Algebraic Geometry

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Contents

Preface

These are lecture notes for the course *Algebraic Geometry 1* offered in Mastermath in Fall 2020. These notes will be updated and expanded during the semester, as the course evolves. So it is better not to print everything at the beginning of the semester.

Students taking this course are encouraged to also follow the Mastermath course *Commutative Algebra*. The combination of the *Algebraic Geometry 1* and *Commutative Algebra* should provide sufficient background and motivation for the M2 course *Algebraic Geometry 2*, offered in Mastermath in the Spring.

The present lecture notes originate in a Mastermath course in algebraic geometry given during the Spring of 2009 by Bas Edixhoven and Lenny Taelman. Those notes were typed, as the course went on, by Michiel Kosters. In later versions, material has been added and revised by David Holmes, Arno Kret, and Martijn Kool.

Over the years, numerous additions, corrections and suggestions were pointed out by lecturers, students and teaching assistants. We want to thank in particular Michiel Vermeulen, Remke Kloosterman, Ariyan Javanpeykar, Samuele Anni, Jan Rozendaal, Robin de Jong, Harm Backx, Max Blans, Boaz Moerman, Joris Nieuwveld, and Kees Kok.

This syllabus is divided roughy into two parts. The first six chapters develop the basic theory of algebraic varieties over algebraically closed fields. The remaining chapters deal with various topics. Some of these are logically independent of each other, and can be skipped or treated in a different order.

The prerequisites for this course are the standard undergraduate algebra courses on groups, rings and fields (see for example the syllabi [Stev] (in Dutch), or [Lang]), and some basic topology.

No prior knowledge of algebraic geometry is necessary. We will occasionally refer to the basic textbook [Hart] on Algebraic Geometry. It is recommended to get hold of this book. The reader is also encouraged to compare this text with the other algebraic geometry syllabi [Moo] and [Looij].

By convention, rings are commutative, contain a 1 and a ring homomorphism $f\colon R \to S$ sends the 1 in R to the 1 in S.

Lecture 1

Affine space and its algebraic sets

The idea of algebraic geometry is to replace (systems of) polynomial equations with geometric objects, called 'varieties' so that it becomes possible to study these polynomial equations in a geometric way. These geometric objects should contain all the essential features of the polynomial equations, and 'equivalent' systems of equations (*e.g.* under a change of variables) should yield isomorphic varieties.

Similar to many (all?) fundamental objects in mathematics, like groups, rings, topological spaces, and so on, varieties will form a category: there is a meaningful way to compare varieties using a concept of 'morphism'. Our goal in the first 5 lectures is to introduce this category of varieties, and along the way introduce many examples.

Let k be an algebraically closed field.

1.1 The Zariski topology

In this section we will make the first step towards the notion of algebraic variety: we will define the *Zariski topology*.

In what follows we will frequently use the notion of k-algebra. By definition a *k-algebra* is a ring A with a given ring morphism $f\colon k \to A$, and a morphism of k-algebras from $f_1\colon k \to A_1$ to $f_2\colon k \to A_2$ is a ring morphism $g\colon A_1 \to A_2$ such that $g \circ f_1 = f_2$. In categorical terms: the category of k-algebras is the category of "rings under k".

Definition 1.1.1 For $n \in \mathbb{Z}_{\geq 0}$ we define *affine n-space*, denoted by $\mathbb{A}^n$, as k^n. Elements of $\mathbb{A}^n$ will be called points. Furthermore, $\mathbb{A}^1$ is called the *affine line* and $\mathbb{A}^2$ is called the *affine plane*.

Let $A = k[x_1, \ldots, x_n]$. We can view an element f of A as a function from $\mathbb{A}^n$ to k by evaluating f at points of $\mathbb{A}^n$. More precisely, we can carry out the following construction. Consider the set $\mathrm{Map}(\mathbb{A}^n, k)$ of all functions $f\colon \mathbb{A}^n \to k$ (so maps of sets). We may add two such functions f, g using the formula $(f+g) = (P \mapsto f(P) + g(P))$ and similarly multiply via $(f \cdot g) = (P \mapsto f(P) \cdot g(P))$. The map $k \to \mathrm{Map}(\mathbb{A}^n, k)$ that sends $a \in k$ to the constant function with value a is a ring morphism. Then under these operations $\mathrm{Map}(\mathbb{A}^n, k)$ is a k-algebra, and we have the following proposition:

Proposition 1.1.2 *The evaluation mapping*

$$\varphi\colon A \to \mathrm{Map}(\mathbb{A}^n, k),\ f \mapsto (\mathbb{A}^n \ni P \mapsto f(P))$$

is an injective morphism of rings.

Proof That φ is a ring morphism can be checked directly. To show that it is injective we argue by induction on n. If $n = 0$, then the map is an isomorphism.

Now assume $n > 0$ and let f be an element in the kernel of φ. Write $f = f_d x_n^d + \cdots + f_1 x_n + f_0$, with $f_i \in k[x_1, \ldots, x_{n-1}]$. The evaluation of f at $(a_1, \ldots, a_n)$ with $a_1, ..., a_{n-1}$ fixed and a_n variable defines a polynomial in one variable which vanishes for all values of a_n. Since k is algebraically closed, it is infinite. In particular, this polynomial has infinitely many zeroes, and hence is the zero polynomial. Thus $f_i(a_1, \ldots, a_{n-1}) = 0$ for all $(a_1, \ldots, a_{n-1})$ and $i = 0, \ldots, d$. We see that the f_i induce the zero-function on $\mathbb{A}^{n-1}$. From the induction hypothesis we obtain $f_i = 0$ for $i = 0, \ldots, d$. Thus $f = 0$. □

Definition 1.1.3 We define the *zero set* of an f in A to be $Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\}$. For $S \subset A$ we define $Z(S) = \{P \in \mathbb{A}^n : \forall f \in S,\ f(P) = 0\}$.

Example 1.1.4 Let S be the subset $\{x_1^2 + x_2^2 - 1, x_1\}$ of $k[x_1, x_2]$. Then $Z(S) = \{(0,1), (0,-1)\} \subset \mathbb{A}^2$.

Remark 1.1.5 Let $S \subset A$ and $I \subset A$ be the ideal generated by S. Then $Z(I) = Z(S)$, by the following argument. Since $S \subset I$, we obviously have $Z(I) \subset Z(S)$. On the other hand, let $P \in Z(S)$ and $f \in I$. Write f as a finite sum $f = \sum_{s \in S} f_s s$ with f_s in A. Then $f(P) = \sum_s f_s(P) s(P) = 0$. Hence P is in $Z(I)$. Therefore we also have the other inclusion $Z(I) \supset Z(S)$.

Definition 1.1.6 A subset $Y \subset \mathbb{A}^n$ is called *algebraic* if there exists some $S \subset A$ such that $Y = Z(S)$. By the previous remark we can replace "some $S \subset A$" by "some ideal $I \subset A$" without changing the meaning.

Example 1.1.7 We consider the case $n = 1$. Since $A = k[x]$ is a principal ideal domain, the algebraic subsets are of the form $Z((f)) = Z(f)$ for some f in A. If $f = 0$ we get $Z(0) = \mathbb{A}^1$. If $f \neq 0$ then f has only a finite number of zeros, hence $Z(f)$ is a finite set. On the other hand for every finite subset Y of $\mathbb{A}^1$ we have $Y = Z(\prod_{a \in Y}(x - a))$. This shows that the algebraic subsets of $\mathbb{A}^1$ are $\mathbb{A}^1$ itself together with the finite subsets of $\mathbb{A}^1$.

Proposition 1.1.8 *Let n be in $\mathbb{Z}_{\geq 0}$.*

i. Let $Y_1, Y_2 \subset \mathbb{A}^n$ be algebraic sets. Then $Y_1 \cup Y_2$ is an algebraic set.

ii. If $\{Y_\alpha\}_\alpha$ is a collection of algebraic subsets of $\mathbb{A}^n$, then $\cap_\alpha Y_\alpha \subset \mathbb{A}^n$ is algebraic.

iii. $\mathbb{A}^n$ is algebraic.

iv. $\emptyset$ is algebraic.

Proof i. We claim that for S_1 and S_2 subsets of A we have $Z(S_1) \cup Z(S_2) = Z(S)$, with

$$S = \{fg : f \in S_1,\ g \in S_2\} \subset A.$$

Obviously we have $Z(S_1) \cup Z(S_2) \subset Z(S)$. For the other inclusion, assume that $P \in Z(S)$ and $P \notin Z(S_1)$. Then there is an f in S_1 such that $f(P) \neq 0$. But we have for all g in S_2 that $0 = (fg)(P) = f(P)g(P)$. Since $f(P) \neq 0$, we get that $g(P) = 0$ for all $g \in S_2$, and hence $P \in Z(S_2)$.

ii. We have $Z(\cup_\alpha S_\alpha) = \cap_\alpha Z(S_\alpha)$.

iii. Note that $Z(\emptyset) = Z(0) = \mathbb{A}^n$.

iv. Note that $Z(1) = \emptyset$. □

Corollary 1.1.9 *The algebraic subsets of $\mathbb{A}^n$ are the* closed *subsets of a topology on $\mathbb{A}^n$. We will call this topology the* Zariski topology.

Remark 1.1.10 By Example 1.1.7 the Zariski topology on $\mathbb{A}^1$ is equal to the co-finite topology on $\mathbb{A}^1$.

Remark 1.1.11 The Zariski topology is very far from Hausdorff: any two non-empty open subsets of $\mathbb{A}^n$ have non-empty intersection. For instance, if $U_1 = Z(f_1)^{\complement} \subset \mathbb{A}^n$, $U_2 = Z(f_2)^{\complement} \subset \mathbb{A}^n$, then $U_1 \cap U_2$ contains $Z(f_1 \cdot f_2)^{\complement} \subset \mathbb{A}^n$. Thus the Zariski topology is very different from the more familiar complex topology on $\mathbb{C}^n$.

Definition 1.1.12 On a subset $Y \subset \mathbb{A}^n$ we define the *Zariski topology* as the induced topology from the Zariski topology on $\mathbb{A}^n$.

1.2 The Nullstellensatz

Let n be in $\mathbb{Z}_{\geq 0}$ and $A = k[x_1, \ldots, x_n]$. Recall that k is an algebraically closed field. In the previous subsection we defined a map:

$$Z : \{\text{subsets of } A\} \to \{\text{closed subsets of } \mathbb{A}^n\}.$$

We would like to "invert" this map Z. Note that Z is surjective, but not injective, not even when we restrict to the set of ideals: for example $Z((x)) = Z((x^2)) \subset \mathbb{A}^1$. The problem comes from the fact that if $f^m(P) = 0$ for some $f \in A$ and $m \geq 1$ then $f(P) = 0$ as well.

Definition 1.2.1 We define the following map:

$$\begin{aligned} I\colon \{\text{subsets of } \mathbb{A}^n\} &\to \{\text{ideals of } A\} \\ Y &\mapsto I(Y) := \{f \in A : \forall P \in Y,\ f(P) = 0\} \end{aligned}$$

Note that the maps I and Z reverse inclusions. If we have $Y_1 \subset Y_2$, then $I(Y_1) \supset I(Y_2)$; and similarly if $S_1 \subset S_2$ then $Z(S_1) \supset Z(S_2)$.

Definition 1.2.2 Let R be a ring. Then $a \in R$ is called *nilpotent* if there exists an integer $n > 0$ such that $a^n = 0$. The ring R is *reduced* if its only nilpotent element is 0.

Examples 1.2.3 Fields and integral domains are reduced, products of reduced rings are reduced, and subrings of reduced rings are reduced. The rings $k[x]/(x^2)$ and $\mathbb{Z}/4\mathbb{Z}$ are not reduced.

Definition 1.2.4 An ideal I in a ring R is *radical* if for all a in R and m in $\mathbb{Z}_{\geq 1}$ such that $a^m \in I$, a is in I.

Remark 1.2.5 Let R be a ring and I an ideal in R. Then I is radical if and only if R/I is reduced.

Example 1.2.6 In the ring $k[x]$ the ideal (x) is radical but (x^2) is not.

Definition 1.2.7 The *radical* of an ideal $I \subset R$ is $\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some } n > 0\}$.

Lemma 1.2.8 *$\sqrt{I}$ is the smallest radical ideal in R that contains I.*

In particular: I is radical if and only if $I = \sqrt{I}$.

Proof Any radical ideal containing I must contain $\sqrt{I}$, so we only need to show that $\sqrt{I}$ is a radical ideal itself. We first show that $\sqrt{I}$ is an ideal. For $x \in \sqrt{I}$ and $a \in R$ we have $ax \in \sqrt{I}$. Moreover, if $a^n \in I$ and $b^m \in I$ then the binomial formula shows that $(a+b)^{n+m}$ is an element of I, hence $a + b \in \sqrt{I}$. This shows that $\sqrt{I}$ is an ideal. To see that $\sqrt{I}$ is radical, note that if $a^m \in \sqrt{I}$, then $a^{nm} \in I$ for some n and hence $a \in \sqrt{I}$. $\square$

Lemma 1.2.9 *If S is a subset of $\mathbb{A}^n$, then $I(S)$ is a radical ideal in $k[x_1, \ldots, x_n]$.*

Proof If $f^n \in I(S)$ then $f(P)^n = 0$ and hence $f(P) = 0$ for all $P \in S$, so $f \in I(S)$. □

We see that the image of the map I in Definition 1.2.1 is contained in the set of radical ideals. Hilbert's famous Nullstellensatz says that when we restrict to this set of ideals, the maps Z and I are inverses of each other.

Theorem 1.2.10 (Nullstellensatz, Hilbert, 1893) *Let n be in $\mathbb{Z}_{\geq 0}$. The maps I and Z restrict to mutually inverse bijections*

$$I \colon \{\text{closed subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals of } A\} : Z.$$

Proof The crux is to show that if $J \subset A$ is a radical ideal, then $I(Z(J)) = J$. The proof relies on quite some commutative algebra, and goes beyond the scope of this course. In [AM, Exc. 7.14] it is explained how to deduce this from the 'weak Nullstellensatz' [AM, Cor. 5.24].

It then suffices to show that every closed subset $Y \subset \mathbb{A}^n$ is in the image of Z. To see this, note that if Y is closed then by definition $Y = Z(J)$ for some ideal $J \subset A$, and since we have $Z(J) = Z(\sqrt{J})$ we conclude that Y is in the image of Z. □

Corollary 1.2.11 *If I is an ideal in $A = k[x_1, \ldots, x_n]$, then $\sqrt{I} = I(Z(I))$.*

1.3 Irreducible topological spaces and prime ideals

Definition 1.3.1 A topological space X is *irreducible* if

i. $X \neq \emptyset$

ii. if $X = Z_1 \cup Z_2$ with Z_1 and Z_2 closed subsets of X then $Z_1 = X$ or $Z_2 = X$.

Remark 1.3.2 Let X be a topological space. Then the second condition in Definition 1.3.1 is equivalent to: every non-empty open subset of X is dense in X. See Exercise 1.6.7.

Examples 1.3.3 The affine line $\mathbb{A}^1$ is irreducible, since its proper closed subsets are finite while $\mathbb{A}^1$ itself is infinite. The real line $\mathbb{R}$ with its usual topology is not irreducible, because $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$.

Remark 1.3.4 Let Y be a non-empty subset of a topological space X. Then Y, with its induced topology, is irreducible if and only if for all closed subsets Z_1 and Z_2 of X with $Y \subset Z_1 \cup Z_2$ one has $Y \subset Z_1$ or $Y \subset Z_2$.

Definition 1.3.5 An *integral domain* is a ring R such that $1 \neq 0$ in R and for a, b in R with $ab = 0$ we must have $a = 0$ or $b = 0$. A *prime ideal* of a ring R is an ideal I of R such that R/I is an integral domain. A *maximal ideal* of a ring R is an ideal I of R such that R/I is a field.

Remark 1.3.6 Let I be an ideal in a ring R. The following two properties are each equivalent with I being prime:

i. $I \neq R$ and for all $a, b \in R$ we have that $ab \in I$ implies $a \in I$ or $b \in I$;

ii. $I \neq R$ and for all ideals $J, K \subset R$ we have that $JK \subset I$ implies $J \subset I$ or $K \subset I$.

Since fields are integral domains, and integral domains are reduced, we have that maximal ideals are prime and prime ideals are radical.

Proposition 1.3.7 *Let $Y \subset \mathbb{A}^n$ be closed. Then:*

i. $I(Y)$ is a maximal ideal if and only if Y consists of a single point;

ii. $I(Y)$ is a prime ideal if and only if Y is irreducible.

Proof Assume that $I(Y)$ is a maximal ideal. Then $Y \neq \emptyset$, since the radical ideal that corresponds to the empty set under the bijection from the Nullstellensatz is A. So by the Nullstellensatz Y is a minimal non-empty algebraic set. Since points are closed, Y is a point. For the converse, assume that Y is a point, say $Y = \{P\}$. The evaluation map $A \to k$, $f \mapsto f(P)$ is surjective, and its kernel is $I(Y)$, by definition. Hence $A/I(Y) \cong k$ is a field, and $I(Y)$ is a maximal ideal.

To prove the second assertion, assume that $I(Y)$ is a prime ideal of A. Then $Y \neq \emptyset$ because $I(Y) \neq A$. Suppose that $Y \subset Z_1 \cup Z_2$ with Z_1 and $Z_2 \subset \mathbb{A}^n$ closed. Then

$$I(Z_1)I(Z_2) \subset I(Z_1) \cap I(Z_2) = I(Z_1 \cup Z_2) \subset I(Y).$$

Hence by Remark 1.3.6 $I(Z_1) \subset I(Y)$ or $I(Z_2) \subset I(Y)$. So $Y \subset Z_1$ or $Y \subset Z_2$. So, Y is irreducible.

On the other hand suppose that Y is irreducible, we show that $I(Y)$ is a prime ideal. First of all $I(Y) \neq A$ because $Y \neq \emptyset$. Suppose that $fg \in I(Y)$. Then $Y \subset Z(fg) = Z(f) \cup Z(g)$. Hence $Y \subset Z(f)$ or $Y \subset Z(g)$ by the irreducibility of Y. So we have that $f \in I(Y)$ or $g \in I(Y)$. So, $I(Y)$ is a prime ideal. □

Corollary 1.3.8 *$\mathbb{A}^n$ is irreducible.*

Proof The ring $A = k[x_1, \dots, x_n]$ is an integral domain, so $(0) \subset A$ is a prime ideal, hence by the Nullstellensatz and the previous proposition $\mathbb{A}^n = Z((0))$ is irreducible. □

Example 1.3.9 Let $f \in k[x_1, \dots, x_n]$ be an irreducible polynomial. As $A = k[x_1, \dots, x_n]$ is a unique factorization domain, the ideal (f) is a prime ideal. Let $Y = Z(f)$. Because (f) is radical, we have $I(Y) = (f)$, and it follows from Proposition 1.3.7 that Y is an irreducible closed subset of $\mathbb{A}^n$.

Definition 1.3.10 Let $Y \subset \mathbb{A}^n$ be a subset. We define $A(Y)$ to be $A/I(Y)$. We have a natural map $k \to A(Y)$. This map makes $A(Y)$ into a k-algebra.

If f and g are elements of A with $f - g \in I(Y)$ then, for all $P \in Y$ we have $f(P) = g(P)$. So elements of the quotient ring $A(Y)$ can be interpreted as functions from Y to k. We note that if Y is irreducible, then $A(Y)$ is an integral domain.

1.4 Decomposition of closed sets in $\mathbb{A}^n$

Definition 1.4.1 A ring R is called *Noetherian* if every ideal of R is finitely generated, or equivalently, if for every chain of ideals $I_1 \subset I_2 \subset \cdots$ there is an r such that $I_r = I_{r+1} = \cdots$.

Theorem 1.4.2 (Hilbert basis theorem) *If R is Noetherian, then so is $R[x]$.*

Proof See [AM, Thm. 7.5]. □

Corollary 1.4.3 *The ring $A = k[x_1, \dots, x_n]$ is Noetherian.*

Definition 1.4.4 A topological space Y is called *Noetherian* if for every chain $Y_1 \supset Y_2 \supset Y_3 \supset \dots$ of closed subsets of Y there is an r such that $Y_r = Y_{r+1} = \cdots$.

Proposition 1.4.5 *$\mathbb{A}^n$ is Noetherian.*

Proof Let $Y_1 \supset Y_2 \supset \cdots$ be a chain of closed subsets. Since A is Noetherian, there is an r such that $I(Y_r) = I(Y_{r+1}) = \cdots$. By the Nullstellensatz we conclude that $Y_r = Y_{r+1} = \cdots$. □

Proposition 1.4.6 *If $Y \subset \mathbb{A}^n$ is closed then there is a finite collection of closed and irreducible subsets $(Y_i)_{1\le i\le t}$ of $\mathbb{A}^n$ such that $Y = Y_1 \cup \ldots \cup Y_t$.*

Proof Assume Y is not a finite union of closed irreducibles, in particular Y is not irreducible. So we can write $Y = Z_1 \cup Z_2$ with $Z_1 \subsetneq Y$, $Z_2 \subsetneq Y$ and Z_1, Z_2 closed. Hence at least one of Z_1, Z_2 is not a finite union of closed irreducibles, say Z_1. Put $Y_1 = Z_1$ and repeat. This gives us an infinite strictly decreasing chain, a contradiction. □

Proposition 1.4.7 *If $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_t$ with Y_i closed, irreducible and with the property that $Y_i \subset Y_j \implies i = j$, then the Y_i are uniquely determined by Y up to ordering. They are then called the* irreducible components *of Y.*

Proof Let $Y \subset \mathbb{A}^n$ be closed. Assume $Y_1' \cup \cdots \cup Y_s' = Y = Y_1 \cup \cdots \cup Y_t$ with Y_i and Y_i' irreducible, closed and $Y_i \subset Y_j \implies i = j$ and $Y_i' \subset Y_j' \implies i = j$. Assume that the two decompositions are different. Without loss of generality we may assume that there is an i with $Y_i \neq Y_j'$ for all j. Then we have $Y_i = Y_i \cap Y = (Y_i \cap Y_1') \cup \cdots \cup (Y_i \cap Y_s')$. Since Y_i is irreducible we obtain $Y_i \subset (Y_i \cap Y_j')$ for some j. So $Y_i \subset Y_j'$. Now repeat the above argument to find a k such that $Y_j' \subset Y_k$. So $Y_i \subset Y_j' \subset Y_k$, hence $Y_i = Y_k$ and $Y_i = Y_j'$, contradiction. □

1.5 Application: the theorem of Cayley-Hamilton

Theorem 1.5.1 (Cayley-Hamilton) *Let k be any field. Let a be an m by m matrix with coefficients in k and let $P_a \in k[x]$ be its characteristic polynomial, then $P_a(a)$ is the zero matrix.*

Lemma 1.5.2 *If a has m distinct eigenvalues, then $P_a(a) = 0$.*

Proof Without loss of generality we may assume that k is algebraically closed. Assume that a has no multiple eigenvalues. Then a is diagonalisable, so $a = qdq^{-1}$ for some invertible matrix q and a diagonal matrix d. We find that $P_a(a) = qP_a(d)q^{-1} = 0$. □

Proof (of Theorem 1.5.1) Put $n = m^2$ and view $\mathbb{A}^n$ as the set of all m by m matrices over k by ordering the coefficients in some way.

Let $Z_1 \subset \mathbb{A}^n$ be the subset of all matrices a that satisfy $P_a(a) = 0$. Note that Z_1 is closed since it is defined by n polynomials in the coefficients of a.

Let $Z_2 \subset \mathbb{A}^n$ be the subset of all matrices a that have multiple eigenvalues. Also Z_2 is closed since $a \in Z_2$ if and only if the discriminant of P_a is zero, and the discriminant of P_a is a polynomial in the entries of a.

The lemma shows that $\mathbb{A}^n = Z_1 \cup Z_2$. Also $\mathbb{A}^n \neq Z_2$ since there exist matrices without multiple eigenvalues. By the irreducibility of $\mathbb{A}^n$ (Corollary 1.3.8) we conclude that $\mathbb{A}^n = Z_1$, which proves the theorem. □

1.6 Exercises

Let k be an algebraically closed field.

Exercise 1.6.1 Let $Y = \{P_1, P_2\} \subset \mathbb{A}^2$ be a set consisting of two distinct points. Give generators for the ideal $I(Y) \subset k[x, y]$.

