.

Now can we use this to construct a true, accurate bisector? Indeed we can, and the observation is as shrewd as it is simple: use two (or actually, three) false bisectors! Observe that, if the angle α is given, the false bisector fixes the size of the angle β. But then, we can also generate an angle α at the other side of the angle β, and then, repeating this again, we can get an other angle of size α, so that, all taken together, we have an angle $\varphi = \alpha + \beta + \alpha + \beta = \psi + \psi$. The angle φ is bisected! Streamlining this procedure a bit, we arrive at the perfect bisector of Figure 12.

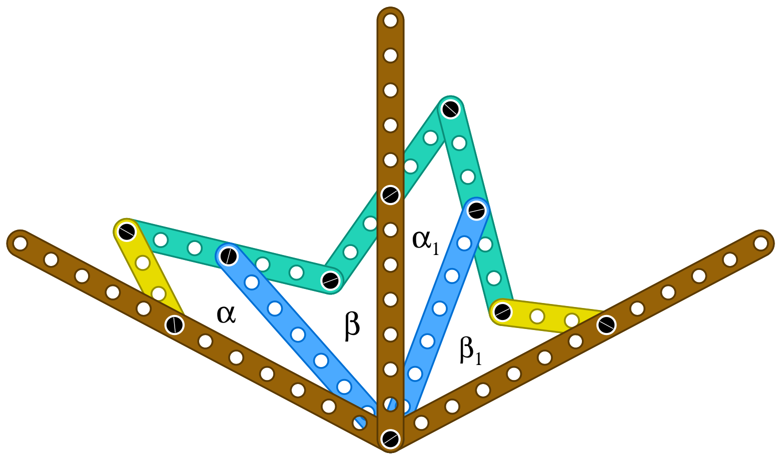

Figure 12: Our new true bisector. It consists of three false bisectors, one relating the angles α and β together yielding the angle ψ, the second relates β to α_1 the same way, so that, for symmetry reasons, $\alpha_1 = \alpha$. The third relates α_1 to β_1, and again, symmetry dictates that $\beta_1 = \beta$, so that the big angle is bisected perfectly: $\psi_1 = \alpha_1 + \beta_1 = \psi$.

Observe, fist of all, that this perfect bisector only requires 7 auxiliary pieces. Secondly, this solution is *not left-right symmetric*! The solution given in *Meccano math I* was also not left-right symmetric, and it appears that this is a fundamental law. We conjecture that *there is no left-right symmetric solution to the bisector problem.* However, symmetries are nevertheless essential in this construction. You see the same pattern repeated at several places, and this symmetry is the key element here.

This solution is so simple that it can be applied ideally to construct devices to split angles into more equal bits. See Figure 13 showing a perfect trisector. Note that these solutions to the bisector and trisector problems contain no loose crossings, while the solution of Figure 6b in *Meccano Math I* did. Also, in this solution, the angles can vary over a larger range, but this is not essential, one can often adjust the range of allowed values for the variable parameters.

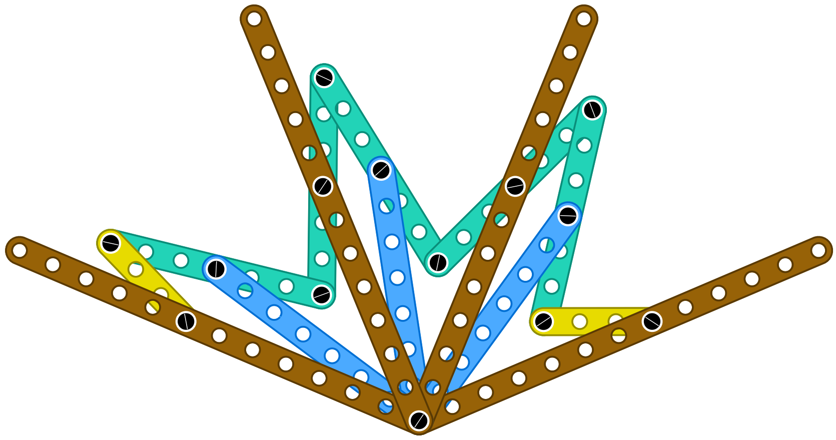

Figure 13: A perfect trisector. It is based on the same principle as that of our bisector, Fig. 12. Note that angles cannot be trisected using a compass and straightedge.