Exercise 1.6.2 Let Y_1 and Y_2 be *disjoint* closed subsets of $\mathbb{A}^n$. Let $I_1 = I(Y_1)$ and $I_2 = I(Y_2)$ be the corresponding ideals in $A = k[x_1, \ldots, x_n]$.

i. Show that $I_1 + I_2 = A$.

ii. Show that $I(Y_1 \cup Y_2) = I_1 I_2$.

iii. Show that $A(Y_1 \cup Y_2) = A(Y_1) \times A(Y_2)$.

Exercise 1.6.3 Let $Y = \{P_1, \ldots, P_r\} \subset \mathbb{A}^n$ be a finite set consisting of r distinct points. Give generators for the ideal $I(Y) \subset k[x_1, \ldots, x_n]$.

Exercise 1.6.4 Let $Y = Z(f_1, \ldots, f_m)$ for some $f_i \in A = k[x_1, \ldots, x_n]$. Assume that the ideal I generated by the f_i is prime. Show that $I(Y) = I$, and conclude that Y is irreducible.

Exercise 1.6.5 ([Hart, I.1.1])

i. Let $Y \subset \mathbb{A}^2$ be the zero set of $y - x^2$. Show that the k-algebra $A(Y)$ is isomorphic to $k[t]$.

ii. Let $Y \subset \mathbb{A}^2$ be the zero set of $xy - 1$. Show that the k-algebra $A(Y)$ is not isomorphic to $k[t]$.

Exercise 1.6.6 Let $Y \subset \mathbb{A}^2$ be the zero set of $x^2 + y^2 - 1$. Show that the k-algebra $A(Y)$ is not isomorphic to $k[t]$ if the characteristic of k is different from 2. What is $A(Y)$ if k is of characteristic 2?

Exercise 1.6.7 Show that the second condition in Definition 1.3.1 is equivalent to: every non-empty open subset of X is dense in X.

Exercise 1.6.8 Let $X \subset \mathbb{A}^n$ be an irreducible closed subset. Show that X, endowed with the Zariski topology, is connected.

Exercise 1.6.9 Let X be a topological space and let $U \subset X$ be an irreducible subspace. Show that the closure $\overline{U}$ of U in X is irreducible.

Exercise 1.6.10 Let X be a topological space and let $Z \subset X$ be a subset. Show that the following two statements are equivalent: (i) Z is closed in X; (ii) there exists a cover $\{U_i\}_{i \in I}$ of X by open subsets U_i of X such that for every $i \in I$, the intersection $Z \cap U_i$ is closed in U_i.

Exercise 1.6.11 For $f \in k[x_1, \ldots, x_n]$ let $D(f) \subset \mathbb{A}^n$ be the complement of $Z(f)$. Show that the $D(f)$ form a basis for the Zariski topology of $\mathbb{A}^n$. The open subsets of the form $D(f)$ are called *principal open subsets*.

Exercise 1.6.12 Show that $\mathbb{A}^n$ with its Zariski topology is compact: every open cover has a finite subcover. (In texts with some french origin, compactness in this sense is often called quasi-compactness, because in french the term "compact" means "Hausdorff and quasi-compact".)

Exercise 1.6.13 Show that the map $\mathbb{A}^n \to \mathbb{A}^1$ defined by a polynomial $f \in k[x_1, \dots x_n]$ is continuous when both $\mathbb{A}^n$ and $\mathbb{A}^1$ are endowed with the Zariski topology.

Exercise 1.6.14 ([Hart, I.1.3]) Let $Y \subset \mathbb{A}^3$ be the common zero set of the polynomials $x^2 - yz$ and $xz - x$. Show that Y is the union of three irreducible components. Describe them and find their prime ideals.

Exercise 1.6.15 ([Hart, I.1.4]) If one identifies $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Exercise 1.6.16 Assume that the characteristic of k is not 3. Show that the common zero set Y in $\mathbb{A}^3$ of the polynomials $x^2 - yz$ and $y^2 - xz$ is the union of four irreducible components. Find their prime ideals.

Exercise 1.6.17 ([Hart, I.1.5]) Show that a k-algebra B is isomorphic to $A(Y)$ for some algebraic set Y in some affine space $\mathbb{A}^n$ if and only if B is a finitely generated k-algebra that is reduced.

Exercise 1.6.18 Let k be any field. Let a, b be m-by-m matrices over k. Show that ab and ba have the same characteristic polynomial. (Hint: when a is invertible, you can use $ba = a^{-1}(ab)a$).

Lecture 2

Projective space and its algebraic sets

In this lecture we construct projective space, and describe its algebraic subsets. As in the previous lecture, we let k be an algebraically closed field.

2.1 $\mathbb{P}^n$ as a set

In this section, we do not need the assumption that k is algebraically closed.

Definition 2.1.1 For $n \in \mathbb{Z}_{\geq 0}$ we define the *projective n-space* $\mathbb{P}^n$ as the quotient of $k^{n+1} - \{0\}$ by the equivalence relation $\sim$, where $a \sim b \iff \exists \lambda \in k^\times$ such that $b = \lambda a$.

Remarks 2.1.2 i. $\sim$ is the equivalence relation given by the action of $k^\times$ on $k^{n+1} - \{0\}$: $(\lambda, a) \mapsto \lambda \cdot a$. So $a \sim b \iff a$ and b are in the same orbit under this action of $k^\times$.

ii. $a \sim b \iff k \cdot a = k \cdot b \iff a$ and b lie on the same line through the origin. So we can view $\mathbb{P}^n$ as the set $\{k{\cdot}a : a \in k^{n+1} - \{0\}\}$ of 1-dimensional k-vector spaces in k^{n+1}.

Remark 2.1.3 If $k = \mathbb{R}$, then $\mathbb{P}^n = S^n/\sim$ where $a \sim b \iff a = \pm b$, so we identify antipodal points.

Notation 2.1.4 Let $q\colon k^{n+1} - \{0\} \to \mathbb{P}^n$ be the quotient map. For $a = (a_0, \ldots, a_n)$ in $k^{n+1} - \{0\}$ we write $q(a_0, \ldots, a_n) = (a_0 : \cdots : a_n)$. These are the so called *homogeneous coordinates*, and the ":" (colons) express the fact that we are dealing with ratios.

Examples 2.1.5 In these examples, we will discuss $\mathbb{P}^n$ for certain n.

i. $\mathbb{P}^0 = (k^1 - \{0\})/\sim = \{(1)\}$, $\mathbb{P}^0$ is a 1-point set.

ii. $\mathbb{P}^1 = \{(a_0, a_1) \in k^2 : (a_0, a_1) \neq (0,0)\}/\sim = \{(a : 1) : a \in k\} \sqcup \{(1 : 0)\} = \mathbb{A}^1 \sqcup \{\infty\}$.

iii. We can generalise the procedure for $n = 1$ to the general case as follows:

$$\begin{aligned}
\mathbb{P}^n &= \{(a_0 : \cdots : a_{n-1} : 1) : a_0, \ldots, a_{n-1} \in k\} \sqcup \{(a_0 : \cdots : a_{n-1} : 0) : 0 \neq (a_0, \ldots, a_{n-1}) \in k^n\} \\
&= \mathbb{A}^n \sqcup \mathbb{P}^{n-1} \\
&= \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.
\end{aligned}$$

Remark 2.1.6 We can even make the decomposition of for example $\mathbb{P}^1$ visible in a picture. For this first draw the affine plane $\mathbb{A}^2$ with coordinates x_0 and x_1. Now $\mathbb{P}^1$ is the set of lines through the origin. We now fix some line not passing through the origin, say the line given by the equation $x_1 = 1$. A point on this line, say $(a_0, 1)$ gives rise to a line through the origin, $Z(x_0 - a_0 x_1)$, and if we vary a_0 we get all the lines through the origin, except the one line which is running parallel to the chosen line (in this case with the equation $x_1 = 0$), this is our point at infinity.

For $i \in \{0, \dots, n\}$, consider the following diagram:

$$
\begin{array}{lcll}
U_i & := & \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid a_i \neq 0\} & = \{(a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n) \mid a_j \in k\} \\
\big\downarrow \varphi_i & & (a_0 : \cdots : a_n) & \\
 & & \big\downarrow & \\
\mathbb{A}^n & & (\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}). &
\end{array}
$$

Notice that φ_i is a bijection, its inverse is given by

$$(b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_n) \mapsto (b_0 : \cdots : b_{i-1} : 1 : b_{i+1} : \cdots : b_n).$$

2.2 $\mathbb{P}^n$ as a topological space

Let $A = k[x_0, \dots, x_n]$, the k-algebra of polynomial functions on $k^{n+1} = \mathbb{A}^{n+1}$.

We have $q\colon \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$, where q is the quotient map previously defined. We give $\mathbb{A}^{n+1} - \{0\}$ the topology induced from the Zariski topology on $\mathbb{A}^{n+1}$: a subset U of $\mathbb{A}^{n+1} - \{0\}$ is open if and only if it is open as a subset of $\mathbb{A}^{n+1}$. We give $\mathbb{P}^n$ the quotient topology induced via q. Let Y be a subset of $\mathbb{P}^n$. Then Y is closed if and only if $q^{-1}Y \subset \mathbb{A}^{n+1} - \{0\}$ is closed, that is, if and only if there exists a closed subset Z of $\mathbb{A}^{n+1}$ such that $q^{-1}Y = Z \cap (\mathbb{A}^{n+1} - \{0\})$. Since $\{0\}$ is closed in $\mathbb{A}^{n+1}$, this is equivalent to $q^{-1}Y \cup \{0\}$ being closed in $\mathbb{A}^{n+1}$.

So we have the following bijection:

$$
\begin{array}{rcl}
\{\text{closed subsets of } \mathbb{P}^n\} & \tilde{\to} & \{\text{closed } k^\times\text{-invariant subsets of } \mathbb{A}^{n+1} \text{ containing } 0\} \\
Y & \mapsto & q^{-1}Y \cup \{0\}
\end{array}
$$

Recall that we have the Nullstellensatz:

$$
\begin{array}{rcl}
\{\text{closed subsets of } \mathbb{A}^{n+1}\} & \overset{1:1}{\longleftrightarrow} & \{\text{radical ideals } I \subset A\} \\
Y & \mapsto & I(Y) \\
Z(I) & \leftarrow\!\shortmid & I
\end{array}
$$

We now ask the following question: what does the property of being $k^\times$-invariant become on the right hand side?

The group $k^\times$ acts on $\mathbb{A}^{n+1}$: an element $\lambda \in k^\times$ acts as the multiplication map $\lambda\cdot\colon \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$, $a \mapsto \lambda{\cdot}a$. Now $k^\times$ also acts on the set of functions from $\mathbb{A}^{n+1}$ to k as follows. Let $a \in \mathbb{A}^{n+1}$. Then $((\lambda\cdot)^* f)(a) := f(\lambda a)$. This means that we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{A}^{n+1} & \xrightarrow{\lambda\cdot} & \mathbb{A}^{n+1} \\
 & \searrow^{(\lambda\cdot)^* f = f \circ \lambda\cdot} & \big\downarrow f \\
 & & k.
\end{array}
$$

The set $\{f\colon \mathbb{A}^{n+1} \to k\}$ of functions from $\mathbb{A}^{n+1}$ to k is a k-algebra: $(f + g)a = fa + ga$ and $(fg)a = (fa){\cdot}(ga)$ (we prefer not to write unnecessary parentheses, such as in $f(a)$). For each λ in $k^\times$ the map $(\lambda\cdot)^*$ from $\{f\colon \mathbb{A}^{n+1} \to k\}$ to itself is a k-algebra automorphism (its inverse is $(\lambda^{-1}\cdot)^*$). For example, we check the additivity. Let f and g be functions $\mathbb{A}^{n+1} \to k$, then

$$((\lambda\cdot)^*(f + g))a = (f + g)(\lambda a) = f(\lambda a) + g(\lambda a) = ((\lambda\cdot)^* f)a + ((\lambda\cdot)^* g)a = ((\lambda\cdot)^* f + (\lambda\cdot)^* g)a.$$

As this is true for all a in $\mathbb{A}^{n+1}$, we have $(\lambda\cdot)^*f + (\lambda\cdot)^*g = (\lambda\cdot)^*(f+g)$.

Recall that $A = k[x_0, \ldots, x_n]$. It is a sub-k-algebra of $\{f\colon \mathbb{A}^{n+1} \to k\}$. We claim that it is preserved by the $k^\times$-action: for f in A and λ in $k^\times$, the function $(\lambda\cdot)^*f$ is again in A. Indeed, for $f = \sum_i f_i x^i$ (multi-index notation) the function $(\lambda\cdot)^*f\colon \mathbb{A}^{n+1} \to k$ is given by

$$a \mapsto \lambda\cdot a \mapsto f(\lambda a) = \sum_{i_0,\ldots,i_n} f_{i_0,\ldots,i_n} \lambda^{i_0} a_0^{i_0} \cdots \lambda^{i_n} a_n^{i_n} = \sum_{i_0,\ldots,i_n} f_{i_0,\ldots,i_n} \lambda^{i_0+\cdots+i_n} a_0^{i_0} \cdots a_n^{i_n}.$$

Hence we see that

$$(\lambda\cdot)^*f = \sum_{i_0,\ldots,i_n} f_{i_0,\ldots,i_n} \lambda^{i_0+\cdots+i_n} x_0^{i_0} \cdots x_n^{i_n} \in A.$$

We conclude that each $(\lambda\cdot)^*\colon A \to A$ is a k-algebra automorphism, with inverse $(\lambda^{-1}\cdot)^*$. So $k^\times$ acts on the k-algebra A.

Now observe that for f in A, λ in $k^\times$, and a in $\mathbb{A}^{n+1}$ we have:

$$a \in Z((\lambda\cdot)^*f) \iff ((\lambda\cdot)^*f)(a) = 0 \iff f(\lambda\cdot a) = 0 \iff \lambda\cdot a \in Z(f) \iff a \in \lambda^{-1}\cdot Z(f).$$

So: $Z((\lambda\cdot)^*f) = \lambda^{-1}\cdot Z(f)$. And for $S \subset A$ we have $Z((\lambda\cdot)^*S) = \lambda^{-1}\cdot Z(S)$. Hence restricting the bijection from the Nullstellensatz on both sides to the $k^\times$-invariant subsets gives the bijection:

$$\begin{array}{ccc} \{\text{closed } k^\times\text{-invariant subsets} & \stackrel{1:1}{\longleftrightarrow} & \{k^\times\text{-invariant radical ideals } \mathfrak{a} \subset A \\ \text{of } \mathbb{A}^{n+1} \text{ containing } 0\} & & \text{with } \mathfrak{a} \subset (x_0, \ldots, x_n) = Ax_0 + \cdots + Ax_n\} \end{array}$$

We now want to know which ideals are $k^\times$-invariant. For this, we first decompose A into eigenspaces for the action of $k^\times$. An eigenspace under the action of $k^\times$ is exactly the set of homogeneous polynomials of a certain degree together with the 0 polynomial: A is graded as a k-algebra. This means that

$$A = \bigoplus_{d\geq 0} A_d, \quad A_d = \bigoplus_{d_0+\cdots+d_n=d} k\cdot x_0^{d_0} \cdots x_n^{d_n}, \quad f \in A_d,\ g \in A_e \implies f\cdot g \in A_{d+e}.$$

The sub-k-vectorspace A_d of A is called the space of homogeneous polynomials of degree d. For $f \in A$ we can write $f = \sum_d f_d$ with $f_d \in A_d$, and such a decomposition is unique. The f_d are called the *homogeneous parts* of f. Then for $\lambda \in k^\times$ we get $(\lambda\cdot)^*f = \sum_d \lambda^d f_d$.

Definition 2.2.1 An ideal $\mathfrak{a}$ is *homogeneous* if for all f in $\mathfrak{a}$ the homogeneous parts f_d are also in $\mathfrak{a}$.

Proposition 2.2.2 *Let $\mathfrak{a} \subset A$ be an ideal. Then $\mathfrak{a}$ is $k^\times$-invariant if and only if $\mathfrak{a}$ is homogeneous.*

Proof $\Leftarrow$: Assume $\mathfrak{a}$ is homogeneous. Let $f \in \mathfrak{a}, \lambda \in k^\times$. Then $(\lambda\cdot)^*f = \sum_d \lambda^d f_d \in \mathfrak{a}$ because $f_d \in \mathfrak{a}$ for all d.

$\Rightarrow$: Assume $\mathfrak{a} \subset A$ is a $k^\times$-invariant ideal. Let $f \in \mathfrak{a}$. Write $f = f_0 + \cdots + f_N$ with $f_i \in A_i$ for some $N \in \mathbb{Z}_{\geq 0}$. Take $\lambda_0, \ldots, \lambda_N \in k^\times$ distinct (we can do this since k is algebraically closed, hence infinite). We have: $\mathfrak{a} \ni (\lambda_i\cdot)^*f = f_0 + \lambda_i f_1 + \cdots + \lambda_i^N f_N$. In matrix form this gives:

$$\begin{pmatrix} (\lambda_0\cdot)^*f \\ \vdots \\ (\lambda_N\cdot)^*f \end{pmatrix} = \begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^N \end{pmatrix} \begin{pmatrix} f_0 \\ \vdots \\ f_N \end{pmatrix}$$

Now use that this Vandermonde matrix is invertible to get $f_0, \ldots, f_N$ in $\mathfrak{a}$ (we can express f_i as a k-linear combination of the $(\lambda_j\cdot)^*f$ in $\mathfrak{a}$). □

Theorem 2.2.3 (Homogeneous Nullstellensatz) *The following maps are inverses:*

$$\begin{array}{rcl} \{\textit{closed subsets of } \mathbb{P}^n\} & \stackrel{1:1}{\longleftrightarrow} & \{\textit{homogeneous radical ideals } \mathfrak{a} \subset A \textit{ with } \mathfrak{a} \subset (x_0, \ldots, x_n)\} \\ Y & \mapsto & I(q^{-1}Y \cup \{0\}) \\ q(Z(\mathfrak{a}) - \{0\}) & \leftarrow\!\shortmid & \mathfrak{a} \end{array}$$

and under this bijection we have that Y is irreducible if and only if $I(q^{-1}Y \cup \{0\})$ is prime and not equal to $(x_0, \ldots, x_n)$.

Proof The proof of the first part follows from the previous observations. The proof of the second part is one of the exercises below. □

2.3 A more direct description of the closed subsets of $\mathbb{P}^n$

Definition 2.3.1 For a homogeneous element f in some $A_d \subset A$ we define

$$Z_{\mathrm{proj}}(f) := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0\}.$$

Note that the condition makes sense, as it is independent of the chosen representative $(a_0, \ldots, a_n)$ of $(a_0 : \cdots : a_n)$. In fact, $Z_{\mathrm{proj}}(f) = q(Z(f) - \{0\})$ where $Z(f) \subset \mathbb{A}^{n+1}$.

The following proposition is a direct consequence of the results of the previous section.

Proposition 2.3.2 *The closed subsets of $\mathbb{P}^n$ are the $Z_{\mathrm{proj}}(T) = \bigcap_{f \in T} Z_{\mathrm{proj}}(f)$ for subsets T of the set $A^{\mathrm{hom}} = \bigcup_{d \geq 0} A_d$ of homogeneous elements of A.*

We first consider a special case: $T \subset A_1$, the case of linear equations. These $Z_{\mathrm{proj}}(T)$ are called linear subspaces of $\mathbb{P}^n$. Using linear algebra you can say a lot about them. For example two lines in $\mathbb{P}^2$ are equal or intersect in exactly one point. A hyperplane is a $Z_{\mathrm{proj}}(f)$ with $0 \neq f \in A_1$. We also have coordinate hyperplanes : $H_i = Z(x_i)$ for $0 \leq i \leq n$. Also we have the standard affine opens: $U_i = \mathbb{P}^n - H_i = \{a \in \mathbb{P}^n : a_i \neq 0\}$.

Proposition 2.3.3 *For $i \in \{0, 1, \ldots, n\}$ the map $\varphi_i \colon U_i \to \mathbb{A}^n$,*

$$(a_0 : \cdots : a_n) \mapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i})$$

is a homeomorphism.

Proof We have already seen that φ_i is bijective. Now consider the following diagram:

$$\begin{array}{ccccc} \mathbb{A}^{n+1} - \{0\} & \supset & q^{-1}U_i = \mathbb{A}^{n+1} - Z(x_i) & & \\ \downarrow q & & \downarrow q & \searrow^{\varphi_i \circ q} & \\ \mathbb{P}^n & \supset & U_i & \xrightarrow{\varphi_i} & \mathbb{A}^n \end{array}$$

We first claim that $\varphi_i \circ q \colon a \mapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i)$ is continuous. It suffices to show that for any f in $k[y_1, \ldots, y_n]$ the set $(\varphi_i \circ q)^{-1}Z(f)$ is closed. So, let f be in $k[y_1, \ldots, y_n]$, of degree at most some d in $\mathbb{Z}_{\geq 0}$. Then, for a in $q^{-1}U_i$, the following conditions are equivalent:

$$\begin{aligned} & a \in (\varphi_i \circ q)^{-1}Z(f) \\ & f((\varphi_i \circ q)a) = 0 \\ & f(a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i) = 0 \\ & a_i^d f(a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i)) = 0 \\ & a \in Z(x_i^d f(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)). \end{aligned}$$

Hence $(\varphi_i \circ q)^{-1}Z(f) = Z(x_i^d f(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)) \cap q^{-1}U_i$. Hence $\varphi_i \circ q$ is continuous. Since U_i has the quotient topology for q, φ_i is continuous.

On the other hand, the map $s_i\colon \mathbb{A}^n \to \mathbb{A}^{n+1} - Z(x_i) = q^{-1}U_i$,

$$(b_0, \ldots, b_{i-1}, b_i, \ldots, b_n) \mapsto (b_0, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_n)$$

is continuous because for any $b = (b_0, \ldots, b_{i-1}, b_i, \ldots, b_n)$ and any f in $k[x_0, \ldots, x_n]$ we have that $f(s_i(b)) = 0$ if and only if $f(b_0, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_n) = 0$, hence $s_i(b) \in Z(f)$ if and only if $b \in Z(f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n))$. Hence $\varphi_i^{-1} = q \circ s_i$ is continuous. □

2.4 How to administrate $\mathbb{P}^n$

On $\mathbb{A}^{n+1}$ we have the coordinate functions $x_0, \ldots, x_n$ and the k-algebra $k[x_0, \ldots, x_n]$ generated by them. Now φ_i is given by n functions on U_i: $x_{i,j}$, $0 \le j \le n$, $j \ne i$, with $x_{i,j} \circ q = x_j/x_i$. So $\varphi_i(P) = (x_{i,0}(P), \ldots, \ldots, x_{i,i-1}(P), x_{i,i+1}(P), \ldots, x_{i,n}(P))$.

Now for $f \in A_d$ we have that $x_i^{-d} f$ is a $k^\times$-invariant function on $q^{-1}U_i$, and it is a polynomial in the $x_{i,j}, j \ne i$. We have: $\varphi_i(Z_{\mathrm{proj}}(f)) = Z(x_i^{-d} f)$.

Example 2.4.1 Let $f = x_1^n - x_0^{n-1}x_2 + x_2^n \in k[x_0, x_1, x_2]_n$. Then:

$$\begin{aligned}
\varphi_0(Z_{\mathrm{proj}}(f) \cap U_0) &= Z(x_{0,1}^n - x_{0,2} + x_{0,2}^n) \\
\varphi_1(Z_{\mathrm{proj}}(f) \cap U_1) &= Z(1 - x_{1,0}^{n-1}x_{1,2} + x_{1,2}^n) \\
\varphi_2(Z_{\mathrm{proj}}(f) \cap U_2) &= Z(x_{2,1}^n - x_{2,0}^{n-1} + 1)
\end{aligned}$$

Vice versa: For i in $\{0, \ldots, n\}$ and $g \in k[\{x_{i,j} : j \ne i\}]$ of degree d you can "homogenise" to go back to $k[x_0, \ldots, x_n]$: just replace $x_{i,j}$ by x_j/x_i and multiply by x_i^d.

2.5 Exercises

We recall: k is an algebraically closed field. We also recall that a topological space X is irreducible if and only if first of all it is not empty and secondly has the property that if U and V are non-empty open subsets of X, then $U \cap V$ is non-empty.

Exercise 2.5.1 Let X and Y be topological spaces, $f\colon X \to Y$ continuous. Assume that X is irreducible and that f is surjective. Show that Y is irreducible.