3. $\sqrt[3]{x}$

In *Meccano Math I*, the exercise was given to engineer a strip of length $\sqrt[3]{x}$ if a strip of length x is given. Ideally, the strip with length x, which could come from some other construction, would occur in this one not more than once. We now show how this can be done. For convenience, we take a fixed integral length a multiplying the lengths in the problem. What is needed then is a construction such that if one pair of bolts is separated by a distance $a \cdot y$, with $y = \sqrt[3]{x}$, elsewhere a pair of bolts necessarily takes the distance $a \cdot y^3$, with $y^3 = x$. We use the trigonometric formula

$$\sin(3\varphi) = 3\sin(\varphi) - 4\sin^3(\varphi) \ ,$$

and choose $y = 2\sin(\varphi)$. The equation is used to derive

$$y^3 = 3y - 2\sin(3\varphi) \ .$$

Now, consider Figure 14.

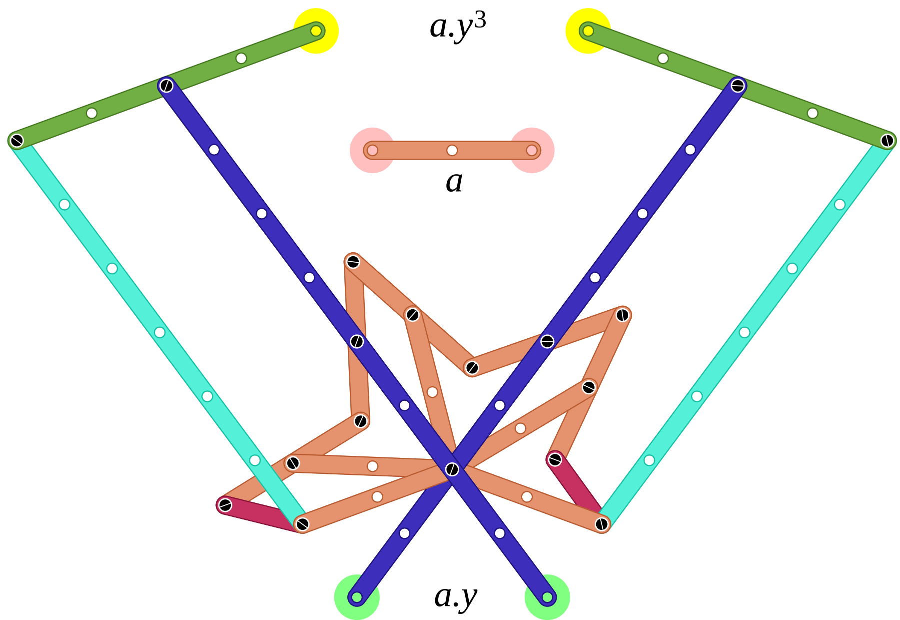

Figure 14: The third-root machine. It uses the angle trisector, see Fig. 13.

Here, $a = 2$, and we see that φ is half the elementary angle of the trisector. Three times the length $a \cdot y$ returns above. The sine of half the triple angle is multiplied by $2a$, also transported to the top and subtracted from the first. The desired length is shown there. As a bonus, we get that the two lengths in question appear parallel to one another. One loose crossing is still there but appears rather insignificant. The original problem is now solved: put the given length x, multiplied by a, at the top, to find $a\sqrt[3]{x}$ below.

A more compact construction of a third-root machine is given in Figure 15.

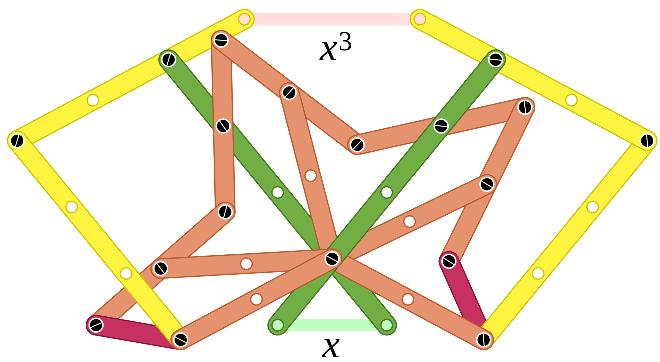

Figure 15: A more compact third-root machine, having $a = 1$.

4. The straightedge

As was stated in *Meccano Math I*, all constructions that can be done using compass and straightedge, can be reproduced with meccano. To prove this was an exercise. The most elegant proof is an explicit construction of both compass and straightedge. The straightedge is the easiest. We use two copies of the straight line producer explained in *I*, see Fig. 5b. Given one fixed point, this produces two bolts that can freely move in one direction, such that the three always stay on one line (within limits; the straightedge has a limited length). See Figure 16. Above (a) the elementary construction, below a slightly more elegant construction. They both show loose crossings, but the second construction has the business part of the straightedge in the clear.