Exercise 2.5.2 Let X and Y be topological spaces, $f\colon X \to Y$ a map, not necessarily continuous, and $Z \subset Y$. Assume that f is *open*: for every open U in X, fU is open in Y. Show that $f\colon f^{-1}Z \to Z$ is open, if Z and $f^{-1}Z$ are equipped with the topologies induced from Y and X.

Exercise 2.5.3 Let X and Y be topological spaces, and let $f\colon X \to Y$ be any map (not necessarily continuous). Assume that f is open and that for every y in Y the subset $f^{-1}\{y\}$ of X, with its induced topology, is irreducible. Assume that Y is irreducible. Show that X is irreducible.

Exercise 2.5.4 Let X be a topological space, and $x \in X$. Assume that $\{x\}$ is not open. Show that $X - \{x\}$ is irreducible if and only if X is irreducible.

Exercise 2.5.5 Let $n \in \mathbb{Z}_{\ge 0}$, and $q\colon \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ the quotient map. Show that q is open and that for all $P \in \mathbb{P}^n$, $q^{-1}\{P\}$ is irreducible.

Exercise 2.5.6 Let $n \in \mathbb{Z}_{\geq 0}$. Let $Y \subset \mathbb{P}^n$ be a closed subset. Let $I \subset A = k[x_0, \ldots, x_n]$ be the ideal of $q^{-1}Y \cup \{0\}$. Show that Y is irreducible if and only if I is prime and not equal to $(x_0, \ldots, x_n)$.

Exercise 2.5.7 Show by means of an example that Proposition 2.2.2 is false if k is not assumed to be infinite.

Exercise 2.5.8 Let $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$ and $P_4 = (1,1)$. Let $Y = \{P_1, P_2, P_3, P_4\}$ and let $I \subset k[x,y]$ be the ideal of Y.

i. Show that the affine coordinate ring $A(Y) = k[x,y]/I$ of Y has dimension 4 as k-vector space. Hint: consider the k-algebra morphism $k[x,y] \to k^4$ sending f to $(f(P_1), f(P_2), f(P_3), f(P_4))$, or use the Chinese Remainder Theorem.

ii. Show that $I = (f,g)$, where $f = x^2 - x$ and $g = y^2 - y$. Hint: show that $(f,g) \subset I$, then that $(1, x, y, xy)$ gives a k-basis for $k[x,y]/(f,g)$ using divisions with remainder, then that the natural morphism $k[x,y]/(f,g) \to A(Y)$ is an isomorphism.

iii. Draw a picture of Y, $Z(f)$ and $Z(g)$.

Exercise 2.5.9 Let $a, b \in k$. Consider $f = x_2x_1^2 - x_0^3 - ax_0x_2^2 + bx_2^3 \in k[x_0, x_1, x_2]$ and its zero set $Y = Z_{\mathrm{proj}}(f) \subset \mathbb{P}^2$. Compute the closed subsets $\varphi_i(Y \cap D_{\mathrm{proj}}(x_i)) \subset \mathbb{A}^2$, with $\varphi_i \colon D_{\mathrm{proj}}(x_i) \to \mathbb{A}^2$ the homeomorphism of Proposition 2.3.3.

Exercise 2.5.10 We assume now that $k \not\supset \mathbb{F}_2$. Let $Z = \{P_1, P_2, P_3\} \subset \mathbb{A}^2$, with the P_i as in Exercise 2.5.8. Let $J \subset k[x,y]$ be the ideal of Z. Our aim is to show that J can be generated by two elements. We view $\mathbb{A}^2$ as a standard open affine subset of $\mathbb{P}^2$ via $(a,b) \mapsto (a:b:1)$. Let $P_4' = (1:1:0) \in \mathbb{P}^2$, and let $Y' = \{P_1, P_2, P_3, P_4'\} \subset \mathbb{P}^2$.

i. Draw a picture of Y', the lines P_1P_2, P_3P_4', P_1P_3 and P_2P_4', and the line at infinity (draw your picture in the affine plane that is the complement of a suitable line in $\mathbb{P}^2$).

ii. Give linear equations for the lines P_1P_2, P_3P_4', P_1P_3 and P_2P_4', and deduce from this your two candidate generators f and g for J.

iii. Show that $J = (f,g)$. Hint: same strategy as in Exercise 2.5.8.ii; $\dim_k A(Z) = 3$; show that $xy \in (f,g)$.

Remark 2.5.11 Later it will be easier for us to show that $J = (f,g)$, by deducing it from the fact that $Z(f) \cap Z(g) = Z$, with "transversal intersection", implying that $I(Z(f) \cap Z(g)) = (f,g)$. More generally, there are standard algorithms based on the concept of Gröbner basis, with which one can compute in quotients such as $k[x,y]/(f,g)$.

Lecture 3

Geometry in projective space

Let k be an algebraically closed field.

3.1 Points and lines in $\mathbb{P}^2$

In this section we do not need the assumption that k is algebraically closed.

A *line* in $\mathbb{P}^2$ is $Z(f)$ where $f = ax + by + cz$ with $(a, b, c) \neq (0, 0, 0)$. Equivalently, a line in $\mathbb{P}^2$ is a closed subset of the form $q(V \setminus \{0\}) \subset \mathbb{P}^2$, where $V \subset k^3$ is a 2-dimensional k-linear subspace. Here q denotes the quotient map of Chapter 2. More generally, subsets of the form $q(V \setminus \{0\}) \subset \mathbb{P}^n$, where $V \subset k^{n+1}$ is a k-linear subspace, are called (projective) linear subspaces of $\mathbb{P}^n$.

A *line* in $\mathbb{A}^2$ is $Z(f)$ where $f = ax + by + c$ with $(a, b) \neq (0, 0)$. Recall that we have a decomposition:

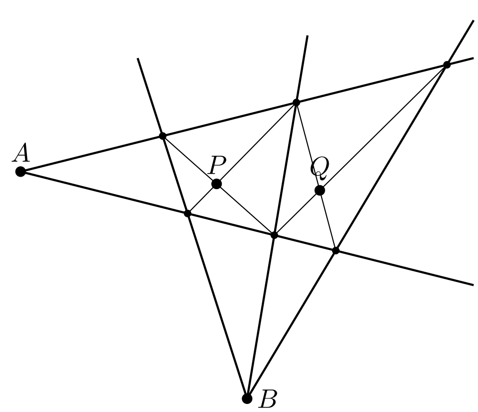
$$\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1,$$

where $\mathbb{A}^2$ is the set of points of the form $(a : b : 1)$, and $\mathbb{P}^1$ is the set of points of the form $(c : d : 0)$ with $(c, d) \neq (0, 0)$. If $(a, b) \neq (0, 0)$ then the line $Z(ax + by + c)$ in the affine plane $\mathbb{A}^2$ can be completed to the line $Z(ax + by + cz)$ in $\mathbb{P}^2$, by adding the 'point at infinity' $(-b : a : 0)$.

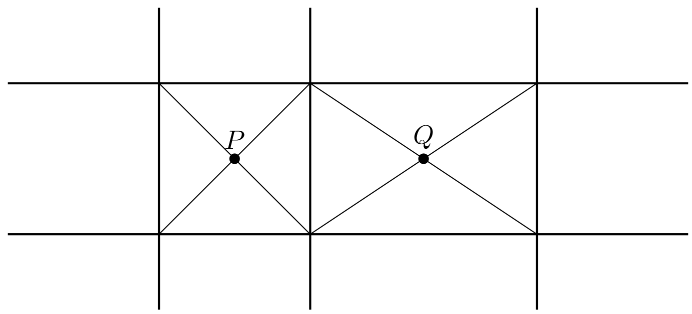
Let $l_1, l_2 \subset \mathbb{A}^2$ be *distinct* lines. Then the intersection $l_1 \cap l_2$ is empty if l_1 and l_2 are parallel, and otherwise it consists of one point. In $\mathbb{P}^2$ the situation is much nicer: two distinct lines always intersect in a unique point. Indeed, if $l_i = q(V_i \setminus \{0\})$ with $V_1, V_2 \subset k^3$ distinct subspaces of dimension 2, then $V_1 \cap V_2$ must be of dimension 1, and hence $l_1 \cap l_2 = q((V_1 \cap V_2) \setminus \{0\})$ is a point in $\mathbb{P}^2$.

Using projective space, many theorems in affine geometry become easier to prove. Here is an example:

Proposition 3.1.1 *In the following configuration (say in $\mathbb{R}^2$), the points A, P, Q lie on a line.*



Proof First consider this problem in $\mathbb{P}^2$ instead of $\mathbb{A}^2$. After a linear change of coordinates we may assume that $A = (1 : 0 : 0)$ and $B = (0 : 1 : 0)$. Indeed, A and B are distinct 1-dimensional subspaces of k^3, hence we can take a basis of k^3 with these lines as the first two coordinate axes. The line AB is then the line at infinity, and therefore the two lines that intersect in A are parallel in $\mathbb{A}^2$ and similarly for the three lines that intersect in B. So we then have the following picture.



But in this case, the result is obvious, and so we are done. □

3.2 Curves in $\mathbb{P}^2$

Remark 3.2.1 From now on, k is again assumed to be algebraically closed.

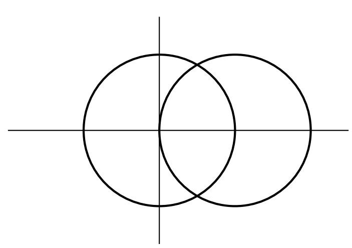
We have seen that the intersection of two distinct lines in $\mathbb{P}^2$ consists of one point. The following classical theorem from projective geometry generalizes this.

Theorem 3.2.2 (Bézout) *Let f_1 and f_2 in $k[x, y, z]$ be homogeneous irreducible polynomials of degrees d_1 and d_2, respectively. Assume $Z(f_1) \neq Z(f_2)$. Then $\#Z(f_1) \cap Z(f_2) = d_1 d_2$ "counted with multiplicity".*

Only later in this course we will be able to define the "multiplicity" occurring in the statement, but let us remark already now that it should be thought of as an "order of contact". So if the multiplicity of an intersection point is higher than one, this means that the curves are "tangent" to one another in that point.

Already the case $d_1 = d_2 = 1$ illustrates that it is important to work in the projective plane, instead of the affine plane. Below is another illustration.

Example 3.2.3 Assume that 2 is nonzero in k. Let $f_1 = x^2 + y^2 - z^2$ and $f_2 = (x - z)^2 + y^2 - z^2$. Here is a (real, affine) picture:



From the picture one can immediately read off two intersection points, namely $(1/2 : \sqrt{3}/2 : 1)$ and $(1/2 : -\sqrt{3}/2 : 1)$, and by putting $z = 0$ we find two more intersection points on the line at infinity: $(1 : \sqrt{-1} : 0)$ and $(1 : -\sqrt{-1} : 0)$. These two points at infinity correspond to the asymptotes. These asymptotes are not visible in the real affine picture, but become visible in $\mathbb{C}^2$.

3.3 Projective transformations

For n in $\mathbb{Z}_{\geq 0}$ we denote the group of invertible n by n matrices with coefficients in k with matrix multiplication by $\mathrm{GL}_n(k)$. It is the automorphism group of the k-vector space k^n.

Let n be in $\mathbb{Z}_{\geq 0}$. Since a linear map sends 0 to 0, the group $\mathrm{GL}_{n+1}(k)$ acts on $k^{n+1} - \{0\}$. Since matrix multiplication commutes with scalar multiplication, this induces an action of $\mathrm{GL}_{n+1}(k)$ on the quotient $\mathbb{P}^n$.

The normal subgroup $k^\times$ of scalar matrices in $\mathrm{GL}_{n+1}(k)$ acts trivially on $\mathbb{P}^n$. Therefore the action of $\mathrm{GL}_{n+1}(k)$ on $\mathbb{P}^n$ induces an action of the quotient $\mathrm{PGL}_{n+1}(k) := \mathrm{GL}_{n+1}(k)/k^\times$ on $\mathbb{P}^n$. An element of $\mathrm{PGL}_{n+1}(k)$ is called a projective transformation.

For f in $k[x_0, \ldots, x_n]$, viewed as function from $\mathbb{A}^{n+1}$ to k, and for g in $\mathrm{GL}_{n+1}(k)$, the function

$$g^*f \colon \mathbb{A}^{n+1} \to k, \; P \mapsto f(gP)$$

is again in $k[x_0, \ldots, x_n]$. This operation is a right-action of $\mathrm{GL}_{n+1}(k)$ on the k-algebra $k[x_0, \ldots, x_n]$. The polynomial g^*f is homogeneous of degree d if and only if f is homogeneous of degree d. Also, given a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ and a point $P \in \mathbb{P}^n$ we have $P \in Z(f)$ if and only if $g^{-1}P \in Z(g^*f)$. It follows that $\mathrm{GL}_{n+1}(k)$, and hence also $\mathrm{PGL}_{n+1}(k)$, act on $\mathbb{P}^n$ by homeomorphisms.

The proof of Proposition 3.1.1 could have started as follows: "There exists a projective transformation $g \in \mathrm{PGL}_3(k)$ such that $gA = (1:0:0)$ and $gB = (0:1:0)$, so we may assume that $A = (1:0:0)$ and $B = (0:1:0)$." Many facts in projective geometry can be proved easily by choosing "an appropriate projective frame" as we will see in the case of Pascal's theorem 3.5.2 and Pappus's theorem (see Exercise 3.6.5).

Definition 3.3.1 A *projective frame* of $\mathbb{P}^n$ is tuple $(P_0, \ldots, P_{n+1})$ of $n+2$ points in $\mathbb{P}^n$ such that no $n+1$ points amongst the P_i lie in a proper linear subspace of $\mathbb{P}^n$. Equivalently, there exists a basis $(v_0, \ldots, v_n)$ of k^{n+1} such that $P_i = q(v_i)$, for all $i = 0, \ldots, n$, and $P_{n+1} = q(v_{n+1})$ where

$$v_{n+1} = \sum_{i=0}^{n} \lambda_i v_i, \quad \lambda_i \neq 0 \text{ for all } i$$

and $q \colon k^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the quotient map.

Taking $(v_0, \ldots, v_n) = (e_0, \ldots, e_n)$ equal to the standard basis and $v_{n+1} = e_0 + \cdots + e_n$, the corresponding $n+2$ points form the so-called *standard frame* of $\mathbb{P}^n$.

Three mutually distinct points of $\mathbb{P}^1$ always form a projective frame (see Exercise 3.6.3). Four mutually distinct points of $\mathbb{P}^2$ form a projective frame if and only if no three of them are collinear, i.e. no three lie on a line (see Exercise 3.6.4). Projective transformations and projective frames are related as follows:

Theorem 3.3.2 *Let $(P_0, \ldots, P_{n+1})$ be the standard frame of $\mathbb{P}^n$. The map*

$$\begin{aligned} \mathrm{PGL}_{n+1}(k) &\longrightarrow \{\text{projective frames of } \mathbb{P}^n\} \\ g &\mapsto (gP_0, \ldots, gP_{n+1}) \end{aligned}$$

is a bijection.

Proof For surjectivity, suppose $(Q_0, \ldots, Q_{n+1})$ is a projective frame of $\mathbb{P}^n$ corresponding to a basis $(v_0, \ldots, v_n)$ of k^{n+1}. Then $Q_{n+1} = q(v_{n+1})$ with

$$v_{n+1} = \sum_{i=0}^{n} \lambda_i v_i,$$

where $\lambda_i \neq 0$ for all i. Define a matrix g by

$$ge_i = \lambda_i v_i, \quad \text{for all } i = 0, \dots, n.$$

Since $(\lambda_0 v_0, \dots, \lambda_n v_n)$ is a basis of k^{n+1}, we have $g \in \mathrm{GL}_{n+1}(k)$. Moreover,

$$ge_{n+1} = \sum_{i=0}^{n} ge_i = \sum_{i=0}^{n} \lambda_i v_i = v_{n+1}.$$

This shows that the image of g in $\mathrm{PGL}_{n+1}(k)$ maps P_i to Q_i, and hence that the projective frame $(Q_0, \dots, Q_{n+1})$ is in the image of the map.

Injectivity is left to the reader (Exc. 3.6.2). □

3.4 Affine transformations

Definition 3.4.1 We define the *group of affine transformations* as follows:

$$\mathrm{Aff}_n = \mathrm{Aff}_n(k) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathrm{GL}_n, b \in k^n \right\} \subset \mathrm{GL}_{n+1}\,.$$

It is the stabiliser in GL_{n+1} of the element x_n in $k[x_0, \dots, x_n]$, and therefore it stabilises all the hyperplanes $Z(x_n - a)$, with $a \in k$. The group Aff_n acts on $\mathbb{P}^n$. This action of Aff_n on $\mathbb{P}^n$ induces a morphism of groups $\mathrm{Aff}_n \to \mathrm{PGL}_{n+1}$. This morphism is injective and its image is the stabiliser in PGL_{n+1} of $Z(x_n)$, the hyperplane at infinity. This means that Aff_n acts on $\mathbb{P}^n - Z(x_n) = \mathbb{A}^n$ and on $Z(x_n) = \mathbb{P}^{n-1}$ as well (Aff_n stabilises $Z(x_n)$ as a subset of $\mathbb{P}^n$, but, if $n > 1$, does not stabilise $Z(x_n)$ point-wise).

Example 3.4.2 Consider the case where $n = 1$. An element of Aff_n has the form $g = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ with $a \in k^\times$ and $b \in k$. Now let $P \in \mathbb{A}^1$ be the point with coordinate $p \in k$. In $\mathbb{P}^1$ this point has homogeneous coordinates $(p : 1)$ and it is mapped by g to $(ap + b : 1)$, so $gP \in \mathbb{A}^1$ has coordinate $ap + b$.

In the same way as before there is a compatible right-action of Aff_n on $k[x_0, \dots, x_{n-1}]$. Explicitly:

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

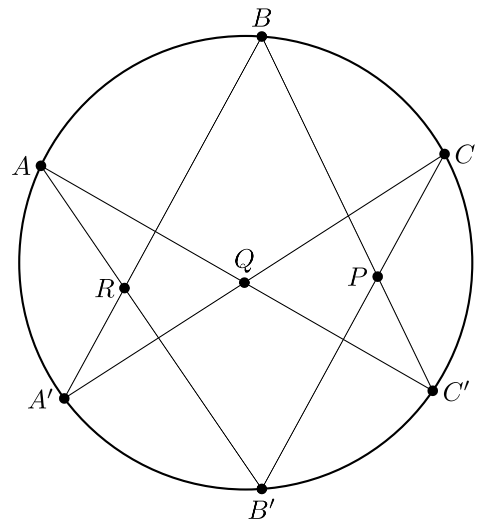
sends the polynomial f, viewed as function on k^n, to $g^* f := (x \mapsto f(ax + b))$. Note that P lies on $Z(f)$ if and only if $g^{-1}P$ lies on $Z(g^* f)$. In particular it follows that Aff_n acts as homeomorphisms on $\mathbb{A}^n$.

Remark 3.4.3 The dimension of Aff_1 is 2, but that of PGL_2 is 3. Hence the projective line has more symmetry than the affine line. In general Aff_n has dimension $n^2 + n$ (we can pick n^2 entries for the linear part, and then we can pick a vector to translate over, this gives an extra n) while PGL_{n+1} has dimension $(n+1)^2 - 1 = n^2 + 2n$.

3.5 Pascal's theorem

In this section, we will prove Pascal's theorem. We first state a Euclidean version of it.

Theorem 3.5.1 *Suppose that X is a circle and $A, B, C, A', B', C' \in X$ are distinct points on this circle. Let $P = B'C \cap BC'$, $Q = AC' \cap A'C$, $R = A'B \cap AB'$, assuming these intersections exist (see the picture below). Then P, Q, R lie on a line.*



To prove this it is convenient to generalize this to a projective statement.

Theorem 3.5.2 (Pascal) *Let $X = Z(g)$ with $g \in k[x_1, x_2, x_3]$ homogeneous of degree 2 and irreducible. Let $A, B, C, A', B', C' \in X$ be distinct points. Let $P = B'C \cap BC'$, $Q = AC' \cap A'C$, $R = A'B \cap AB'$. Then P, Q, R lie on a line.*

Proof Note that no three of the six points A, B, C, A', B', C' can lie on a line, for otherwise this would contradict Bézout's theorem (together with the irreducibility of X).

So, without loss of generality we may assume that $A' = (1 : 0 : 0)$, $B' = (0 : 1 : 0)$, $C' = (0 : 0 : 1)$. We now write down the equation for X:

$$g = g_{11}x_1^2 + g_{22}x_2^2 + g_{33}x_3^2 + g_{12}x_1x_2 + g_{13}x_1x_3 + g_{23}x_2x_3.$$

Since $A' = (1 : 0 : 0)$ lies on this quadric, we see that $g(1, 0, 0) = g_{11} = 0$. In the same manner, one obtains $g_{22} = g_{33} = 0$. So

$$g = g_{12}x_1x_2 + g_{13}x_1x_3 + g_{23}x_2x_3.$$

Note that none of g_{12}, g_{13}, g_{23} are zero, for otherwise our g would be reducible. After applying the projective transformation

$$\begin{pmatrix} g_{23} & 0 & 0 \\ 0 & g_{13} & 0 \\ 0 & 0 & g_{12} \end{pmatrix} \in \mathrm{PGL}_3$$

we may assume that

$$g = x_1x_2 + x_2x_3 + x_3x_1.$$

Note that A', B' and C' are fixed under this transformation.

Now let A, B, C be the points $(a_1 : a_2 : a_3)$, $(b_1 : b_2 : b_3)$ and $(c_1 : c_2 : c_3)$, respectively. Let us compute the coordinates of the point $P = B'C \cap BC'$. The line $B'C$ is given by $c_3x_1 = c_1x_3$, and BC' is given by $b_2x_1 = x_2b_1$. So we find that $P = (1 : b_2/b_1 : c_3/c_1)$. Note that b_1 is not zero, since B lies on X and B is distinct from B' and C', similarly a_i, b_i, c_i are all non-zero. By symmetry, we find that $Q = (a_1/a_2 : 1 : c_3/c_2)$ and $R = (a_1/a_3 : b_2/b_3 : 1)$. To check that P, Q and R lie on a line, it is enough to show that

$$\det \begin{pmatrix} 1 & b_2/b_1 & c_3/c_1 \\ a_1/a_2 & 1 & c_3/c_2 \\ a_1/a_3 & b_2/b_3 & 1 \end{pmatrix} = 0.$$

But this is true. The sum of the rows is zero, this follows since A, B and C lie on our quadric. For example, for the first coordinate:

$$1 + \frac{a_1}{a_2} + \frac{a_1}{a_3} = \frac{a_2a_3 + a_1a_3 + a_1a_2}{a_2a_3} = \frac{g(a_1, a_2, a_3)}{a_2a_3} = 0.$$

$\square$

Note that the above proof shows that any two irreducible conics in $\mathbb{P}^2$ are projectively equivalent. Say that two closed subsets X, Y of $\mathbb{P}^n$ are projectively equivalent if there exists $g \in \mathrm{PGL}_{n+1}$ such that $Y = g(X)$.

3.6 Exercises

Exercise 3.6.1 Consider $Y_1 = Z(y - x^2)$ and $Y_2 = Z(xy - 1)$ in $\mathbb{A}^2$. Denote by $i : \mathbb{A}^2 \to \mathbb{P}^2$ the map $(a, b) \mapsto (a : b : 1)$. Let X_1 and X_2 be the closures of iY_1 and iY_2, respectively.

i. Give equations for X_1 and X_2.

ii. Describe $X_2 - iY_2$ and $X_1 - iY_1$.

iii. Show that there is no affine transformation α such that $\alpha Y_1 = Y_2$.

iv. Show that there is a projective transformation β such that $\beta X_1 = X_2$.

Exercise 3.6.2 Complete the proof of Theorem 3.3.2.

Exercise 3.6.3 Let P_1, P_2 and P_3 be three distinct points in $\mathbb{P}^1$. Show (without using Theorem 3.3.2) that there is a unique projective transformation that maps P_1 to $(1 : 0)$, P_2 to $(0 : 1)$, and P_3 to $(1 : 1)$.

Exercise 3.6.4 Let P_1, P_2, P_3 and P_4 be four points in $\mathbb{P}^2$ such that there is no line in $\mathbb{P}^2$ containing three of them. Show (without using Theorem 3.3.2) that there is a unique projective transformation that maps P_1 to $(1 : 0 : 0)$, P_2 to $(0 : 1 : 0)$, P_3 to $(0 : 0 : 1)$, and P_4 to $(1 : 1 : 1)$.

Exercise 3.6.5 Let L, L' be two distinct lines in $\mathbb{P}^2$ and let $A, B, C \in L$, $A', B', C' \in L'$ such that $A, B, C, A', B', C', L \cap L'$ are mutually distinct. For any two distinct points $P, Q \in \mathbb{P}^2$, we denote by PQ the line through P and Q.

i. Prove that there exists a projective transformation mapping A, B, A', B' to $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : 1 : 1)$ respectively.

ii. Write down equations for the lines $L, L', AB', A'B, AC', A'C, BC', B'C$.

iii. Prove that $AB' \cap A'B$, $AC' \cap A'C$, and $BC' \cap B'C$ are collinear. This statement is known as Pappus's theorem.

Exercise 3.6.6 ([Hart, 2.14]) Given positive integers r and s consider the map

$$((a_1, \ldots, a_r), (b_1, \ldots, b_s)) \to (a_1b_1 : a_1b_2 : \cdots : a_rb_s)$$

from $(\mathbb{A}^r - \{0\}) \times (\mathbb{A}^s - \{0\})$ to $\mathbb{P}^{rs-1}$.

i. Show that the map factors through $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$;

Denote the resulting map from $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$ to $\mathbb{P}^{rs-1}$ by Ψ.

ii. Show that Ψ is injective;

iii. Show that the image of Ψ is closed in $\mathbb{P}^{rs-1}$.

The map Ψ is called the *Segre embedding* of $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$ in $\mathbb{P}^{rs-1}$.

Exercise 3.6.7 Consider the Segre embedding $\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Let $Q \subset \mathbb{P}^3$ be the image of Ψ.

i. Give equations for Q.

ii. Show that for all $P \in \mathbb{P}^1$ that each of the images of $\{P\} \times \mathbb{P}^1$ and of $\mathbb{P}^1 \times \{P\}$ are lines in $\mathbb{P}^3$ lying on Q. (A *line* in $\mathbb{P}^n$ is the common zero locus of $n-1$ independent homogeneous linear equations).

iii. Show that all lines in $\mathbb{P}^3$ lying on Q can be obtained in this way (hint: choose points (A_1, A_2) and (B_1, B_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ and verify that the line through $\Psi((A_1, A_2))$ and $\Psi((B_1, B_2))$ lies on Q if and only if $A_1 = B_1$ or $A_2 = B_2$).

iv. For any pair of lines L_1, L_2 lying on Q determine their intersection $L_1 \cap L_2$.

Exercise 3.6.8 Consider the Segre embedding $\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Let $Q \subset \mathbb{P}^3$ be the image of Ψ.

i. Describe all closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ with the product topology.

ii. Show that Ψ is not continuous when $\mathbb{P}^1 \times \mathbb{P}^1$ is equipped with the product topology and Q is equipped with the induced topology from $\mathbb{P}^3$.

In fact, as we will see later, the induced topology on Q, and *not* the product topology, is the "right one" for the product $\mathbb{P}^1 \times \mathbb{P}^1$.

Lecture 4

Regular functions and algebraic varieties

As usual, k is an algebraically closed field.

4.1 Regular functions on closed subsets of $\mathbb{A}^n$

It is now time to make geometric objects of the closed subsets of $\mathbb{A}^n$ and $\mathbb{P}^n$ that we have seen so far: until now they are just topological spaces, and moreover, their topology is quite weird. Before we give formal definitions, we have an informal discussion.

The difference between topology, differential geometry and say complex analytic geometry comes from the kind of functions that one allows: continuous functions, differentiable functions and complex analytic functions. In algebraic geometry, the allowed functions are called "regular." We want to specify, for $Y \subset \mathbb{A}^n$ closed, and $U \subset Y$ open, which functions $f\colon U \to k$ are regular. We then denote by $\mathcal{O}_Y(U)$ the set of regular functions on U, and will call Y, together with all $\mathcal{O}_Y(U)$, the algebraic variety $(Y, \mathcal{O}_Y)$. Just like continuity, differentiability, and analyticity, regularity of a function should be a local property. This means that $f\colon U \to k$ is regular if and only if it is regular at every $P \in U$.

Definition 4.1.1 Let $n \in \mathbb{Z}_{\geq 0}$, $Y \subset \mathbb{A}^n$ closed, $U \subset Y$ open (for the induced topology on Y), and $f\colon U \to k$ a function. Then, for $P \in U$, f is called *regular at* P if there is an open subset $V \subset \mathbb{A}^n$ with $P \in V$, and elements $g, h \in k[x_1, \ldots, x_n]$ such that for all $Q \in V$, $h(Q) \neq 0$ and for all $Q \in U \cap V$: $f(Q) = g(Q)/h(Q)$.

A function $f\colon U \to k$ is called *regular* if it is regular at all $P \in U$.

The set of regular functions on $U \subset Y$ is denoted by $\mathcal{O}_Y(U)$. It is a k-algebra for point-wise addition and multiplication.

The following example shows that for $Y \subset \mathbb{A}^n$ closed, $U \subset Y$ open, and $f\colon U \to k$ regular, there need not exist g and h in $k[x_1, \ldots, x_n]$ such that f is given by g/h on all of U. This illustrates the occurrence of V in Definition 4.1.1.

Example 4.1.2 Consider the closed subset $Y = Z(xw - yz) \subset \mathbb{A}^4$ where $A = k[x, y, z, w]$. We consider $U = Y \backslash Z(y, w)$, and the function

$$f\colon U \to k$$

$$P = (a, b, c, d) \mapsto \begin{cases} x(P)/y(P) = a/b & P \in D(y) \cap U \\ z(P)/w(P) = c/d & P \in D(w) \cap U. \end{cases}$$

Observe that $Z(y,w) = \mathbb{A}^4 - (D(y) \cup D(w))$, hence $U = (U \cap D(y)) \cup (U \cap D(w))$, and on $D(y) \cap D(w) \cap U$ we have $x/y = z/w$ (since $xw = zy$), so f is indeed well-defined, and regular.

Lemma 4.1.3 *Let X be a topological space, and Y a subset of X. Then Y is closed if and only if X can be covered by open subsets $U_i \subset X$ such that for all i, $Y \cap U_i$ is closed in U_i.*

Proof Assume that Y is closed; just take the covering $\{X\}$. Conversely, if, for all i, $Y \cap U_i$ is closed in U_i then every point in the complement $X - Y$ has an open neighbourhood in $X - Y$, hence Y is closed in X. □

Lemma 4.1.4 *Let $Y \subset \mathbb{A}^n$ be closed, $U \subset Y$ be open, and $f \in \mathcal{O}_Y(U)$. Then $f\colon U \to k = \mathbb{A}^1$ is continuous.*

Proof Since k has the co-finite topology, it is enough to show that for any $a \in k$, $f^{-1}\{a\} \subset U$ is closed. By the previous lemma it is enough to give for every $P \in U$ an open $V \subset \mathbb{A}^n$ with $P \in V$ such that $f^{-1}\{a\} \cap V$ is closed in $U \cap V$. So let $P \in U$ be given and take an open $V \subset \mathbb{A}^n$ and g and h in $k[x_1, \ldots, x_n]$ as in Definition 4.1.1. Then for $Q \in U \cap V$ the condition $f(Q) = a$ is equivalent to $Q \in Z(g - ah)$. So $f^{-1}\{a\} \cap V = Z(g - ah) \cap (U \cap V)$, hence closed in $U \cap V$. □

Remark 4.1.5 Not all continuous $f\colon \mathbb{A}^1 \to k$ are regular. For example every permutation of k is a homeomorphism of $\mathbb{A}^1$.

Corollary 4.1.6 *Let $Y \subset \mathbb{A}^n$ be closed and irreducible, $U \subset Y$ open, f and g in $\mathcal{O}_Y(U)$ such that $f|_V = g|_V$ for some open nonempty $V \subset U$. Then $f = g$.*

Proof Note that $f - g$ is regular, hence continuous by the previous lemma. So the subset

$$Z = \{P \in U \mid f(P) - g(P) = 0\} \subset U$$

is closed in U. By the assumption, Z contains V. We see that U is the union of the closed sets Z and $U - V$, and since U is irreducible and $U - V \neq U$ we find that we must have $Z = U$. □

For $f \in k[x_1, \ldots, x_n]$ we set $D(f) := \{P \in \mathbb{A}^n : f(P) \neq 0\} = \mathbb{A}^n - Z(f)$, so the $D(f)$ are open in the Zariski topology. Observe that $D(1) = \mathbb{A}^n$ and $D(0) = \emptyset$.

Lemma 4.1.7 *The set of all $D(f)$ is a basis for the Zariski topology on $\mathbb{A}^n$.*

The proof was left to the reader in Exercise 1.6.11. Note that the function $1/f\colon D(f) \to k$, $P \mapsto 1/f(P)$ is given as a fraction whose denominator has no zero on $D(f)$.

Theorem 4.1.8 *Let n be in $\mathbb{Z}_{\geq 0}$ and let $Y \subset \mathbb{A}^n$ be closed. Then the k-algebra morphism φ from $A := k[x_1, \ldots, x_n]$ to $\mathcal{O}_Y(Y)$ that sends a polynomial to the function that it defines is surjective and has kernel $I(Y)$. Hence it induces an isomorphism from $A/I(Y) = A(Y)$ to $\mathcal{O}_Y(Y)$.*

Proof By definition $\ker(\varphi) = \{f \in A : \forall P \in Y,\ f(P) = 0\} = I(Y)$. So we only need to prove the surjectivity of φ, the rest follows immediately.

Let $f \in \mathcal{O}_Y(Y)$. We want to show that f is in $\mathrm{im}(\varphi)$. Consider the ideal

$$J = \{h \in A : \varphi(h) f \in \mathrm{im}(\varphi)\}.$$

in the ring A. We need to show that 1 lies in J, or, equivalently, that $J = A$. Note that $I(Y) \subset J$ since for $h \in I(Y)$ we have $\varphi(h) f = 0 \cdot f = 0$.

Suppose that $J \neq A$. Take $\mathfrak{m} \subset A$ a maximal ideal such that $J \subset \mathfrak{m}$. By the Nullstellensatz there is a $P \in \mathbb{A}^n$ such that $\mathfrak{m} = \mathfrak{m}_P$, the maximal ideal corresponding to P. As $I(Y) \subset J \subset \mathfrak{m}_P$, we have $P \in Y$. Since f is a regular function on Y we can take an open neighbourhood $V \subset \mathbb{A}^n$ of P and $g, h \in A$ such that for all $Q \in V \cap Y$ we have $h(Q) \neq 0$ and $h(Q)f(Q) = g(Q)$

Now by Lemma 4.1.7, we may assume without loss of generality that $V = D(h')$ for some $h' \in A$. But then we have $h'(Q)h(Q)f(Q) = h'(Q)g(Q)$ for *all* $Q \in Y$, since h' vanishes on the complement $Y \cap Z(h')$ of $Y \cap V$. We conclude that $\varphi(h'h)f = \varphi(h'g)$, and hence that $\varphi(h'h)f$ lies in the image of φ, and that $h'h \in J$.

But we have $h'(P) \neq 0$ and $h(P) \neq 0$ by construction, hence hh' does not lie in $\mathfrak{m}_P$, a contradiction. Hence $J = A$ and we are done. □

As a final example, we give an explicit description of the regular functions on a principal open subset $D(f) \subset \mathbb{A}^n$.

Proposition 4.1.9 *Let $f \in A = k[x_1, \ldots, x_n]$, and set $U = D(f)$. Then a function on U is regular if and only if it can be written as g/f^r for some $g \in A$ and $r \geq 0$.*

Proof Clearly g/f^r defines a regular function on U.

Conversely, let φ be a regular function on U. Let $J \subset A$ be the ideal consisting of those $g \in A$ such that $g\varphi \in A$. We need to show that $f^r \in J$ for some $r \geq 0$.

Let P be a point in U. Then there exists an open neighbourhood $V \subset U$ of P and $g, h \in A$ such that h is non-zero on V and $\varphi = g/h$ on V. In particular, we have $\varphi h = g$ on V. By Corollary 4.1.6 we conclude that $\varphi h = g$ on all of U, and therefore $h \in J$.

We have shown that for every $P \in D(f)$ there is a function h_P with $h_P \in J$ and $h_P(P) \neq 0$. Hence we have $Z(J) \cap D(f) = \emptyset$, or in other words f vanishes on $Z(J)$. By the Nullstellensatz, this implies that $f^r \in J$ for some $r \geq 0$, which is what we had to prove. □

4.2 Regular functions on closed subsets of $\mathbb{P}^n$

We now define $\mathcal{O}_Y$ for Y a closed subset of $\mathbb{P}^n$. First we do this for $\mathbb{P}^n$ itself. Let $A = k[x_0, \ldots, x_n]$.

Definition 4.2.1 Let $U \subset \mathbb{P}^n$ be open, $f\colon U \to k$, $P \in U$. Then f is called *regular at* P if there exists a $d \geq 0$, and $g, h \in A_d$ such that for every $Q = (a_0 : \cdots : a_n)$ in a neighbourhood of P we have

$$h(a_0, \ldots, a_n) \neq 0, \text{ and } f(Q) = \frac{g(a_0, \ldots, a_n)}{h(a_0, \ldots, a_n)}.$$

The function f is called *regular* if f is regular at all $P \in U$. Notation: $\mathcal{O}_{\mathbb{P}^n}(U) = \{f\colon U \to k : f \text{ is regular}\}$

Note that the quotient in the definition is well-defined (independent on the choice of homogeneous coordinates of Q) since g and h are homogeneous of the *same* degree. Informally, we will say "$f = g/h$ on a neighbourhood of P", but keep in mind that g and h are themselves not well-defined functions on such a neighbourhood.

Definition 4.2.2 Let $Y \subset \mathbb{P}^n$ be closed, $V \subset Y$ open, $f\colon V \to k$, and $P \in V$. Then f is called *regular at* P if there exists an open $U \subset \mathbb{P}^n$ and $g \in \mathcal{O}_{\mathbb{P}^n}(U)$ such that $P \in U$ and for all $Q \in V \cap U : f(Q) = g(Q)$.

Remark 4.2.3 For $Y \subset \mathbb{A}^n$ closed we could have done the same thing: first define $\mathcal{O}_{\mathbb{A}^n}$ and then continue as above.

Remark 4.2.4 In Corollary 5.3.5 we will show that any regular function $f\colon \mathbb{P}^n \to k$ is constant, and more generally that for a projective variety X, any $f \in \mathcal{O}_X(X)$ is locally constant. So in contrast with affine varieties, projective varieties X have very few regular functions defined on all of X. (Note however that functions in $\mathcal{O}_X(U)$ for $U \subsetneq X$ need *not* be locally constant).

4.3 k-spaces and algebraic varieties

From this section on, we will use some terminology from category theory. Some references for categories, functors, and equivalence of categories are the section "categories and functors" in [Lang], or the lecture notes [Tae]. See also the video recordings of the 2017 Mastermath online course "Intensive Course on Categories and Modules".

In our development of the theory of algebraic varieties, we are now at a point where we really must introduce morphisms between them. For example, we want to compare $U_i \subset \mathbb{P}^n$ with $\mathbb{A}^n$ via the map $\varphi_i\colon U_i \to \mathbb{A}^n$ and we would like to call φ_i an isomorphism, so both φ_i and φ_i^{-1} should be morphisms. We know that φ_i and φ_i^{-1} are continuous. The idea is then to ask for a morphism to be a continuous function that, by composition, sends regular functions to regular functions. We formalize this as follows.

Definition 4.3.1 Let k be a field (not necessarily algebraically closed). A *k-space* is a pair $(X, \mathcal{O}_X)$, with X a topological space, and for every $U \subset X$ open, $\mathcal{O}_X(U) \subset \{f\colon U \to k\}$ a sub-k-algebra such that:

i. for all $V \subset U$ (both open) and for all f in $\mathcal{O}_X(U)$, $f|_V$ is in $\mathcal{O}_X(V)$;

ii. for all U open and for all $f\colon U \to k$, f is in $\mathcal{O}_X(U)$ if and only if for all $P \in U$ there is an open $U_P \subset U$ such that $P \in U_P$ and $f|_{U_P}$ is in $\mathcal{O}_X(U_P)$.

We call $\mathcal{O}_X$ the *sheaf of regular functions*. The two conditions in Definition 4.3.1 mean that the "regularity" condition is a local condition: a function is regular if and only if it is regular locally. Note that each $\mathcal{O}_X(U)$ contains the constant functions, since we have required $\mathcal{O}_X(U)$ to be a sub-k-algebra of the k-algebra of all functions $f\colon U \to k$.

Examples 4.3.2

i. Let Y be a closed subset of $\mathbb{A}^n$ or $\mathbb{P}^n$. Then the pair $(Y, \mathcal{O}_Y)$ defined in the previous two sections is a k-space.

ii. Let X be a topological space, and for $U \subset X$ open let $\mathcal{C}_X^0(U)$ be the set of continuous functions $f\colon U \to \mathbb{R}$. Then $(X, \mathcal{C}_X^0)$ is an $\mathbb{R}$-space.

iii. Let X be a manifold, and for $U \subset X$ open let $\mathcal{C}_X^\infty(U)$ be the set of smooth functions $f\colon U \to \mathbb{R}$. Then $(X, \mathcal{C}_X^\infty)$ is an $\mathbb{R}$-space.

iv. Let X be a Riemann surface, and for $U \subset X$ open let $\mathcal{O}_X^{\mathrm{an}}(U)$ be the set of holomorphic functions $f\colon U \to \mathbb{C}$. Then $(X, \mathcal{O}_X^{\mathrm{an}})$ is a $\mathbb{C}$-space.

Definition 4.3.3 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be k-spaces. A *morphism* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a map $\varphi\colon X \to Y$ such that:

i. φ is continuous;

ii. for all $U \subset Y$ open, for all $f \in \mathcal{O}_Y(U)$, the function $\varphi^* f := f \circ \varphi\colon \varphi^{-1}U \to k$ is in $\mathcal{O}_X(\varphi^{-1}U)$.

The k-spaces and their morphisms form a category: k-Spaces. This gives us the notion of isomorphism: a morphism φ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is an isomorphism if there is a morphism ψ from $(Y, \mathcal{O}_Y)$ to $(X, \mathcal{O}_X)$ with $\psi \circ \varphi = \mathrm{id}_{(X,\mathcal{O}_X)}$ and $\varphi \circ \psi = \mathrm{id}_{(Y,\mathcal{O}_Y)}$. In more concrete terms: for k-Spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a map $\varphi \colon X \to Y$ is an isomorphism if and only if φ is a homeomorphism and has the property that, for all $f \colon Y \to k$ and all P in X, f is regular at $\varphi(P)$ if and only if $f \circ \varphi$ is regular at P.

For $(X, \mathcal{O}_X)$ a k-space and U an open subset of X we define $\mathcal{O}_X|_U$, the restriction of $\mathcal{O}_X$ to U, by: for $V \subset U$ open, $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$. We can now define what algebraic varieties are.

Definition 4.3.4 Let k be an algebraically closed field.

An *algebraic variety over* k is a k-space $(X, \mathcal{O}_X)$ such that for all $x \in X$ there is an open $U \subset X$ with $x \in U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic (in k-Spaces) to a $(Y, \mathcal{O}_Y)$ with $Y \subset \mathbb{A}^n$ closed for some n, and $\mathcal{O}_Y$ the sheaf of regular functions.

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are algebraic varieties over k, a *morphism* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism in k-Spaces. The category of algebraic varieties over k is denoted $\mathrm{Var}(k)$; by definition, it is a full subcategory of k-Spaces.

An algebraic variety is called *affine* if it is isomorphic to a $(Y, \mathcal{O}_Y)$ with Y a closed subset of some $\mathbb{A}^n$ and $\mathcal{O}_Y$ its sheaf of regular functions.

Remark 4.3.5 Our notion of variety in $\mathrm{Var}(k)$ is more general than that in the first chapter of [Hart]. The category of varieties defined by Hartshorne is equivalent to the full sub-category of irreducible and quasi-projective varieties in $\mathrm{Var}(k)$.

The main difference with [Hart], however, is of a more philosophical nature. In [Hart], varieties come by definition with an embedding in an affine or projective space, although morphisms are defined in such a way that the embedding need not be preserved. In the approach above, morphisms are defined 'internally', without reference to some (arbitrarily chosen) embedding.

A similar approach is followed by Varadarajan [Var] (for manifolds) and by Springer [Spri] and Mumford [Mum] (for algebraic varieties).

We believe this approach is also useful as a stepping stone towards the theory of schemes.

Proposition 4.3.6 *Let $n \in \mathbb{Z}_{\geq 0}$, $i \in \{0, \ldots, n\}$, $U_i \subset \mathbb{P}^n$ as before, and $\varphi_i \colon U_i \to \mathbb{A}^n$ the map $(a_0 : \cdots : a_n)$ to $(a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i)$. Then φ_i is an isomorphism of k-spaces. Hence $\mathbb{P}^n$ is an algebraic variety.*

Proof We have already seen that φ_i and its inverse are continuous. It remains to be shown that the conditions "regular at P" and "regular at $\varphi_i(P)$" correspond, that is, for $f \colon U \to k$ with U a neighbourhood of $\varphi_i(P)$, f is regular at $\varphi_i(P)$ if and only if $\varphi_i^* f$ is regular at P. In what follows, we use the notation as in Section 2.4.

Let P be in U_i, and $U \subset \mathbb{A}^n$ open containing $\varphi_i(P)$, and $f \colon U \to k$ a function. Then f is regular at $\varphi_i(P)$ if and only if there exist $g, h \in k[\{x_{i,j} : j \neq i\}]$ such that $h(\varphi_i(P)) \neq 0$ and $f = g/h$ in a neighbourhood of $\varphi_i(P)$.

The function $\varphi_i^* f$ is regular at P if and only if there exist $d \in \mathbb{Z}_{\geq 0}$ and $g', h' \in k[x_0, \ldots, x_n]_d$ such that $h'(P) \neq 0$ and $\varphi_i^* f = g'/h'$ in a neighbourhood of P.

Suppose that f is a regular function at $\varphi_i(P)$, locally given by g/h. Let $d = \max(\deg(g), \deg(h))$ and notice that for a in a neighbourhood of P

$$\begin{aligned}
(\varphi_i^*(g/h))(a_0 : \cdots : a_n) &= g(\varphi_i(a_0 : \cdots : a_n))/h(\varphi_i(a_0 : \cdots : a_n)) \\
&= g((a_j/a_i)_{j \neq i})/h((a_j/a_i)_{j \neq i}) \\
&= a_i^d g((a_j/a_i)_{j \neq i})/a_i^d h((a_j/a_i)_{j \neq i}) \\
&= (g'/h')(a_0 : \cdots : a_n)
\end{aligned}$$

where $g' = x_i^d g((x_j/x_i)_{j\neq i})$ and $h' = x_i^d h((x_j/x_i)_{j\neq i})$ are in $k[x_0, \ldots, x_n]_d$. Hence $\varphi_i^* f$ is regular at P.

Suppose now that $\varphi_i^* f$ is regular at P, locally given as g'/h' with g' and h' in $k[x_0, \ldots, x_n]_d$ for some d. Then f is locally given by g/h with $g = x_i^{-d} g'$ and $h = x_i^{-d} h'$ in $k[x_{i,j} : j \neq i]$ (recall that $x_{i,j} = x_j/x_i$), showing that f is regular at $\varphi_i(P)$. □

Corollary 4.3.7 *Let $Y \subset \mathbb{P}^n$ be closed, then $(Y, \mathcal{O}_Y)$ is an algebraic variety.*

Proof Consider the isomorphisms of algebraic varieties $\varphi_i : U_i \to \mathbb{A}^n$ of the above proposition. Set $Y_i = U_i \cap Y$ and $Z_i = \varphi_i(Y_i)$. Then $(Y_i, \mathcal{O}_Y|_{Y_i})$ is a k-space, and $(Z_i, \mathcal{O}_{Z_i})$ is an affine algebraic variety.

By Definition 4.2.2, a function f on an open part of Y_i is regular at P if there is some neighbourhood V of P in U_i and a regular function $\tilde{f}$ on V such that $f|_{V\cap Y_i} = \tilde{f}|_{V\cap Y_i}$.

Regularity on the affine variety Z_i has the analogous property: a function g on an open part of Z_i is regular at P if there is some neighbourhood W of P in $\mathbb{A}^n$ and a regular function $\tilde{g}$ on W such that $g|_{W\cap Z_i} = \tilde{g}|_{W\cap Z_i}$.