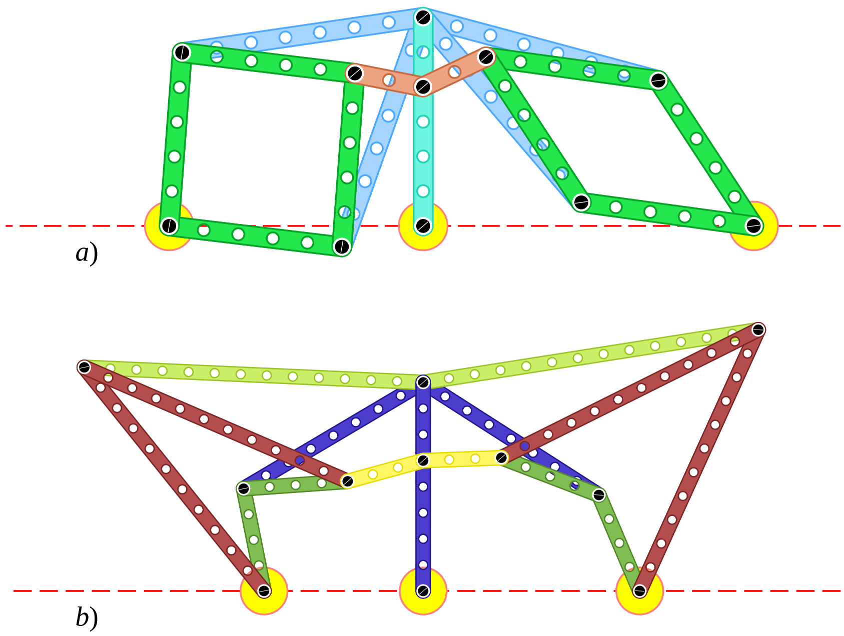

Figure 16: The straightedge. *a*) The elementary construction; *b*) A slightly more elegant variety.

These straightedges are to be used just like one would use ordinary straightedges: Pinpoint two of its active points on the points given in a construction; the third point can still move freely, along the line given by the others. A second straightedge, or a compass (to be given next) can then be used to fix the third point completely; it will be on the intersection of two given lines, or a line and a circle.

5. The compass

Finally, we should construct the compass. Given two points, our machine must pick up the distance between these two points, and then draw a circle around a third given point, with the given length as its radius. It is not hard to demonstrate that all this is possible, but the complete construction invariably turns out to be rather ugly, with a large number of loose crossings. So-far, what we constructed was a machine with two compartments: one rotator and one parallel transporter. The rotator takes the given two points and allows the second point to be reproduced rotated around the first. This can be achieved by taking two perfect bisectors. The parallel transporter takes the two points and transports the figure to an arbitrary new position, but without rotating it. In combination, these two ingredients do all that a compass would do. See Figure 17.

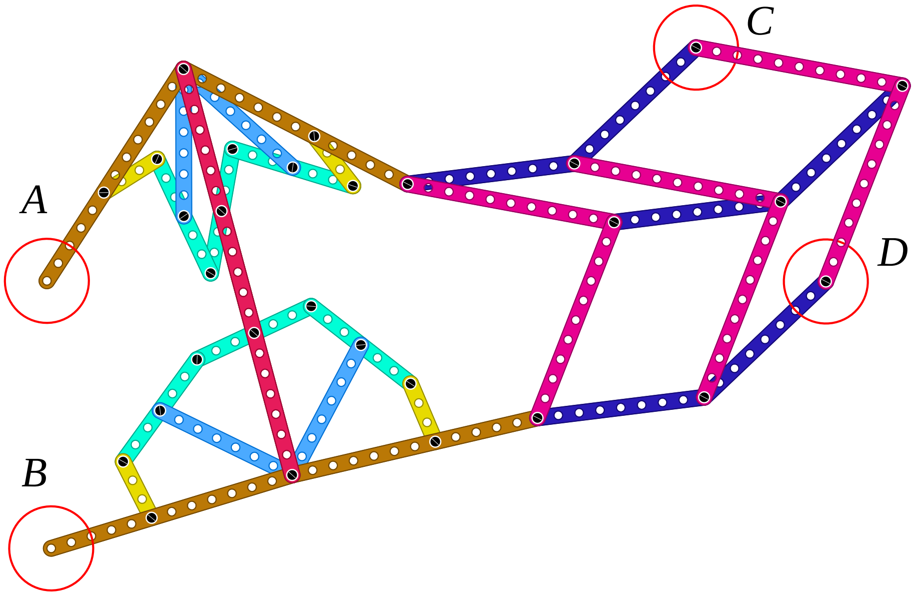

Figure 17: The compass. Points A, B, C and D can freely moved with respect to one another (within certain limits), but the construction is such that, while rotations are free, the distance between C and D must stay identical to that between A and B. Use was made of the bisector, Fig. 12.

We are waiting, however, for a more elegant idea, because what we sketched here emerges as something so complex, that one cannot help suspecting that it can be done in a simpler way. We challenge readers to come with some good idea.