From these observations, we conclude that the isomorphism $\varphi_i : U_i \to \mathbb{A}^n$ restricts to an isomorphism of k-spaces $Y_i \to Z_i$. Since the Z_i are affine algebraic varieties, we conclude that Y is covered by affine algebraic varieties Y_i, and hence that Y is an algebraic variety. □

Definition 4.3.8 An algebraic variety is called *projective* if it is isomorphic to a $(Y, \mathcal{O}_Y)$ with Y a closed subset of some $\mathbb{P}^n$ and $\mathcal{O}_Y$ its sheaf of regular functions. A variety is called *quasi-projective* if it is isomorphic to an open subvariety of a projective variety.

We will now prove some things which will be useful later.

Proposition 4.3.9 *Let X be an algebraic variety, $U \subset X$ an open subset with its induced topology and regular functions, let $Y \subset \mathbb{A}^n$ closed, and let $\psi : U \to Y$ a map of sets. For i in $\{1, \ldots, n\}$ let $\psi_i = \mathrm{pr}_i \circ \psi$, hence for all P in U, $\psi(P) = (\psi_1(P), \ldots, \psi_n(P))$. Then ψ is a morphism if and only if for all i, ψ_i is in $\mathcal{O}_U(U)$.*

Proof Assume that ψ is a morphism. Let i be in $\{1, \ldots, n\}$. The restriction of the function $x_i : \mathbb{A}^n \to k$ to Y is in $\mathcal{O}_Y(Y)$ and we denote it still by x_i. Then $\psi_i = \psi^*(x_i)$ is in $\mathcal{O}_U(U)$.

Assume that all ψ_i are regular. We have to show that ψ is a morphism. We start with showing that ψ is continuous. For f in $k[x_1, \ldots, x_n]$, $\psi^* f$ is the function $P \mapsto f(\psi_1(P), \ldots, \psi_n(P))$, hence $\psi^* f = f(\psi_1, \ldots, \psi_n)$, the image in $\mathcal{O}_U(U)$ of f under the k-algebra morphism that sends x_i to ψ_i. Hence for all f in $k[x_1, \ldots, x_n]$ we have:

$$\psi^{-1} Z(f) = \{P \in U : f(\psi(P)) = 0\} = (\psi^* f)^{-1}\{0\}.$$

Now $\psi^* f \in \mathcal{O}_U(U)$ is continuous, because continuity is a local property and by Lemma 4.1.4 $\psi^* f$ is continuous at every P in X.

Now we show that ψ is a morphism. Let $V \subset Y$ be open and $f \in \mathcal{O}_Y(V)$. We must show that $\psi^* f : \psi^{-1} V \to k$ is in $\mathcal{O}_U(\psi^{-1} V)$. This is a local property by the second part of Definition 4.3.1. We must show that for all P in $\psi^{-1} V$, $\psi^* f$ is regular at P. So let P be in $\psi^{-1}(V)$. Write $f = g/h$ in a neighbourhood of $\psi(P)$, with g and h in $k[x_1, \ldots, x_n]$. Then $\psi^* f = g(\psi_1, \ldots, \psi_n)/h(\psi_1, \ldots, \psi_n)$ in a neighbourhood of P, hence a quotient of the two elements $g(\psi_1, \ldots, \psi_n)$ and $h(\psi_1, \ldots, \psi_n)$ in $\mathcal{O}_X(U)$, with $(h(\psi_1, \ldots, \psi_n))P = h(\psi(P)) \neq 0$. Hence, by Definition 4.1.1, $\psi^* f$ is regular at P. □

Corollary 4.3.10 *Let X be an algebraic variety and $f : X \to k$ a function. Then $f \in \mathcal{O}_X(X)$ if and only if $f : X \to \mathbb{A}^1$ is a morphism of algebraic varieties.*

We have the following theorem, which is needed for the exercises below. The proof will be given in the next lecture, see Corollary 5.1.5.

Theorem 4.3.11 *Let $Y \subset \mathbb{A}^n$ be closed, $h \in k[x_1, \ldots, x_n]$, and let V be the intersection $Y \cap D(h)$. Then $(V, \mathcal{O}_Y|_V)$ is an affine variety, that is, isomorphic to a closed subset of some $\mathbb{A}^m$ with its regular functions.*

4.4 Sheaves of abelian groups

As we are using the notion of sheaf, it is a good idea to discuss this a bit more, and that is the goal of this section. For simplicity, we restrict ourselves (mostly) to the case of sheaves of abelian groups.

Definition 4.4.1 Let X be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on X is a collection of data consisting of

i. for each open subset $U \subset X$ an abelian group $\mathcal{F}(U)$, called the group of *sections* of $\mathcal{F}$ on U,

ii. for each inclusion $U \subset V$ between opens of X, a morphism of abelian groups $r_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$,

such that for $U \subset X$ open we have $r_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and for all $U, V, W \subset X$ open with $U \subset V \subset W$ we have $r_{W,U} = r_{V,U} \circ r_{W,V}$.

For X a topological space, $\mathcal{F}$ a presheaf of abelian groups on X, $U \subset V$ an inclusion of opens of X, and s in $\mathcal{F}(V)$, we will often write $s|_U$ for $r_{V,U}(s)$, and call it "the restriction of s from V to U."

Example 4.4.2 i. For any k-space X, hence for all X in Examples 4.3.2, the sheaf of regular functions is a presheaf.

ii. Let A be an abelian group and X a topological space. Define, for each open $U \subset X$, $A_X^{\mathrm{p}}(U) := A$, and for any inclusion $U \subset V$ of opens of X, let $r_{V,U}$ be the identity map $A \to A$. This presheaf A_X^{p} is called the *constant presheaf on X attached to A*.

iii. Let $X \subset \mathbb{A}^n$ be a subset equipped with the induced topology. Define, for each open $U \subset X$, $\mathcal{O}_X^{\mathrm{p}}(U)$ to be the ring of functions $f \colon U \to k$ such that there exists $g, h \in k[x_1, \ldots, x_n]$ such that h does not vanish on U and, for all $a \in U$, $f(a) = g(a)/h(a)$. For $U \subset V$ an inclusion of opens in X we let $r_{V,U}$ be the restriction map of functions. Then $\mathcal{O}_X^{\mathrm{p}}$ is a presheaf.

In categorical language, presheaves are contravariant functors, as follows. For a topological space X, we define its category $\mathrm{Open}(X)$ of its open subsets.

Definition 4.4.3 Let X be a topological space. Then $\mathrm{Open}(X)$ is the category whose objects are the open subsets of X. For U and V open subsets of X, $\mathrm{Hom}_{\mathrm{Open}(X)}(U, V)$ is the empty set if $U \not\subset V$, and the singleton set $\{j_{U,V}\}$, where $j_{U,V} \colon U \hookrightarrow V$ is the inclusion, if $U \subset V$. Composition of morphisms in $\mathrm{Open}(X)$ is composition of inclusions.

With this definition, to give a presheaf $\mathcal{F}$ on X is exactly the same as giving a contravariant functor $\mathcal{F}$ from $\mathrm{Open}(X)$ to the category of abelian groups: for every inclusion $j \colon U \to V$ in $\mathrm{Open}(X)$, the restriction map is $\mathcal{F}(j) \colon \mathcal{F}(V) \to \mathcal{F}(U)$. We will use the two notions interchangeably.

Definition 4.4.4 Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups on X. A *morphism* $f \colon \mathcal{F} \to \mathcal{G}$ is a natural transformation of functors from $\mathcal{F}$ to $\mathcal{G}$. Concretely, this means that for each U in $\mathrm{Open}(X)$, we have a morphism of abelian groups $f(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$, such that for all inclusions $j \colon U \subset V$ in $\mathrm{Open}(X)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \\ \big\downarrow{\scriptstyle \mathcal{F}(j)} & & \big\downarrow{\scriptstyle \mathcal{G}(j)} \\ \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \end{array}$$

commutes (where the vertical maps are the restriction mappings).

For X a topological space, the presheaves of abelian groups on X with their morphisms form a category. By replacing the notion of abelian group by rings, or k-vector spaces, or k-algebras, we obtain the categories of presheaves of rings, of k-vector spaces, of k-algebras, and so on. For example, if $(X, \mathcal{O}_X)$ is a k-space, then $\mathcal{O}_X$ is a presheaf of k-algebras.

Definition 4.4.5 Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups on X is called a *sheaf of abelian groups* if for all $U \subset X$ open, and all open covers $U = \bigcup_{i\in I} U_i$ the map

$$\mathcal{F}(U) \to \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) : \forall i, j \in I,\ s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\}, \quad s \mapsto (s|_{U_i})_{i\in I}$$

is a bijection. By definition, a morphism of sheaves is a morphism of presheaves. In particular the sheaves of abelian groups on X form a full subcategory of the category of presheaves of abelian groups on X.

Remark 4.4.6

i. The mapping in Definition 4.4.5 is a bijection if and only if it is injective and surjective. Injective means that two sections $s, t \in \mathcal{F}(U)$ are equal if and only if their restrictions to the given open cover are the same. Surjectivity and injectivity together mean that to give a section $s \in \mathcal{F}(U)$ it is equivalent to give sections $s_i \in \mathcal{F}(U_i)$, such that, for all i and j, s_i and s_j agree on $U_i \cap U_j$.

ii. Note that the empty set has the empty cover (the index set is empty). The empty product is the trivial group $\{0\}$, so for a sheaf $\mathcal{F}$ of abelian groups we have $\mathcal{F}(\emptyset) = \{0\}$.

   Let us clarify this a bit, set-theoretically. If I is a set, and, for every $i \in I$, A_i is an abelian group, the product $\prod_{i\in I} A_i = \{a\colon I \to \cup_{i\in I} A_i : \forall_{i\in I}\, a(i) \in A_i\}$. For $I = \emptyset$ this product has a unique element, the unique map $f\colon \emptyset \to \emptyset$ (whose graph is the empty set).

Example 4.4.7

i. For every k-space $(X, \mathcal{O}_X)$, the presheaf $\mathcal{O}_X$ is a sheaf (see Exercise 4.7.13).

ii. Let X be a non-empty topological space and A a non-trivial abelian group. Then the constant presheaf A_X^{p} on X attached to A is not a sheaf, see Exercise 4.7.14.

iii. Let X be a topological space, and A an abelian group. Then the *constant sheaf* with value A is the presheaf A_X given by $U \mapsto A_X(U) = \{s\colon U \to A : s \text{ is locally constant}\}$.

iv. Consider the closed variety $Z = Z(xy - zw) \subset \mathbb{A}^4$ and let X be the open subvariety $Z \cap (D(y) \cup D(w))$ of Z. Let $\mathcal{O}_X^{\mathrm{p}}$ be the presheaf of k-algebras on X from Example 4.4.2. Then $\mathcal{O}_X^{\mathrm{p}}$ is not a sheaf (see Example 4.1.2).

There is a natural way to attach to a presheaf of abelian groups a sheaf of abelian groups, and the easiest way to describe this is via stalks and germs.

For X a topological space, $\mathcal{F}$ a presheaf on X, and $x \in X$, we consider the colimit of the $\mathcal{F}(U)$ where U ranges over the neighbourhoods of x and where the morphisms $\mathcal{F}(U) \to \mathcal{F}(V)$ are the restriction maps. Here, it is not necessary to know what a colimit is, as the notion will be made explicit in this easy case where the index category is a directed set. But if you want to know, see the Mastermath "Intensive course on categories and modules."

Definition 4.4.8 Let X be a topological space, $\mathcal{F}$ a presheaf of abelian groups on X, and $x \in X$. Then we define the *stalk of $\mathcal{F}$ at x* as the quotient set

$$\mathcal{F}_x := \{(U, s) : x \in U \subset X \text{ open and } s \in \mathcal{F}(U)\}/\sim,$$

where $(U_1, s_1) \sim (U_2, s_2)$ if and only if there is an open neighbourhood U_3 of x contained in $U_1 \cap U_2$ such that $s_1|_{U_3} = s_2|_{U_3}$ in $\mathcal{F}(U_3)$. For $x \in U \subset X$ open and $s \in \mathcal{F}(U)$ we denote the image of (U, s) in $\mathcal{F}_x$ by s_x and call it the *germ of s at x*. If $\mathcal{F}$ is a presheaf of abelian groups (k-algebras,...) then $\mathcal{F}_x$ is an abelian group (a k-algebra,...).

Examples 4.4.9 i. Let $n \geq 1$, $X = \mathbb{A}^n$ and $x = 0$. Then $\mathcal{O}_{X,x}$ is the sub-k-algebra of $k(x_1, \ldots, x_n)$ consisting of the fractions f/g with f and g in $k[x_1, \ldots, x_n]$ with $g(0) \neq 0$.

ii. If $X = \mathbb{R}^n$ with the sheaf $\mathcal{C}^\infty_X$ of $\mathcal{C}^\infty$-functions, then $\mathcal{C}^\infty_{X,0}$ is called the $\mathbb{R}$-algebra of germs of smooth functions at the origin.

iii. If $X = \mathbb{C}$ with the sheaf $\mathcal{O}^{\mathrm{an}}_X$ of holomorphic complex functions, then $\mathcal{O}^{\mathrm{an}}_{X,0}$ is the $\mathbb{C}$-algebra of power series $\sum_{n\geq 0} a_n z^n$ with radius of convergence > 0 (this radius depends on the power series).

Proposition 4.4.10 *Let X be a topological space and let $\mathcal{F}$ be a presheaf of abelian groups on X. Then there exists a sheaf of abelian groups $\mathcal{F}^+$ on X and a morphism of presheaves $u\colon \mathcal{F} \to \mathcal{F}^+$, such that for every sheaf of abelian groups $\mathcal{G}$ and for every $f\colon \mathcal{F} \to \mathcal{G}$ there exists a unique $f^+\colon \mathcal{F}^+ \to \mathcal{G}$ making the triangle*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{u} & \mathcal{F}^+ \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^+} \\ & & \mathcal{G} \end{array}$$

commute.

Proof One way to construct such an $\mathcal{F}^+$ is as follows. For $U \subset X$ open one defines $\mathcal{F}^+(U)$ as the set of all collections

$$(s(x) \in \mathcal{F}_x)_{x \in U}$$

satisfying: for every $x \in U$ there is an open neighbourhood $V \subset U$ of x and a section $t \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s(y) = t_y$. From this construction, it is not hard to verify the universal property in the proposition. □

Definition 4.4.11 We call the morphism $u\colon \mathcal{F} \to \mathcal{F}^+$ a *sheafification* of $\mathcal{F}$. (Often the morphism u is suppressed from the notation.)

4.5 Local rings and function field

Definition 4.5.1 A ring is called a *local ring* if it has precisely one maximal ideal. For a local ring R, the maximal ideal is denoted $\mathfrak{m}_R$, and the field $R/\mathfrak{m}_R$, called the residue field, is denoted k_R.

Lemma 4.5.2 *A ring R is a local ring if and only if the set of non-units $R \setminus R^\times$ forms an ideal of R.*

Proof Assume that R is local with maximal ideal $\mathfrak{m}_R$. Then $x \in R$ is a unit if and only if $(x) = R$ if and only if (x) is not contained in $\mathfrak{m}_R$. For the converse, note that if $I \subsetneq R$ is a proper ideal in a ring, then $I \subset R \setminus R^\times$, so that if $R \setminus R^\times$ is an ideal it must be the unique maximal ideal. □

Definition 4.5.3 Let $(X, \mathcal{O}_X)$ be a variety and $P \in X$. The stalk $\mathcal{O}_{X,P}$ of X at P is called the *local ring* of X at P.

The stalk $\mathcal{O}_{X,P}$ is a k-algebra by construction. As the name suggests, it is indeed a local ring:

Proposition 4.5.4 *Let $(X, \mathcal{O}_X)$ be a variety and P a point of X. Then $\mathcal{O}_{X,P}$ is a local ring.*

Proof By definition we have

$$\mathcal{O}_{X,P} := \{(U, f) : U \subset X \text{ is open, } P \in U, \text{ and } f \in \mathcal{O}_X(U)\}/\sim,$$

where $(U, f) \sim (V, g)$ if and only if there is an open $W \subset U \cap V$ with $P \in W$ such that $f = g$ on W. We claim that sending each such pair (U, f) to $f(P)$ is compatible with this equivalence relation. Indeed, if $(U, f) \sim (V, g)$, then f and g have the same restriction to some neighbourhood of P, hence $f(P) = g(P)$. Hence we get a well-defined *evaluation map* $\mathcal{O}_{X,P} \to k$, sending $[(U, f)]$ (the equivalence class of (U, f)) to $f(P)$. This evaluation map is a ring homomorphism, and it is surjective (consider the constant functions). We let $\mathfrak{m}_{X,P}$ be its kernel, then $\mathfrak{m}_{X,P}$ is a maximal ideal. Moreover, if $[(U, f)] \notin \mathfrak{m}_{X,P}$, then $f(P) \neq 0$, and by Exercise 4.7.3 the element $[(U, f)] = [(U \setminus Z(f), f)]$ is invertible in $\mathcal{O}_{X,P}$. By Lemma 4.5.2 we conclude that $\mathcal{O}_{X,P}$ is local. □

For the student also taking the course on Commutative Algebra, we show that the local ring $\mathcal{O}_{X,P}$ can also be seen as a localisation (see [AM, §3]).

Proposition 4.5.5 *Let $(X, \mathcal{O}_X)$ be an affine variety and $P \in X$. Then $\mathcal{O}_{X,P}$ is the localisation of $\mathcal{O}_X(X)$ at the maximal ideal $\mathfrak{m} \subset \mathcal{O}_X(X)$ corresponding to P.*

Proof Consider the natural map $\mathcal{O}_X(X) \to \mathcal{O}_{X,P}$, $f \mapsto [(X, f)]$. If $f(P)$ is non-zero (that is, $f \notin \mathfrak{m}$), then there exists an open $P \in U$ with f non-zero on U, and hence $[(X, f)] = [(U, f)]$ is invertible in $\mathcal{O}_{X,P}$. It follows from the universal property of localisation [AM, Prop. 3.1] that the map factors over a ring homomorphism

$$\varphi \colon \mathcal{O}_X(X)_{\mathfrak{m}} \to \mathcal{O}_{X,P}.$$

We will show that this map is an isomorphism.

First recall that every element of $\mathcal{O}_X(X)_{\mathfrak{m}}$ is of the form f/g with $f, g \in \mathcal{O}_X(X)$ and $g(P) \neq 0$, and that $f/g = 0$ in $\mathcal{O}_X(X)_{\mathfrak{m}}$ if and only if there exists an $h \in \mathcal{O}_X(X)$ with $h(P) \neq 0$ and $fh = 0$. (Note that we have not assumed that $\mathcal{O}_X(X)$ is an integral domain).

We now show that the map φ is injective. Let $f, g \in \mathcal{O}_X(X)$ with $g(P) \neq 0$ and assume $\varphi(f/g) = 0$ in $\mathcal{O}_{X,P}$. Then there exists an $h \in \mathcal{O}_X(X)$ with $h(P) \neq 0$ such that g does not vanish on $D(h)$ and $f|_{D(h)} = 0$. In particular, $fh = 0$ everywhere on X. But since $h \notin \mathfrak{m}$, this implies that $f/g = 0$.

To show that φ is surjective, consider an element $[(U, f)]$ in $\mathcal{O}_{X,P}$. Without loss of generality, we may assume $X \subset \mathbb{A}^n$ for some n. Then by definition of regular function, there exist $g, h \in k[x_1, \dots, x_n]$ with $h(P) \neq 0$ and $f = g/h$ on some open $V \subset U$. Since $h \notin \mathfrak{m}$ we have that g/h defines an element of the localisation $\mathcal{O}_X(X)_{\mathfrak{m}}$, and $\varphi(g/h) = [(V, f)] = [(U, f)]$ in $\mathcal{O}_{X,P}$. □

A variety $(X, \mathcal{O}_X)$ is called *irreducible* if the topological space X is irreducible.

Definition 4.5.6 Let $(X, \mathcal{O}_X)$ be an irreducible variety. The *function field* of X is

$$K(X) := \{(U, f) : U \subset X \text{ open dense, and } f \in \mathcal{O}_X(U)\}/\sim,$$

where $(U, f) \sim (V, g)$ if and only if $f = g$ on $U \cap V$. Elements of $K(X)$ are called *rational functions* on X. For every dense open $U \subset X$, the k-algebra morphism $\mathcal{O}_X(U) \to K(X)$, that sends f to the equivalence class of (U, f), is injective. Therefore we will often simply write f for that image.

If U and V are open dense, then so is $U \cap V$. This allows us to equip $K(X)$ with the structure of a k-algebra, with for example multiplication defined by

$$[(U, f)] \cdot [(V, g)] = [(U \cap V, fg)].$$

In more categorical terms: $K(X) = \operatorname{colim}_U \mathcal{O}_X(U)$, where the colimit is taken in the category of k-algebras, and is indexed by the category of dense open subsets $U \subset X$ and inclusions.

Proposition 4.5.7 *Let X be an irreducible variety. Then $K(X)$ is a field.*

Proof Let $[(U, f)] \in K(X)$ be non-zero. Then U is irreducible and $Z(f) \subset U$ is a proper closed subset. Since X is irreducible, the open $V := U \setminus Z(f)$ is dense in X. By Excercise 4.7.3 we have that f^{-1} is a regular function on V and hence $[(U, f)] = [(V, f)]$ is invertible in $K(X)$. □

Proposition 4.5.8 *Let $(X, \mathcal{O}_X)$ be an irreducible affine variety. Then $K(X)$ is the fraction field of the integral domain $\mathcal{O}_X(X)$.*

Proof This follows similarly to the proof of Proposition 4.5.5. □

Proposition 4.5.9 *Let X be an irreducible variety and $U \subset X$ a non-empty open subset. Then the map*

$$K(X) \to K(U),\ [(V, f)] \mapsto [(V \cap U, f)]$$

is an isomorphism.

Note that U is irreducible, and that the intersection $V \cap U$ in the proposition is non-empty, and hence dense in U.

Proof Being a homomorphism of fields, the map is automatically injective. To see that it is surjective, observe that $[(V, f)] \in K(U)$ is the image of $[(V, f)] \in K(X)$. □

4.6 Dimension

Dimension theory in algebraic geometry is subtle, and a full treatment requires quite a bit more commutative algebra than we can assume in this course. The reader is encouraged to take the results of this section for granted, and come back to the underlying commutative algebra at a later stage in their studies.

Definition 4.6.1 If Y is an irreducible topological space, then the *dimension of* Y, denoted $\dim(Y)$, is the biggest integer m, such that there is a chain $Y = Y_m \supsetneq Y_{m-1} \supsetneq \cdots \supsetneq Y_0$ with $Y_i \subset Y$ irreducible and closed (in Y); if such an integer does not exist then $\dim(Y) := \infty$.

Example 4.6.2 $\dim \mathbb{A}^1 = 1$, since the longest chain is $\mathbb{A}^1 \supsetneq \{\mathrm{pt}\}$. Similarly $\dim \mathbb{P}^1 = 1$.

Definition 4.6.3 Let $k \subset K$ be a field extension. The *transcendence degree* of K over k is the largest number of k-algebraically independent elements of K.

Examples 4.6.4 The transcendence degree of an algebraic extension $k \subset K$ is zero. The transcendence degree of $\mathbb{Q} \subset \mathbb{R}$ is infinite. The transcendence degree of $k(x_1, \ldots, x_n)$ (the fraction field of $k[x_1, \ldots, x_n]$) over k is n.

Theorem 4.6.5 *Let X be an irreducible algebraic variety, then $\dim X$ equals the transcendence degree of $K(X)$ over k.*

Proof This follows from [AM, Theorem 11.25]. □

Corollary 4.6.6 $\dim \mathbb{A}^n = n$.

Corollary 4.6.7 *If X is an irreducible variety and $U \subset X$ is non-empty open, then $\dim U = \dim X$.*

Corollary 4.6.8 $\dim \mathbb{P}^n = n$.

Proposition 4.6.9 *Let $Y \subset \mathbb{A}^n$ be closed . Then Y is irreducible of dimension $n-1$ if and only if $Y = Z(f)$ for some irreducible $f \in k[x_1, \ldots, x_n]$.*

Proof Let $Y \subset \mathbb{A}^n$ be closed, irreducible, of dimension $n-1$. Then $I(Y)$ is a non-zero prime ideal of A and every non-zero prime ideal P of A with $P \subset I(Y)$ is equal to $I(Y)$. Let g be a non-zero element of $I(Y)$. Then g is not a unit (because $I(Y)$ is not A), and since $I(Y)$ is prime, some irreducible factor h of g must be contained in $I(Y)$. But then (h) is a non-zero prime ideal contained in $I(Y)$, and hence $(h) = I(Y)$, so that $Y = Z(h)$.

Conversely, let $f \in A := k[x_1, \ldots, x_n]$ be irreducible. Then (f) is prime (by unique factorisation), and hence $Y := Z(f)$ is irreducible. We need to show that $\dim Y = n-1$. Re-ordering the variables if necessary, we may assume that f contains a term divisible by x_n. By Gauss' lemma, the polynomial f is irreducible in $k(x_1, \ldots, x_{n-1})[x_n]$, and we have

$$K(Y) = k(x_1, \ldots, x_{n-1})[x_n]/(f).$$

In particular, the elements $x_1, \ldots, x_{n-1}$ of $K(Y)$ are algebraically independent over k, so that by Theorem 4.6.5 we have $\dim Y \geq n-1$. The dimension of Y cannot be greater than $n-1$, since that would imply $\dim \mathbb{A}^n > n$. We conclude Y has dimension $n-1$. □

Warning 4.6.10 One may be tempted to believe that something more general is true: that for every closed irreducible $Y \subset \mathbb{A}^n$ of dimension d there are $f_1, \ldots, f_{n-d} \in A$ so that $Y = Z((f_1, \ldots, f_{n-d}))$. This is *wrong*. A counter-example is given by

$$Y = \{(s^3, s^2t, st^2, t^3) \colon s, t \in k\} \subset \mathbb{A}^4.$$

We have that Y is closed, irreducible, of dimension 2 (verify this!). It can be defined using three equations, for example as

$$Y = Z(x_0x_3 - x_1x_2,\ x_1^2 - x_0x_2,\ x_2^2 - x_1x_3),$$

but one can show that it can not be defined using only two equations.

There is a similar result for closed subsets of $\mathbb{P}^n$:

Proposition 4.6.11 *Let $Y \subset \mathbb{P}^n$ be closed. Then Y is irreducible of dimension $n-1$ if and only if $Y = Z(f)$ for some irreducible homogeneous $f \in k[x_0, \ldots, x_n]$.*

Proof Similar to the proof of Proposition 4.6.9 □

Definition 4.6.12 A *hypersurface in* $\mathbb{A}^n$ is a closed irreducible subset $Y \subset \mathbb{A}^n$ of dimension $n-1$. A *hypersurface in* $\mathbb{P}^n$ is a closed irreducible subset $Y \subset \mathbb{A}^n$ of $\mathbb{P}^n$.

4.7 Exercises

Exercise 4.7.1 Let $n \in \mathbb{Z}_{\geq 0}$. For $f \in k[x_0, \ldots, x_n]$ homogeneous of some degree $d \geq 0$ let

$$D_+(f) := \{a \in \mathbb{P}^n \mid f(a) \neq 0\}.$$

Show that the set of all $D_+(f)$ is a basis for the topology on $\mathbb{P}^n$.

Exercise 4.7.2 Let $\mathrm{pt} = \mathbb{A}^0$. Let X be a variety. Show that all maps of sets $\mathrm{pt} \to X$ and $X \to \mathrm{pt}$ are morphisms.

Exercise 4.7.3 Let X be an algebraic variety and $U \subset X$ an open subset. Let $f \in \mathcal{O}_X(U)$ such that $f(P) \neq 0$ for all $P \in U$. Show that f is a unit in the ring $\mathcal{O}_X(U)$.

Exercise 4.7.4 Let X be a variety, and $U \subset X$ an open subset, equipped with the induced topology.

i. Show that $(U, \mathcal{O}_X|_U)$ is a k-space and that the inclusion map $j\colon U \to X$ is a morphism.

ii. Let $(Z, \mathcal{O}_Z)$ be a k-space and $f\colon Z \to U$ a map of sets. Show that f is a morphism if and only if $j \circ f$ is a morphism.

iii. Use Theorem 4.3.11 to show that $(U, \mathcal{O}_X|_U)$ is a variety.

We call U an *open subvariety* of X.

Exercise 4.7.5 Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be k-spaces, and $f\colon X \to Y$ a map of sets. Show that f is a morphism if and only if for each $x \in X$ there are open subsets $U \subset X$ and $V \subset Y$ such that $x \in U$, $fU \subset V$, and $f|_U\colon (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ is a morphism.

Exercise 4.7.6 Let $n \in \mathbb{Z}_{>0}$, and let $a_1, \ldots, a_n$ be distinct elements of k. Show that the set

$$\{x^i : i \in \mathbb{Z}_{\geq 0}\} \cup \{(x - a_j)^{-l} : j \in \{1, \ldots, n\} \text{ and } l \in \mathbb{Z}_{>0}\}$$

is a basis of the k-vector space $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 - \{a_1, \ldots, a_n\})$. Give also a basis for $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 - \{a_1, \ldots, a_n\})$. (Hint: prove that every regular function on a non-empty open subset of $\mathbb{A}^1$ is uniquely represented as g/h with g and h in $k[x]$ relatively prime and h monic.)

Exercise 4.7.7 Let X be a variety, and $Y \subset X$ a closed subset, equipped with the induced topology. For $V \subset Y$ open, $f\colon V \to k$, and $P \in V$, we define f to be regular at P if and only if there is an open $U \subset X$ and a $g \in \mathcal{O}_X(U)$ such that $P \in U$, and for all $Q \in V \cap U$, $f(Q) = g(Q)$. Notation: $\mathcal{O}_Y(V)$. Show that $(Y, \mathcal{O}_Y)$ is a variety and that the inclusion map $i\colon Y \to X$ is a morphism. We call Y a *closed subvariety* of X. Let $(Z, \mathcal{O}_Z)$ be a variety and $f\colon Z \to Y$ a map of sets. Show that f is a morphism if and only if $i \circ f$ is a morphism.

Exercise 4.7.8 Let X be an affine variety, and let Y be a closed subvariety (see Exercise 4.7.7).

i. Show that Y is affine. Hint: embed X into some $\mathbb{A}^n$ as a closed subvariety; then show that Y is a closed subvariety of $\mathbb{A}^n$.

ii. Show that the restriction map from $\mathcal{O}_X(X)$ to $\mathcal{O}_Y(Y)$ is surjective.

iii. Show that the map that sends Y to the kernel of the restriction map is a bijection from the set of closed subvarieties of X to the set of radical ideals of $\mathcal{O}_X(X)$.

Exercise 4.7.9 Let k be an algebraically closed field. Give an example of a k-space $(X, \mathcal{O}_X)$ such that not all $f \in \mathcal{O}_X(X)$ are continuous as maps from X to $\mathbb{A}^1$.

Exercise 4.7.10 Consider the projective algebraic variety $Y = Z_{\mathrm{proj}}(x_1^2 - x_0x_2) \subset \mathbb{P}^2$. Show that the function

$$\varphi\colon \mathbb{P}^1 \to Y,\ (a_0 : a_1) \mapsto (a_0^2 : a_0a_1 : a_1^2)$$

is an isomorphism of algebraic varieties. (Hint: show that it is a bijection, and then verify that both φ and φ^{-1} are morphisms by passing to standard affine opens, and using Proposition 4.3.9.)

Exercise 4.7.11 Let $X \subset \mathbb{A}^2$ be the zero set of the polynomial $x^2 - y^3$. Consider the map of sets $\varphi\colon \mathbb{A}^1 \to X$ given by $t \mapsto (t^3, t^2)$. Show that φ is a morphism of algebraic varieties. Show that φ is bijective. Show that φ is not an isomorphism of algebraic varieties.

Exercise 4.7.12 Give an example of two affine varieties X, Y and a morphism $\varphi\colon X \to Y$ such that the image of φ is not locally closed in Y. (A subset Z of a topological space is called *locally closed* if Z is the intersection of an open subset and a closed subset of X.)

Exercise 4.7.13 Show that for $(X, \mathcal{O}_X)$ a k-space, $\mathcal{O}_X$ is indeed a sheaf.

Exercise 4.7.14 Let X be a topological space and A a non-trivial abelian group. Show that A_X^{p} is not a sheaf.

Exercise 4.7.15 Show that the $\mathbb{R}$-algebra $\mathcal{C}^\infty_{\mathbb{R},0}$ is not an integral domain.

Exercise 4.7.16 Let X be a variety, $U \subset X$ an open subset and $P \in U$. Show that the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{U,P}$ are isomorphic. (Here U is considered as an algebraic variety as in Exercise 4.7.4).

Exercise 4.7.17 Let X be an irreducible variety and $P \in X$. Show that $\mathcal{O}_{X,P}$ is an integral domain, and that the function field $K(X)$ is isomorphic to the fraction field of $\mathcal{O}_{X,P}$.

Exercise 4.7.18 Let X be a topological space, and A an abelian group. Show that the natural morphism $i\colon A_X^{\mathrm{p}} \to A_X$ induces isomorphisms on stalks and that it is the sheafification.

Exercise 4.7.19 Let X be an irreducible topological space, and A an abelian group. Show that for all $U \neq \emptyset$ open in X, the natural morphism $i\colon A = A_X^{\mathrm{p}}(U) \to A_X(U)$ is an isomorphism.

Exercise 4.7.20 Let X be a closed subset of an affine space $\mathbb{A}^n$, let $\mathcal{O}_X^{\mathrm{p}}$ be the presheaf on X from Example 4.4.2(iii).

i. Show that for every open $U \subset X$ the algebra $\mathcal{O}_X^{\mathrm{p}}(U)$ is a subalgebra of $\mathcal{O}_X(U)$. Use these inclusions to construct a morphism of presheaves of k-algebras $i\colon \mathcal{O}_X^{\mathrm{p}} \to \mathcal{O}_X$.

ii. Show that for every $x \in X$ the map on stalks $i_x\colon \mathcal{O}_{X,x}^{\mathrm{p}} \to \mathcal{O}_{X,x}$ is a bijection.

iii. Conclude that $i\colon \mathcal{O}_X^{\mathrm{p}} \to \mathcal{O}_X$ is the (or “a”, if you prefer) sheafification of $\mathcal{O}_X^{\mathrm{p}}$.

Exercise 4.7.21 Let $\mathcal{F}$ be a sheaf of abelian groups on a space X. Let $\{U_1, U_2\}$ be an open cover of X. Show that the sequence

$$0 \to \mathcal{F}(X) \xrightarrow{\iota} \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \xrightarrow{\delta} \mathcal{F}(U_1 \cap U_2)$$

with $\iota(s) = (s|_{U_1}, s|_{U_2})$ and $\delta(s_1, s_2) = s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2}$ is exact.

Exercise 4.7.22 Let X be a topological space, let $\mathcal{F}$ be a sheaf of abelian groups on X, let $U \subset X$ be open and s and t be in $\mathcal{F}(U)$ such that s and t have the same germs at all $x \in U$: for all $x \in U$, $s_x = t_x$ in $\mathcal{F}_x$. Prove that $s = t$.

Exercise 4.7.23 Let X be a topological space, and let $x \in X$. Show that each morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups on X induces a morphism $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$, and that this gives a functor from the category of presheaves of abelian groups on X to the category of abelian groups.

Exercise 4.7.24 Let X be a topological space, and $\mathcal{F}$ and $\mathcal{G}$ sheaves of abelian groups on X, and $\varphi\colon \mathcal{F} \to \mathcal{G}$ a morphism, such that for all x in X the morphism $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism. Prove that φ is an isomorphism.

Exercise 4.7.25 Let X be a topological space, $\varphi\colon \mathcal{F} \to \mathcal{G}$ a morphism of presheaves of abelian groups, with $\mathcal{G}$ a sheaf, such that for every x in X the morphism $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism. Prove that the natural morphism $\psi\colon \mathcal{F}^+ \to \mathcal{G}$ is an isomorphism. Hence, in this case, $\varphi\colon \mathcal{F} \to \mathcal{G}$ is also a sheafification of $\mathcal{F}$.

Exercise 4.7.26 Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X.

i. Construct the presheaves of abelian groups $\ker(f)$ and $\operatorname{im}(f)$ by setting $\ker(f)(U)$ equal to the kernel of $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ and

$$\operatorname{im}(f)(U) := \{s \in \mathcal{G}(U) : \forall x \in U\, \exists V \subset U \text{ open with } x \in V\, \exists t \in \mathcal{F}(V) \text{ such that } f_V(t) = s|_V\}.$$

Show that $\ker(f)$ is a subsheaf of $\mathcal{F}$ and $\operatorname{im}(f)$ is a subsheaf of $\mathcal{G}$. [1] (A *subsheaf* $\mathcal{E}$ of $\mathcal{G}$ is a sheaf of abelian groups such that $\mathcal{E}(U) \subset \mathcal{G}(U)$ is an abelian subgroup and the restriction maps of $\mathcal{E}$ are induced by those from $\mathcal{G}$.)

ii. We call f *injective* when $\ker(f) = 0$, i.e. $\ker(f)$ is the zero sheaf and *surjective* when $\operatorname{im}(f) = \mathcal{G}$.

Consider a sequence of morphisms of sheaves of abelian groups $0 \to \mathcal{E} \stackrel{\varphi}{\to} \mathcal{F} \stackrel{\psi}{\to} \mathcal{G} \to 0$ on X. It is called *exact* when φ is injective, $\operatorname{im}(\varphi) = \ker(\psi)$, and ψ is surjective.

Prove that $0 \to \mathcal{E} \stackrel{\varphi}{\to} \mathcal{F} \stackrel{\psi}{\to} \mathcal{G} \to 0$ is exact if and only if $0 \to \mathcal{E}_P \stackrel{\varphi_P}{\to} \mathcal{F}_P \stackrel{\psi_P}{\to} \mathcal{G}_P \to 0$ is an exact sequence of abelian groups for all $P \in X$.

[1] One can also define the presheaf $\operatorname{im}(f)^{\mathrm{p}}$ of f by setting $\operatorname{im}(f)^{\mathrm{p}}(U) := f(U)(\mathcal{F}(U))$. Its sheafification is naturally isomorphic to the sheaf $\operatorname{im}(f)$ described in this exercise.

Lecture 5

The category of algebraic varieties

5.1 Affine varieties

Definition 5.1.1 A variety $(Y, \mathcal{O}_Y)$ is called *affine* if there is an $n \in \mathbb{Z}_{\geq 0}$ and $Z \subset \mathbb{A}^n$ closed such that $(Y, \mathcal{O}_Y) \cong (Z, \mathcal{O}_Z)$ where $\mathcal{O}_Z$ is the sheaf of regular functions on Z defined in § 4.1.

Suppose $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of k-spaces. Then we obtain a map φ^* from $\mathcal{O}_Y(Y)$ to $\mathcal{O}_X(X)$, $f \mapsto f \circ \varphi$. This φ^* is a morphism of k-algebras, for example, for every P in X,

$$(\varphi^*(f+g))P = (f+g)(\varphi P) = f(\varphi P) + g(\varphi P) = (\varphi^* f)P + (\varphi^* g)P = (\varphi^* f + \varphi^* g)P.$$

This procedure is a contravariant functor from the category k-Spaces to that of k-algebras, sending an object $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X)$, and a morphism $\varphi\colon X \to Y$ to $\varphi^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Indeed, for $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\psi\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ in k-Spaces, we get $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proposition 5.1.2 *Let X be a variety and Y an affine variety. Then the map*

$$\mathrm{Hom}_{\mathrm{Var}(k)}(X, Y) \to \mathrm{Hom}_{k\text{-algebras}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)), \quad \varphi \mapsto \varphi^*$$

is a bijection.

Proof We may and do assume that Y is a closed subset of $\mathbb{A}^n$, with its sheaf of regular functions, as Y is isomorphic to such a k-space. We construct an inverse of $\varphi \mapsto \varphi^*$. So let $h\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ be a k-algebra morphism. We have $k[x_1, \ldots, x_n] \to \mathcal{O}_Y(Y)$, surjective, and with kernel $I := I(Y)$, see Theorem 4.1.8. Let $\tilde{h}\colon k[x_1, \ldots, x_n] \to \mathcal{O}_X(X)$ be the composition of this morphism with h. Let $\psi_i := \tilde{h}(x_i)$. Let $\psi\colon X \to \mathbb{A}^n$ be the map $P \mapsto (\psi_1(P), \ldots, \psi_n(P))$.

We claim that $\psi(X)$ is contained in Y. As $Y = Z(I(Y))$, it suffices to show that for all $f \in I(Y)$, and all $P \in X$, $f(\psi(P)) = 0$. Now let $P \in X$. Consider the following diagram of morphisms of k-algebras

$$\begin{array}{ccc} k[x_1, \ldots, x_n] & \xrightarrow{\tilde{h}} & \mathcal{O}_X(X) \\ & \searrow^{\mathrm{eval}_{\psi(P)}} & \downarrow \mathrm{eval}_P \\ & & k, \end{array}$$

where $\mathrm{eval}_{\psi(P)}$ sends an $f \in k[x_1, \ldots, x_n]$ to its value at the point $\psi(P) \in \mathbb{A}^n$, and where eval_P sends a function $g \in \mathcal{O}_X(X)$ to its value at the point $P \in X$. For each i we have $\mathrm{eval}_P(\tilde{h}(x_i)) = \psi_i(P)$ and $\mathrm{eval}_{\psi(P)}(x_i) = \psi_i(P)$. Since the x_i generate the k-algebra $k[x_1, \ldots, x_n]$, this shows that the triangle commutes. But then, for any $f \in I(Y) \subset k[x_1, \ldots, x_n]$ we have

$$f(\psi(P)) = \mathrm{eval}_{\psi(P)}(f) = \mathrm{eval}_P(\tilde{h}(f)) = \mathrm{eval}_P(0) = 0,$$

which proves our claim.

By Proposition 4.3.9, the map ψ defines a morphism $\psi\colon X \to Y$.

We will now check that the two given maps between the Hom-sets are inverse to each other. We write ψ_h for the map $\psi : X \to Y$ obtained in the previous part of this proof for $h\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

Let $\varphi\colon X \to Y$ be a morphism in $\mathrm{Var}(k)$. Then we have, for all $P \in X$:

$$\psi_{\varphi^*}(P) = ((\varphi^* x_1)P, \ldots, (\varphi^* x_n)P) = (x_1(\varphi P), \ldots, x_n(\varphi P)) = \varphi(P).$$

This shows that $\psi_{\varphi^*} = \varphi$.

For h in $\mathrm{Hom}_{k-\mathrm{algebras}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ and $P \in X$ we have:

$$(\psi_h^* x_i)(P) = x_i(\psi_h P) = x_i((hx_1)P, \ldots, (hx_n)P) = (hx_i)P,$$

where we write x_i for the image of x_i under $k[x_1, \ldots, x_n] \to \mathcal{O}_Y(Y)$. Hence $(\psi_h)^*$ and h have the same value on each x_i, hence are equal (the x_i generate $\mathcal{O}_Y(Y)$). □

Let $(X, \mathcal{O}_X)$ be an affine variety, closed in some $\mathbb{A}^n$. Then $\mathcal{O}_X(X) = A(X)$ by Theorem 4.1.8. Hence the k-algebra $\mathcal{O}_X(X)$ is reduced and finitely generated.

Theorem 5.1.3 *The functor*

$$\begin{array}{rcl} \{\textit{affine varieties}\}^{\mathrm{opp}} & \to & \{\textit{reduced } k\textit{-algebras of finite type}\} \\ (X, \mathcal{O}_X) & \mapsto & \mathcal{O}_X(X) \\ \varphi & \mapsto & \varphi^* \end{array}$$

is an equivalence of categories.

Proof General category theory tells us: a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective. By Proposition 5.1.2 we see that the functor is fully faithful, and Exercise 1.6.17 shows that it is essentially surjective. □

This theorem tells us that "the only categorical difference between the two categories is the direction of the arrows".

We will now start proving Theorem 4.3.11.

Theorem 5.1.4 *Let $f \in k[x_1, \ldots, x_n]$. Then $(D(f), \mathcal{O}_{\mathbb{A}^n}|_{D(f)})$ is an affine variety.*

Proof Consider the closed subset $Z := Z(x_{n+1}f - 1) \subset \mathbb{A}^{n+1}$. Then we have the following maps:

$$D(f) \to Z, \quad (a_1, \ldots, a_n) \mapsto \left(a_1, \ldots, a_n, \frac{1}{f(a_1, \ldots, a_n)}\right)$$

and

$$Z \to D(f), \quad (a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n)$$

These maps are inverses of each other. By Proposition 4.3.9, the map $D(f) \to Z$ is a morphism since it is given by the regular functions $x_1, \ldots, x_n, 1/f$. Again by Proposition 4.3.9, the map $Z \to D(f)$ composed with the inclusion $j\colon D(f) \to \mathbb{A}^n$ is a morphism, because it is given by the regular functions $x_1, \ldots, x_n$. By Exercise 4.7.4, part 2, the map $Z \to D(f)$ is a morphism. So $D(f)$ is an affine variety and $\mathcal{O}_{D(f)}(D(f)) \cong k[x_1, \ldots, x_{n+1}]/(x_{n+1}f - 1)$. □

From this we obtain the following corollary, which was earlier stated as Theorem 4.3.11:

Corollary 5.1.5 *Let $X = Z(g_1, \dots, g_r) \subset \mathbb{A}^n$ be a closed subset, and let f be in $k[x_1, \dots, x_n]$. Then $(X \cap D(f), \mathcal{O}_{X\cap D(f)})$ is an affine variety isomorphic to $Z(g_1, \dots, g_r, x_{n+1}f - 1) \subset \mathbb{A}^{n+1}$ with its regular functions.*

Together with Exercise 1.6.11 we also conclude:

Corollary 5.1.6 *Every variety has a basis for the topology consisting of affine open subvarieties.*

5.2 Products of varieties

In this section we construct the product $X \times Y$ of varieties X and Y.

We will first construct products in the affine case. Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be closed. Let $I = I(X)$ and let $f_1, \dots, f_a$ in $k[x_1, \dots, x_m]$ be a system of generators. Similarly, let $J = I(Y)$ with system of generators $g_1, \dots, g_b$ in $k[y_1, \dots, y_n]$. We identify $\mathbb{A}^m \times \mathbb{A}^n$ with $\mathbb{A}^{m+n}$, and its coordinate ring with $k[x_1, \dots, x_m, y_1, \dots, y_n]$. We have

$$X \times Y = Z(f_1, \dots, f_a, g_1, \dots, g_b) \subset \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$$

so that $X \times Y$ is closed in $\mathbb{A}^{m+n}$. But in fact something stronger holds:

Lemma 5.2.1 *The ideal $I(X \times Y)$ in $k[x_1, \dots, x_m, y_1, \dots, y_n]$ is generated by $\{f_1, \dots, f_a, g_1, \dots, g_b\}$.*

Proof It suffices to show that the k-algebra morphism

$$\varphi\colon k[x_1, \dots, x_m, y_1, \dots, y_n]/(f_1, \dots, f_a, g_1, \dots, g_b) \longrightarrow A(X \times Y)$$

is an isomorphism. The morphism φ is surjective because the images of x_i and y_j generate $A(X \times Y)$. Let us show that φ is injective. We have

$$k[x_1, \dots, x_m, y_1, \dots, y_n]/(f_1, \dots, f_a) = A(X)[y_1, \dots, y_n].$$

Let h be in $A(X)[y_1, \dots, y_n]$ such that its image in $A(X \times Y)$ is zero. Write $h = \sum_{i=1}^{p} a_i h_i$ with a_i in $A(X)$ and h_i in $k[y_1, \dots, y_n]$. By choosing a basis of the sub-k-vector space of $A(X)$ generated by the a_i, and expressing the a_i as k-linear combinations of the elements of that basis, we may and do assume that the a_i are k-linearly independent. For each Q in Y we have $0 = \sum_{i=1}^{p} a_i h_i(Q)$ in $A(X)$, implying that for each Q in Y and each i we have $h_i(Q) = 0$ in k. So, for each i, h_i is in $I(Y)$, and therefore h is in the ideal of $A(X)[y_1, \dots, y_n]$ that is generated by $g_1, \dots, g_b$. □

Definition 5.2.2 For closed subvarieties $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ as above, we let $\mathcal{O}_{X\times Y}$ be the sheaf of regular functions on $X \times Y$ induced from those on $\mathbb{A}^{m+n}$. This makes $X \times Y$ into an affine variety.

Example 5.2.3 Consider $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$. Note that the Zariski topology is larger than the product topology. For example, the diagonal in $\mathbb{A}^2$ is not closed in the product topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$.

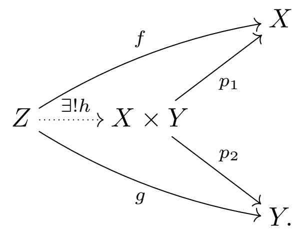
Remark 5.2.4 In the situation of Definition 5.2.2 we have:

$$\mathcal{O}_{X\times Y}(X \times Y) = k[x_1, \dots, x_m, y_1, \dots, y_n]/(f_1, \dots, f_a, g_1, \dots, g_b) = \mathcal{O}_X(X) \otimes_k \mathcal{O}_Y(Y),$$

where $\otimes_k$ denotes the tensor product of k-algebras (see [AM, p. 30]). In particular, the tensor product on the right is reduced. This is not a priori obvious, and in fact if the field k is not perfect, then the tensor product of two reduced k-algebras need not be reduced!

Remark 5.2.5 The projections $p_1 \colon X \times Y \to X$, and $p_2 \colon X \times Y \to Y$ are morphisms. This follows from Proposition 4.3.9.

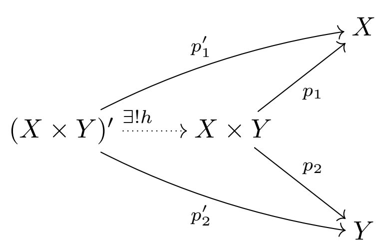
Theorem 5.2.6 *(Universal property of the product) Let X and Y be affine varieties and Z a variety. Let $f \colon Z \to X$ and $g \colon Z \to Y$ be morphisms. Then there exists a unique morphism $h \colon Z \to X \times Y$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$. This means that we have the following commutative diagram:*



Proof For h as a map of sets, there is a unique solution, namely for $P \in Z$ we set $h(P) = (f(P), g(P))$. This map is a morphism by Proposition 4.3.9. $\square$

Corollary 5.2.7 *The topology on $X \times Y$ and the sheaf $\mathcal{O}_{X \times Y}$ do not depend on the embeddings of X and Y in affine spaces.*

Proof Suppose we have a second product with the same universal property, say $(X \times Y)'$ with projections p_1' and p_2', obtained from a second pair of closed embeddings of X and Y in affine spaces. This means that $(X \times Y)'$ is, as a set, $X \times Y$, but with maybe another topology and another sheaf of regular functions. We apply the universal property in the following situation:



and conclude that the identity map of sets of $X \times Y$ to itself is a morphism of varieties from $(X \times Y)'$ to $X \times Y$. By symmetry, the same holds for the identity map of sets from $X \times Y$ to $(X \times Y)'$. $\square$

Corollary 5.2.8 *Let X and Y be affine varieties, and $U \subset X$ and $V \subset Y$ be affine open subvarieties. Then the subset $U \times V$ of $X \times Y$ is open, and the morphism of varieties $U \times V \to X \times Y$ induced by the inclusions is an isomorphism to the open subvariety $U \times V$ of $X \times Y$ (that is, the subset $U \times V$ equipped with the induced topology and the sheaf of functions $\mathcal{O}_{X \times Y}|_{U \times V}$).*

Proof We prove that the subset $U \times V$ of $X \times Y$ is open. Its complement is the union of $p_1^{-1}(X - U)$ and $p_2^{-1}(Y - V)$, and these two sets are closed because p_1 and p_2 are continuous.

We denote by $(U \times V)'$ the open subvariety of $X \times Y$ with subset $U \times V$. Let Z be any variety. By Exercise 4.7.4, to give a morphism $h \colon Z \to (U \times V)'$ is equivalent to giving a morphism $h \colon Z \to X \times Y$ with image contained in $U \times V$. By the universal property of $X \times Y$ and its two projectsions, and again Exercise 4.7.4 (applied to $U \subset X$ and to $V \subset Y$), this is equivalent to giving morphisms $f \colon Z \to U$ and $g \colon Z \to V$. Therefore, $(U \times V)'$, with its projections, has the same universal property as $U \times V$, and therefore the identity map between them is an isomorphism. $\square$

Now let X and Y be arbitrary varieties. We construct the product variety $X \times Y$ as follows. It does not matter how we do this, as long as the result satisfies the universal property of a product, because then it is unique up to unique isomorphism.

As a *set*, we take $X \times Y$.

The second step is to give $X \times Y$ a *topology*. Let T be the set of subsets W that are open in some $U \times V$ where $U \subset X$ and $V \subset Y$ are open and affine and where $U \times V$ has the Zariski topology as defined above.

We claim that for all W and W' in T the intersection $W \cap W'$ is an of element of T. Let W and W' be in T, and let (P, Q) be in $W \cap W'$. Let U and U' be affine opens of X, and V and V' be affine opens of Y, such that W is open in $U \times V$ and W' is open in $U' \times V'$. Then there are g in $\mathcal{O}_X(U)$ and h in $\mathcal{O}_Y(V)$ such that $P \in D(g) \subset U \cap U'$ and $Q \in D(h) \subset V \cap V'$, hence $(P, Q) \in D(g) \times D(h)$. By Corollary 5.2.8, $D(g) \times D(h)$ is an open subvariety of $U \times V$ as well as of $U' \times V'$, showing that $W \cap W' \cap (D(g) \times D(h))$ is an open neigborhood of (P, Q) in $U \times V$ as well as in $U' \times V'$. Hence $W \cap W'$ is open in $U \times V$ (and in $U' \times V'$), and therefore an element of T.

Hence the unions of subsets of T are the open sets for a topology on $X \times Y$ that we call the Zariski topology, and T is a basis for that topology.

The third step is to provide the sheaf of *regular functions*. We only need to do this on the basis T above (since a function is regular if and only if it is locally regular). A function $W \to k$ (with W as above) is regular if it is regular as a function on W as open subset of $U \times V$ with U open affine in X and V open affine in Y. We show that this does not depend on U and V as long as $U \times V$ contains W. The question is local at the points of W. So let (P, Q) be in W, U, U' affine opens in X containing P, and V, V' affine opens in Y containing Q. Then above we have produced g in $\mathcal{O}_X(U)$ and h in $\mathcal{O}_Y(V)$ such that $D(g) \times D(h)$ is an open subvariety of $U \times V$ as well as of $U' \times V'$. This means that $U \times V$ and $U' \times V'$ induce the same topology and the same regular functions on opens of $D(g) \times D(h)$.

Theorem 5.2.9 *The projections p_1 and p_2 are morphisms and the product $X \times Y$ with its projections has the universal property (as in the affine case).*

Proof Apply Exercise 4.7.5 to see that we only need to prove it locally. The local case follows by Theorem 5.2.6. ☐

Theorem 5.2.10 *The product of projective varieties is a projective variety.*

Proof Exercise 5.5.13. ☐

5.3 Complete varieties

Complete varieties are analogues in algebraic geometry of compact spaces in topology. The definition is motivated by the following theorem in topology, which we will not prove (but see Exercise 5.5.10 for one direction).

Theorem 5.3.1 *A topological space X is compact if and only if for every topological space Y the projection $\pi\colon X \times Y \to Y$ is closed.*

Recall that a map of topological spaces is called closed if it maps closed subsets to closed subsets.

Definition 5.3.2 An algebraic variety X is *complete* if for every variety Y the projection $\pi\colon X \times Y \to Y$ is a closed map.

The product $X \times Y$ in the definition is the product in the category of algebraic varieties. In particular, $X \times Y$ does *not* have the product topology (unlike the product of topological spaces in Theorem 5.3.1).

Example 5.3.3 $X = \mathbb{A}^1$ is *not* complete. For example, take $Y = \mathbb{A}^1$ and consider the closed subset $Z = \{(a,b) \in X \times Y \mid ab = 1\}$ of $X \times Y \cong \mathbb{A}^2$. Then $\pi(Z) = \{b \in Y \mid b \neq 0\}$, which is not closed.

The main result of this section is the following theorem.

Theorem 5.3.4 *Projective varieties are complete.*

Before giving the proof, we give an important consequence.

Corollary 5.3.5 *Let X be a projective variety and $f\colon X \to k$ a regular map. Then f is locally constant.*

In particular, if X is connected and projective, then $\mathcal{O}_X(X) = k$.

Proof By Corollary 4.3.10, the map f defines a morphism $f\colon X \to \mathbb{A}^1$. Using either Proposition 4.3.9 or the universal property of the product, one shows that

$$g\colon X \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1,\ (P,Q) \mapsto (f(P),Q)$$

is a morphism of varieties. The subset $\Delta \subset \mathbb{A}^1 \times \mathbb{A}^1$ is closed (it is given by the equation $x_1 = x_2$ on $\mathbb{A}^2$), and hence by continuity of the morphism g also its inverse image

$$Z := g^{-1}(\Delta) = \{(P, f(P))\colon P \in X\} \subset X \times \mathbb{A}^1$$

is closed. By Theorem 5.3.4 the image of Z under $\pi\colon X \times \mathbb{A}^1 \to \mathbb{A}^1$ is closed. But we have $\pi(Z) = f(X)$, so we conclude that the image of $f\colon X \to \mathbb{A}^1$ is closed. In particular: either $f(X) = \mathbb{A}^1$ or $f(X)$ is finite and discrete.

We claim that the image of f cannot be $\mathbb{A}^1$. Indeed, consider the map $f'\colon X \to \mathbb{P}^1$, $P \mapsto (f(P) : 1)$. The same argument as above (with $\mathbb{A}^1$ replaced by $\mathbb{P}^1$) shows that also the image of f' is closed. But if the image of f is $\mathbb{A}^1$, then the image of f' is $U_1 \subset \mathbb{P}^1$, which is not closed.

We have shown that the image of $f\colon X \to \mathbb{A}^1$ is finite and discrete. Since f is continuous, it follows that f is locally constant. □

The remainder of this chapter is devoted to proving Theorem 5.3.4. The essential point is the following special case.

Proposition 5.3.6 *Let Y be an affine variety, and let $n \geq 0$. Then the projection $\pi\colon \mathbb{P}^n \times Y \to Y$ is a closed map.*

Proof Let $A = A(Y)$ be the coordinate ring of Y, and consider the ring $A[x_0, \dots, x_n]$. Denote by $A[x_0, \dots, x_n]_d$ the subspace of polynomials that are homogeneous of degree d (in the x_i). Note that an $f \in A[x_0, \dots, x_n]_d$ has a well-defined zero-locus $Z(f) \subset \mathbb{P}^n \times Y$.

Let $Z \subset \mathbb{P}^n \times Y$ be a closed subset. Consider the homogeneous ideal $I = \oplus_d I_d \subset A[x_0, \dots, x_n]$, where

$$I_d := \big\{f \in A[x_0, \dots, x_n]_d : Z \subset Z(f)\big\}.$$

Then one has $Z = Z(I)$ (verify this!).

Now let $P \in Y \backslash \pi(Z)$, and let $\mathfrak{m} = \mathfrak{m}_P \subset A$ be the corresponding maximal ideal. Then $\mathfrak{m}A[x_0, \dots, x_n]$ is a homogeneous ideal in $A[x_0, \dots, x_n]$, and $Z(\mathfrak{m}A[x_0, \dots, x_n]) = \pi^{-1}(P)$.

We claim that there is an $N > 0$ such that $x_i^N \in I + \mathfrak{m}A[x_0, \dots, x_n]$ for all i.

Fix an i and consider the standard open $U_i \subset \mathbb{P}^n$. Then $U_i \times Y \subset \mathbb{P}^n \times Y$ is affine with coordinate ring $A[x_{ij}\colon j \neq i]$. The closed subset $Z_i := Z \cap U_i \times Y$ is given by the ideal $I(Z_i)$ generated by the polynomials

$$f(x_0/x_i, \dots, x_n/x_i) \in A[x_{ij}\colon j \neq i], \quad f \in I,$$

where we identify x_{ij} with x_j/x_i.

By our assumption on P we have that Z_i and $U_i \times \{P\}$ are disjoint, and hence

$$A[x_{ij} \colon j \neq i] = I(Z_i) + \mathfrak{m}A[x_{ij} \colon j \neq i]$$

as ideals in $A[x_{ij} \colon j \neq i]$. In particular, we have

$$1 = f(x_0/x_i, \dots, x_n/x_i) + \sum_t m_t a_t$$

for some $f \in I$, $m_t \in \mathfrak{m}$ and $a_t \in A[x_{ij} \colon j \neq i]$. Multiplying by $x_i^{N_i}$ for a suitably large N_i to clear denominators, we find $x_i^{N_i} \in I + \mathfrak{m}A[x_0, \dots, x_n]$.

Taking $N = \max_i N_i$, we find $x_i^N \in I + \mathfrak{m}A[x_0, \dots, x_n]$ for all i, as we had claimed.

Now choose an integer $d \geq (n+1)N$. Then any monomial of degree d in the variables x_i is divisibly by x_i^N for some i, and therefore contained in $I + \mathfrak{m}A[x_0, \dots, x_n]$. We conclude that

$$A[x_0, \dots, x_n]_d = I_d + \mathfrak{m}A[x_0, \dots, x_n]_d.$$

Now consider the finitely generated A-module $M := A[x_0, \dots, x_n]_d/I_d$ (generated by the classes of the monomials of degree d). The above identity implies $\mathfrak{m}M = M$ and Nakayama's Lemma [AM, Cor. 2.5] implies that there is an $h \in 1 + \mathfrak{m} \subset A$ such that $hM = 0$, or equivalently, such that $hA[x_0, \dots, x_n]_d \subset I_d$. In particular, $hx_i^d \in I_d$ for all i. Since $\cap_i Z(x_i) = \emptyset$, we find that Z is contained in the zero locus of h (considered as a homogeneous element of $A[x_0, \dots, x_n]$ of degree 0). This implies that $Y \setminus \pi(Z)$ contains the open neighbourhood $D(h) \subset Y$ of P, and hence $\pi(Z)$ is closed. □

We can now finish the proof of Theorem 5.3.4.

Proof Assume that the projective variety X is embedded as a closed subset in $\mathbb{P}^n$. Then any closed subset $Z \subset X \times Y$ is closed in $\mathbb{P}^n \times Y$, and it suffices to show its image under $\mathbb{P}^n \times Y \to Y$ is closed. So without loss of generality we may assume $X = \mathbb{P}^n$.

By Lemma 4.1.3, to show that the image of some closed subset $Z \subset \mathbb{P}^n \times Y$ in Y is closed, it suffices to show that every $P \in Y$ has an open neighbourhood U such that the image of $Z \cap \mathbb{P}^n \times U$ in U is closed. Using Corollary 5.1.6 we see that we may further reduce to the case where Y is affine, which was settled in the preceding proposition. □

5.4 Separated varieties

Separated varieties are analogues in algebraic geometry of Hausdorff spaces in topology. Their definition is motivated by the following lemma in topology.

Lemma 5.4.1 *Let X be a topological space, and $\Delta \subset X \times X$ be the diagonal, that is, Δ is the subset $\{(x,x) : x \in X\} \subset X \times X$. We give $X \times X$ the product topology. Then X is Hausdorff if and only if $\Delta \subset X \times X$ is closed.*

Proof Let $x, y \in X$ with $x \neq y$. Then $(x,y) \notin \Delta$ has an open neighbourhood U with $U \cap \Delta = \emptyset$ if and only if there are $V \subset X$, $W \subset X$ open with $x \in V, y \in W$ with $V \times W \cap \Delta = \emptyset$ (since the sets of the form $V \times W$ with $V, W \subset X$ open form a basis of the product topology). Note that $(V \times W) \cap \Delta = \emptyset$ if and only if $V \cap W = \emptyset$. □

Definition 5.4.2 A variety X is *separated* if $\Delta = \{(P,P) : P \in X\}$ is closed in $X \times X$ (product of varieties).

The name 'separated' comes from the French *séparé* for Hausdorff (in topology).

Proposition 5.4.3 *Affine varieties are separated.*

Proof Let X be a closed subset of $\mathbb{A}^n$. Note that $\Delta_{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{A}^n$ is closed, since it is the zero set of $(x_1 - y_1, \ldots, x_n - y_n)$. Therefore, also $\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$, and hence closed in $X \times X$. □

In fact, something much more general holds:

Proposition 5.4.4 *Quasi-projective varieties are separated.*

Proof See Exercise 5.5.13. □

See Exercise 5.5.16 for an example of a variety which is not separated.

Proposition 5.4.5 *Let X be a separated variety, and let U and $V \subset X$ be open and affine. Then $U \cap V$ is open and affine.*

Proof Consider the following diagram:

$$\begin{array}{ccccc} U \cap V & \xrightarrow{\sim} & (U \times V) \cap \Delta_X & \subset & X \times X \\ \cap & & \cap & & \\ X & \xrightarrow{\sim} & \Delta_X & \subset & X \times X. \end{array}$$

The map from $X \to \Delta_X$ sends a point P to (P, P), and one can show that this is an isomorphism (using the universal property, use the identity morphisms on X and for the inverse use a projection). This isomorphism restricts to an isomorphism on $U \cap V \to (U \times V) \cap \Delta_X$. Now $(U \times V) \cap \Delta_X$ is closed in the affine variety $U \times V$, hence it is affine. □

Example 5.4.6 Here is a counter example to Proposition 5.4.5 in case you drop the assumption that X is separated: Glue two copies X_0, X_1 of the affine plane $\mathbb{A}^2$ along $X_0 \supset \mathbb{A}^2 \backslash \{(0,0)\} \subset X_1$ (see Excercises 5.5.14 and 5.5.16), to obtain the affine plane where the origin is doubled. In this variety X_0, X_1 are affine opens, and their intersection equals $\mathbb{A}^2 \backslash \{(0,0)\}$. The variety $\mathbb{A}^2 \backslash \{(0,0)\}$ is not affine.

5.5 Exercises

Exercise 5.5.1 Show that $\mathbb{P}^n$ is not affine if $n > 0$. (Use Corollary 5.3.5.)

Exercise 5.5.2 Let $f\colon X \to Y$ be a morphism of affine varieties and assume that the corresponding morphism of k-algebras $f^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is surjective.

i. Show that $\ker(f^*)$ is a radical ideal in $\mathcal{O}_Y(Y)$.

ii. Let Z be the closed subvariety of Y that corresponds to the ideal $\ker(f^*)$ under the correspondence of Exercise 4.7.8. Show that $f^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ induces an isomorphism $\mathcal{O}_Z(Z) \to \mathcal{O}_X(X)$.

iii. Show that f factors as $i \circ f'$ with $i\colon Z \to Y$ the inclusion and $f'\colon X \to Z$ an isomorphism. Hint: use Theorem 5.1.3.

Exercise 5.5.3 Let $f\colon X \to Y$ be a morphism of affine varieties and assume that the corresponding morphism of k-algebras $f^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective. Show that fX is dense in Y. Give an example with $fX \neq Y$.

Exercise 5.5.4 Consider the open subvariety $X = \mathbb{A}^2 - \{0\}$ of $\mathbb{A}^2$. Denote the embedding by $i\colon X \to \mathbb{A}^2$. We denote the coordinate ring of $\mathbb{A}^2$ by $k[x_1, x_2]$.

i. Consider the open subsets $U_1 = D(x_1) \subset X$ and $U_2 = D(x_2) \subset X$. Verify that $X = U_1 \cup U_2$ and show that $\mathcal{O}_X(X) = \mathcal{O}_X(U_1) \cap \mathcal{O}_X(U_2)$, with the intersection taken inside $K(X)$.

ii. Show that $\mathcal{O}_X(X) = k[x_1, x_2]$ and that $i^*\colon \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_X(X)$ is an isomorphism of k-algebras.

iii. Deduce that X is not an affine variety.

Exercise 5.5.5 Assume $\mathrm{char}(k) \neq 2$.

i. Give an isomorphism between $\mathbb{P}^1$ and $Z(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$.

ii. Parametrise all rational solutions to the equation $x^2 + y^2 = z^2$.

iii. Parametrise all integer solutions to the equation $x^2 + y^2 = z^2$.

Exercise 5.5.6 Let q and n be positive integers.

i. Show that
$$f\colon \mathbb{P}^n \to \mathbb{P}^n, \quad (a_0 : \cdots : a_n) \mapsto (a_0^q : \cdots : a_n^q)$$
is a morphism of varieties.

ii. Assume from here on that k has characteristic $p > 0$ and that $q = p^d$ for some integer $d > 0$. Show that f is bijective but not an isomorphism of varieties.

iii. Find all $P \in \mathbb{P}^n$ such that $f(P) = P$.

Exercise 5.5.7 Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be closed subsets and let $f\colon X \to Y$ be a morphism of affine varieties. Show that f extends to a morphism of varieties $\overline{f}\colon \mathbb{A}^m \to \mathbb{A}^n$. Note that this an analogue of the "Tietze extension theorem" from topology. The analogous statement for projective varieties does not hold: in a future exercise you are asked to give closed subsets $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ for some m, n and a morphism $f\colon X \to Y$ of varieties such that f does not extend to a morphism of varieties $\overline{f}\colon \mathbb{P}^m \to \mathbb{P}^n$.

Exercise 5.5.8 Let X and Y be varieties.

i. Let $P \in X$. Show that the map $Y \to X \times Y$, $Q \mapsto (P, Q)$ is a morphism.

ii. Assume $\mathcal{O}_X(X) = k$. Show that $\mathcal{O}_{X\times Y}(X \times Y) \cong \mathcal{O}_Y(Y)$.

Exercise 5.5.9 Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ and $S \subset \mathbb{A}^\ell$ be closed subsets. Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of algebraic varieties.

i. Show that the subset
$$X \times_S Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$
of $\mathbb{A}^{n+m}$ is closed (hint: extend f and g to maps $\bar{f}$ and $\bar{g}$ as in Exercise 5.5.7). This gives $X \times_S Y$ the structure of an affine algebraic variety.

ii. Show that the projections $p\colon X \times_S Y \to X$ and $q\colon X \times_S Y \to Y$ are morphisms of algebraic varieties.

iii. Show that $X \times_S Y$ is the fibre product of f and g in the category of algebraic varieties. In other words, show that for every variety Z and for all morphisms $\varphi\colon Z \to X$, $\psi\colon Z \to Y$ satisfying $f \circ \varphi = g \circ \psi$ there is a unique morphism $h\colon Z \to X \times_S Y$ with $p \circ h = \varphi$ and $q \circ h = \psi$.

Exercise 5.5.10 Let X be a compact topological space, and Y an arbitrary topological space. Show that the projection $\pi\colon X \times Y \to Y$ is closed.

Exercise 5.5.11 Show that the product of two complete varieties is complete. Show that a closed subvariety of a complete variety is complete.

Exercise 5.5.12 Show that $\mathbb{A}^n$ is complete if and only if $n = 0$.

Exercise 5.5.13 Let $\Psi\colon \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{mn-1}$ be the Segre map (of sets):

$$((a_1 : \cdots : a_m), (b_1 : \cdots : b_n)) \mapsto (a_1 b_1 : \cdots : a_m b_n).$$

Let $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ be closed.

i. Show that Ψ is a morphism of varieties.

ii. Show that $\Psi(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$ is closed in $\mathbb{P}^{mn-1}$.

iii. Show that Ψ is an isomorphism from the product variety $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ to the projective variety $\Psi(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$.

iv. Show that Ψ restricts to an isomorphism from the product variety $X \times Y$ to the projective variety $\Psi(X \times Y)$.

v. Show that the diagonal $\Delta_{\mathbb{P}^{n-1}}$ is closed in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

vi. Show that quasi-projective varieties are separated.

Exercise 5.5.14 Let X_1 and X_2 be varieties and $U_1 \subset X_1$ and $U_2 \subset X_2$ open subvarieties. Let $\varphi\colon U_1 \tilde{\to} U_2$ be an isomorphism of varieties.

Let X be the topological space obtained from X_1 and X_2 by identifying U_1 with U_2 using the homeomorphism φ. Formally $X = (X_1 \coprod X_2)/\sim$ where $\sim$ is the smallest equivalence relation satisfying $x \sim \varphi(x)$ for all $x \in U_1$.

Let $\iota_1\colon X_1 \to X$ and $\iota_2\colon X_2 \to X$ be the natural maps. For an open subset $U \subset X$, define $\mathcal{O}_X(U)$ as the ring of functions $f\colon U \to k$ such that $f \circ \iota_1 \in \mathcal{O}_{X_1}(\iota_1^{-1}U)$ and $f \circ \iota_2 \in \mathcal{O}_{X_2}(\iota_2^{-1}U)$.

Show that $(X, \mathcal{O}_X)$ is an algebraic variety.

Exercise 5.5.15 In Exercise 5.5.14, take $X_1 = X_2 = \mathbb{A}^1$ and $U_1 = U_2 = \mathbb{A}^1 - \{0\}$. Consider the 'glueing map'

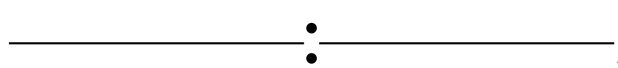
$$\varphi\colon U_1 \tilde{\to} U_2,\ a \mapsto a^{-1}.$$

Show that the resulting variety X is isomorphic to $\mathbb{P}^1$.

Exercise 5.5.16 In Exercise 5.5.14, take $X_1 = X_2 = \mathbb{A}^1$ and $U_1 = U_2 = \mathbb{A}^1 - \{0\}$. Consider the 'glueing map'

$$\varphi\colon U_1 \tilde{\to} U_2,\ a \mapsto a.$$

The resulting algebraic variety X is an 'affine line with two origins':


.

Show that X is a non-separated variety.

Exercise 5.5.17 Show that the product of two separated varieties is separated.

Exercise 5.5.18 Let $f\colon X \to Y$ be a morphism of varieties, and let

$$Z := \{(x, f(x)) \mid x \in X\} \subset X \times Y$$

be its graph.

i. Assume that Y is separated. Show that Z is closed.

ii. Assume that Y is separated and that X is complete. Show that f is closed.

Lecture 6

Smoothness, tangent spaces and 1-forms

6.1 Smooth varieties

To define smoothness, we need the concept of partial derivatives of polynomials. For n in $\mathbb{Z}_{\geq 0}$ and f in $k[x_1, \ldots, x_n]$ the partial derivatives $\partial f/\partial x_i$ in $k[x_1, \ldots, x_n]$ are defined formally, that is, the partial derivatives $\partial/\partial x_i \colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ are k-linear, satisfy the Leibniz rule and satisfy $\partial(x_j)/\partial x_i = 1$ if $j = i$ and is zero otherwise. For example, for $m \in \mathbb{Z}_{\geq 0}$, $\partial(x_1^m)/\partial x_1 = m x_1^{m-1}$ (which is 0 if $m = 0$). This is a purely algebraic operation on $k[x_1, \ldots, x_n]$, and there is no need to take limits of any kind. But note that in characteristic p we have $\partial(x^p)/\partial x = p x^{p-1} = 0$.

Our definition of smoothness is inspired by the implicit function theorem from real analysis, which more or less states that the zero set of $n - d$ smooth functions on $\mathbb{R}^n$ can be described as the graph of a function of the last d variables when the Jacobian matrix with respect to the first $n-d$ variables is invertible.

Theorem 6.1.1 (Implicit function theorem) *Let $U \subset \mathbb{R}^n$ be a Euclidean open subset and let $f_1, \ldots, f_{n-d} : U \to \mathbb{R}$ be C^∞-functions. Let $p \in U$ be such that*

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i,j \leq n-d}$$

is invertible. Then there exist

- *open subsets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^{n-d}$ such that $p \in B \times A \subset U$*
- *a C^∞-function $h = (h_1, \ldots, h_{n-d}) : A \to B$*

such that for all $(x_1, \cdots, x_n) \in B \times A$ we have that

$$f_i(x_1, \cdots, x_n) = 0 \text{ for all } 1 \leq i \leq n-d \iff x_i = h_i(x_{n-d+1}, \ldots, x_n) \text{ for all } 1 \leq i \leq n-d.$$

Back to algebraic varieties over k.

Definition 6.1.2 Let X be a variety. For P in X, X is *smooth* at P if there is an open subvariety U of X containing P and an isomorphism $\varphi\colon U \tilde{\to} Z(f_1, \ldots, f_{n-d}) \subset \mathbb{A}^n$ for some $d \leq n$ and $f_1, \ldots, f_{n-d}$, such that the rank of the $n - d$ by n matrix over k

$$\left(\frac{\partial f_i}{\partial x_j}(\varphi P)\right)_{i,j}$$

equals $n - d$, i.e. is maximal. The variety X is *smooth* if it is smooth at all its points. Points of X where X is not smooth are called *singular* and we denote the subset of singular points of X by X_{sing}.

Although Definition 6.1.2 coincides with our intuitive notion of smoothness, and often can be used to show that a point of a variety is smooth, it is not very useful for showing that a point of a variety is singular. The latter requires us to prove that that the Jacobian matrix does not have rank $n-d$ on *any* affine chart $\varphi\colon U \xrightarrow{\sim} Z(f_1,\ldots,f_{n-d}) \subset \mathbb{A}^n$. We will assume the following two theorems on smoothness.

Theorem 6.1.3 *Let $P \in X$ be a point of a variety that lies on two irreducible components of X. Then P is a singular point of X.*

Example 6.1.4 In each of the following examples, $X \subset \mathbb{A}^n$ is reducible with two smooth irreducible components Y_1, Y_2. By Theorem 6.1.3, $X_{\text{sing}} = Y_1 \cap Y_2 \subset X$.

(a) $X = Z(xy) \subset \mathbb{A}^2$. We see that X is the union of the x and y axes, and it appears to have a 1-dimensional tangent space at all points except at the origin (where it is 2-dimensional). Later we will see more about the connection between the tangent space and smoothness (Proposition 6.2.4). The Jacobian matrix is (y, x), which has rank 1 on $X \setminus \{(0,0)\}$ and rank 0 on $\{(0,0)\}$.

(b) $X = Z(xy, yz) \subset \mathbb{A}^3$. This time X is the union of the plane $Z(y)$ and the line $Z(x,z)$ (in particular, X has two irreducible components of different dimensions). The Jacobian matrix is given by

$$\begin{pmatrix} y & x & 0 \\ 0 & z & y \end{pmatrix},$$

which has rank 1 on $Z(y) \setminus \{(0,0,0)\}$, rank 2 on $Z(x,z) \setminus \{(0,0,0)\}$, and rank 0 on $\{(0,0,0)\}$.

Theorem 6.1.5 *Let $X \subset \mathbb{A}^n$ be closed, irreducible and of dimension d and let $P \in X$. Suppose $I(X) = (f_1,\ldots,f_r)$. Then $\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j}$ has rank $\leq n-d$ with equality if and only if X is smooth at P. Moreover, X has a smooth point.*

Example 6.1.6 The affine space $\mathbb{A}^d$ is smooth. Indeed, it is given by zero equations as subset of $\mathbb{A}^d$.

Example 6.1.7 In each of the following examples, $X = Z(f) \subset \mathbb{A}^n$ with $f \in k[x_1,\ldots,x_n]$ an irreducible polynomial. Hence X is irreducible of codimension 1 by Proposition 4.6.9. By Theorem 6.1.5, we have

$$X_{\text{sing}} = Z\Big(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big) \subset X.$$

i. $X = Z(y^2 - x^2(1+x)) \subset \mathbb{A}^2$. Then $X_{\text{sing}} = \{(0,0)\}$ is called a *node*.

ii. $X = Z(y^2 - x^3) \subset \mathbb{A}^2$. Then $X_{\text{sing}} = \{(0,0)\}$ is called a *cusp*.

iii. $X = Z(x^2 + y^2 - z^2) \subset \mathbb{A}^3$. Then $X_{\text{sing}} = \{(0,0,0)\}$ is called an *ordinary double point*.

iv. $X = Z(x^2 + y^2 z^3 - z^4) \subset \mathbb{A}^3$. Then $X_{\text{sing}} = Z(x,z)$ is 1-dimensional.

v. $X = Z(xw - yz) \subset \mathbb{A}^4$. Then $X_{\text{sing}} = \{(0,0,0,0)\}$ is called a *conifold singularity*.

In the examples above, we saw that the set of singular points forms a strict closed subset. This is true in general by the following corollary of Theorem 6.1.5.

Corollary 6.1.8 *Let X be a non-empty variety, then $X_{\text{sing}} \subset X$ is a strict closed subset.*

Proof By Lemma 4.1.3, it suffices to show that every $P \in X_{\mathrm{sing}}$ has an open neighbourhood U such that $X_{\mathrm{sing}} \cap U$ is closed in U. We may without loss of generality assume that X is affine, and in particular that X has only finitely many irreducible components, say $X_1, \ldots, X_n$.

Then we have

$$X_{\mathrm{sing}} = \bigcup_{i \neq j} (X_i \cap X_j) \cup \bigcup_i X_{i,\mathrm{sing}}.$$

Indeed, if a point $P \in X$ lies on more than one component then it is a singular point of X by Theorem 6.1.3. In the case that P lies on a unique component X_i, then P has an open neighbourhood $U \subset X$ with $U \subset X_i$, and with Exercise 6.7.2 we conclude that P is a smooth point of X if and only if it is a smooth point of X_i.

The intersections $X_i \cap X_j$ are closed in X, so to prove the Corollary it suffices to show that $X_{i,\mathrm{sing}}$ is closed in X_i. In other words, we have reduced the problem to the case of an affine irreducible X.

So we may take $X \subset \mathbb{A}^n$ closed and irreducible of dimension d with $I(X) = (f_1, \ldots, f_r)$. By Theorem 6.1.5,

$$X_{\mathrm{sing}} = \left\{ P \in X \,:\, \mathrm{rk} \left(\frac{\partial f_i}{\partial x_j}(P) \right)_{i,j} < n - d \right\}$$

in particular, X_{sing} is the common zero locus of the f_i and all $(n-d) \times (n-d)$ minors of $\left(\frac{\partial f_i}{\partial x_j}(P) \right)_{i,j}$. Hence $X_{\mathrm{sing}} \subset X$ is closed (and strict because smooth points exist by Theorem 6.1.5). $\square$

6.2 Tangent spaces of embedded affine varieties

In this section we give a rather geometric and intuitive definition of tangent space at a point $P \in X$ of a closed subset X of $\mathbb{A}^n$. In the following sections, we will translate this into a more algebraic definition, and use this to show that our definition does not depend on the embedding of X in an affine space, and to generalize the definition to arbitrary varieties.

So let X be a closed subset of $\mathbb{A}^n$, with ideal $I \subset A = k[x_1, \ldots, x_n]$. Let P be a point of X.

Let $v \in k^n$ and $f \in I$. Then the function $\lambda \mapsto f(P + \lambda v)$ is a polynomial in λ that vanishes at $\lambda = 0$.

Definition 6.2.1 The *tangent space* of X at P is

$$T_X(P) := \{ v \in k^n : \forall f \in I,\ \lambda \mapsto f(P + \lambda v) \text{ has order} \geq 2 \text{ at } 0 \}.$$

Intuitively, the tangent consists of those 'directions' v for which the line $\{P + \lambda v : \lambda \in k\}$ is 'tangent' to X.

The Taylor series of $f(P + \lambda v)$ as a function of λ starts as

$$f(P + \lambda v) = f(P) + \lambda \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(P) \cdot v_j + \cdots,$$

from which we see that the tangent space coincides with

$$T_X(P) = \{ v \in k^n : \forall f \in I,\ \sum_j \frac{\partial f}{\partial x_j}(P) \cdot v_j = 0 \}.$$

In particular, $T_X(P)$ is a sub-vector space of k^n.

Lemma 6.2.2 *Assume $I = (f_1, \ldots, f_r)$. Then*

$$T_X(P) = \{ v \in k^n : \forall i,\ \sum_j \frac{\partial f_i}{\partial x_j}(P) \cdot v_j = 0 \}.$$

Proof Clearly $T_X(P)$ is contained in the right hand side. For the reverse inclusion, let v be such that $\sum_j (\partial f_i/\partial x_j)(P)\cdot v_j = 0$ for all i. Let $f = \sum s_i f_i \in I$. Using the product rule, and the vanishing of the f_i at P, we find

$$\frac{\partial f}{\partial x_j}(P) = \sum_i s_i(P)\frac{\partial f_i}{\partial x_j}(P),$$

and hence

$$\sum_j \frac{\partial f}{\partial x_j}(P)v_j = \sum_i s_i(P)\sum_j \frac{\partial f_i}{\partial x_j}(P)v_j = 0.$$

We conclude that $v \in T_X(P)$. □

Example 6.2.3 Assume that $k \not\supset \mathbb{F}_2$. Let $X = Z(x^2+y^2-z^2) \subset \mathbb{A}^3$; note that $x^2+y^2-z^2$ is irreducible, so $I(X) = (x^2+y^2-z^2)$. It is a good idea to make a drawing of X: it is a cone. Let $P = (a,b,c) \in X$. Then we obtain:

$$T_X(P) = \{(u,v,w) \in k^3 : 2au + 2bv - 2cw = 0\}.$$

So $\dim T_X(P) = 2$ if $P \neq 0$, and $\dim T_X(0) = 3$.

Proposition 6.2.4 *Assume that $X \subset \mathbb{A}^n$ is a closed irreducible subset of dimension d. Then $\dim T_X(P) \geq d$ with equality if and only if X is smooth at P.*

Proof From Lemma 6.2.2 we see that

$$T_X(P) = \ker\left(\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \cdots & \frac{\partial f_1}{\partial x_n}(P) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}(P) & \cdots & \frac{\partial f_r}{\partial x_n}(P)\end{pmatrix} : k^n \to k^r\right)$$

and the proposition then follows from Theorem 6.1.5. □

6.3 Intrinsic definition of the tangent space

In this section we will get a description of the tangent space of an embedded affine variety that does not depend on the embedding; it will depend only on the coordinate ring. The notation is as in Definition 6.2.1. We let $\mathfrak{m} = \mathfrak{m}_a \subset A$ be the maximal ideal of $a = (a_1,\dots,a_n) \in X \subset \mathbb{A}^n$, so $\mathfrak{m} = (x_1-a_1,\dots,x_n-a_n)$.

We start by considering the map:

$$\langle\cdot,\cdot\rangle : \mathfrak{m}\times T_{\mathbb{A}^n}(a) \to k, \quad (f,v) \mapsto \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j}\right)(a)\cdot v_j.$$

Lemma 6.3.1 *The map $\langle\cdot,\cdot\rangle$ is bilinear and induces a perfect pairing $\langle\cdot,\cdot\rangle : \mathfrak{m}/\mathfrak{m}^2 \times T_{\mathbb{A}^n}(a) \to k$ of k-vector spaces ("perfect" means that each side is identified with the dual of the other side).*

Proof The map $(f,v) \mapsto \langle f,v\rangle$ is visibly linear in f and in v, and hence is bilinear. Now $\langle\cdot,\cdot\rangle$ gives a map $\mathfrak{m} \to T_{\mathbb{A}^n}(a)^\vee$, $f \mapsto \langle f,\cdot\rangle$. The kernel of this map is $\{f \in \mathfrak{m} : \forall i,\ (\partial f/\partial x_i)(a) = 0\}$. By translation, we may assume that $a = 0$. Let f be in the kernel. We write $f = \sum_i f_i$, with f_i homogeneous of degree i. Since $f(0) = 0$, the constant term f_0 is zero and since all the partial derivatives at 0 vanish, f_1 is zero as well. This shows that $f \in (x_1,\dots,x_n)^2 = \mathfrak{m}^2$. So we have an injection $\mathfrak{m}/\mathfrak{m}^2 \to T_{\mathbb{A}^n}(0)^\vee$. Note that $T_{\mathbb{A}^n}(0) = k^n$ and that $(\overline{x_1},\dots,\overline{x_n})$ is a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, so since the dimensions agree, our map is surjective and hence it is an isomorphism. □

Let $B := A/I = \mathcal{O}_X(X)$, let $\overline{\mathfrak{m}} = (\overline{x_1 - a_1}, \ldots, \overline{x_n - a_n})$ be the maximal ideal in B of a. In what follows we use the bijection between the set of ideals of B and the set of ideals of A containing I, given by sending an ideal of B to its inverse image in A, and sending an ideal in A containing I to its image in B. For example, $\overline{\mathfrak{m}}$ is the image in B of $\mathfrak{m}$, and $\mathfrak{m}$ is the inverse image in A of $\overline{\mathfrak{m}}$. The image of $\mathfrak{m}^2$ in B equals $\overline{\mathfrak{m}}^2$. So $I + \mathfrak{m}^2$ is an ideal of A that contains I and whose image in B is $\overline{\mathfrak{m}}^2$, hence it is the inverse image in A of $\overline{\mathfrak{m}}^2$. The Snake Lemma (or the statement that, for $J_2 \subset J_1$ ideals of A containing I, $A \to B$ induces an isomorphism $J_1/J_2 \to \overline{J_1}/\overline{J_2}$) shows that $A \to B$ induces an isomorphism of k-vector spaces $\mathfrak{m}/(I + \mathfrak{m}^2) \to \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$:

$$
\begin{array}{ccccc}
I & \hookrightarrow & I + \mathfrak{m}^2 & \twoheadrightarrow & \overline{\mathfrak{m}}^2 \\
\| & & \downarrow & & \downarrow \\
I & \hookrightarrow & \mathfrak{m} & \twoheadrightarrow & \overline{\mathfrak{m}} \\
 & & \downarrow & & \downarrow \\
 & & \mathfrak{m}/(I + \mathfrak{m}^2) & \xrightarrow{\simeq} & \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 .
\end{array}
\tag{6.3.2}
$$

Proposition 6.3.3 *The pairing $\langle \cdot, \cdot \rangle$ of Lemma 6.3.1 induces a perfect pairing $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \times T_X(a) \to k$.*

Proof By Lemma 6.3.1, we have the perfect pairing $\langle \cdot, \cdot \rangle : \mathfrak{m}/\mathfrak{m}^2 \times T_{\mathbb{A}^n}(a) \to k$. By definition:

$$T_X(a) = \{v \in k^n : \langle \overline{f}, v \rangle = 0 \text{ for all } \overline{f} \in (I + \mathfrak{m}^2)/(\mathfrak{m}^2) \subset \mathfrak{m}/\mathfrak{m}^2.\}$$

So we get a perfect pairing between $T_X(a)$ and the quotient $(\mathfrak{m}/\mathfrak{m}^2)/((I + \mathfrak{m}^2)/\mathfrak{m}^2) = \mathfrak{m}/(I + \mathfrak{m}^2)$, which is $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ by the isomorphism above. Here we have used that if $\langle \cdot, \cdot \rangle$ is a perfect pairing between finite dimensional k-vector spaces V and W, and V' is a subspace of V, then we get an induced perfect pairing between V/V' and $V'^{\perp} = \{w \in W : \forall v \in V', \langle v, w \rangle = 0\}$. □

Lemma 6.3.4 *Let X be an affine variety, $P \in X$, and let $U = D(f)$ be a principal affine open neighbourhood of P. Let $\mathfrak{m} \subset \mathcal{O}_X(X)$ and $\mathfrak{n} \subset \mathcal{O}_X(U)$ be the maximal ideals corresponding to P. Then the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$.*

Proof Since $f - f(P) \in \mathfrak{m}$, we have that multiplication by f on the $\mathcal{O}_X(X)$-module $\mathfrak{m}/\mathfrak{m}^2$ is given by multiplication by $f(P)$. In particular, multiplication by f on $\mathfrak{m}/\mathfrak{m}^2$ is invertible.

We first show that $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is surjective. Let $g \in \mathfrak{n}$. Then there exists some $n \geq 0$ such that $g = \tilde{g}/f^n$ for $\tilde{g} \in \mathfrak{m}$. But then the class of $\tilde{g}/f(P)^n$ in $\mathfrak{m}/\mathfrak{m}^2$ is mapped to the class of g in $\mathfrak{n}/\mathfrak{n}^2$.

For injectivity, let $g \in \mathfrak{m}$ and assume that its image lands in $\mathfrak{n}^2$. Then $g = \sum s_i t_i$ for some s_i, t_i in $\mathfrak{n}$. For n sufficiently large, we have that $s_i = \tilde{s}_i/f^n$ and $t_i = \tilde{t}_i/f^n$. We find that $g = \sum \tilde{s}_i \tilde{t}_i / f(P)^{2n}$, which lies in $\mathfrak{m}^2$. □

Remark 6.3.5 A more conceptual way to phrase the above argument uses localisation of rings. The $\mathcal{O}_X(X)$-modules $\mathfrak{n}$ and $\mathfrak{n}^2$ are the localisation of $\mathfrak{m}$ and $\mathfrak{m}^2$ at the multiplicative system $S = \{f^n : n \geq 0\}$. Since localisation is exact, we find that $\mathfrak{n}/\mathfrak{n}^2$ is the localisation of $\mathfrak{m}/\mathfrak{m}^2$ at S. But the elements of S act as isomorphisms on $\mathfrak{m}/\mathfrak{m}^2$, so the localisation map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is an isomorphism.

Corollary 6.3.6 *Let X be an affine variety and $P \in X$. Let $\mathcal{O}_{X,P}$ be the local ring of X at P, and $\mathfrak{m}_{X,P} \subset \mathcal{O}_{X,P}$ its maximal ideal. Then there is a natural isomorphism $T_X(P) \to (\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2)^{\vee}$ of k-vector spaces.*

Proof This can be shown using the previous lemma, and using the fact that the principal affine open neighbourhoods of P form a neighbourhood basis of P. □

This motivates the following definition of the tangent space at a point on an arbitrary variety.

Definition 6.3.7 For X a variety, $P \in X$, we define $T_X(P) = (\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2)^\vee$.

Definition 6.3.8 Let $\varphi\colon X \to Y$ be a morphism of varieties, and let P be in X. Then the k-algebra morphism $\varphi^*\colon \mathcal{O}_{Y,\varphi(P)} \to \mathcal{O}_{X,P}$ induces a k-linear map $\varphi^*\colon \mathfrak{m}_{Y,\varphi(P)}/\mathfrak{m}_{Y,\varphi(P)}^2 \to \mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$. The dual of this map is the *tangent map of φ at P*

$$T_\varphi(P)\colon T_X(P) \longrightarrow T_Y(\varphi(P)).$$

This behaves well for composition of varieties: if $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are morphisms of varieties, and $P \in X$, then the following diagram commutes:

$$\begin{array}{ccc} T_X(P) & \xrightarrow{T_\varphi(P)} & T_Y(\varphi(P)) \\ & \searrow^{T_{\psi\circ\varphi}(P)} & \downarrow{\scriptstyle T_\psi(\varphi(P))} \\ & & T_Z(\psi(\varphi(P))). \end{array}$$

6.4 Derivations and differentials

Definition 6.4.1 Let A be a k-algebra and M an A-module. A *derivation* is a k-linear map $D\colon A \to M$ such that for all $f, g \in A$ the 'Leibniz identity'

$$D(fg) = fD(g) + gD(f)$$

holds. We denote the set of those derivations by $\mathrm{Der}(A, M)$.

If D is a derivation then $D(1) = D(1\cdot 1) = 1\cdot D(1) + 1\cdot D(1)$. Hence $D(1) = 0$ and by k-linearity we see for $c \in k$ that $D(c) = 0$.

Example 6.4.2 Let X be an affine variety, $A := \mathcal{O}_X(X)$, and let $P \in X$. Then the map "evaluation at P", $A \to k$, $f \mapsto f(P)$, makes $k = A/\mathfrak{m}_P$ into an A-module. For every $v \in T_X(P)$ the map

$$\partial_v\colon A \to k,\ f \mapsto \langle \overline{f - f(P)}, v\rangle$$

is a derivation (Exercise 6.7.9), and the map

$$T_X(P) \to \mathrm{Der}(A, k),\ v \mapsto \partial_v$$

is an isomorphism (Exercise 6.7.9).

Example 6.4.3 Let A be a k-algebra, and let M and N be A-modules. If $D\colon A \to M$ is a derivation, and $f\colon M \to N$ is a morphism of A-modules, then $f \circ D\colon A \to N$ is again a derivation.

Proposition 6.4.4 *Let A be a k-algebra. There exists an A-module Ω_A^1 and a derivation $d\colon A \to \Omega_A^1$ such that for any A-module M and any derivation $D\colon A \to M$ there exists a unique A-linear map φ making the following diagram commute:*

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega_A^1 \\ \downarrow{\scriptstyle D} & \swarrow_{\varphi} & \\ M & & \end{array}$$

Since it is given by a universal property, the pair (Ω^1_A, d) is uniquely determined (up to unique isomorphism). We will call Ω^1_A the *module of Kähler differentials* of A.

Proof Let N be the free A-module with basis the symbols da for all a in A: $N = \bigoplus_{a \in A} Ada$. Let $N' \subset N$ be the submodule generated by the relations $d(\lambda a) = \lambda{\cdot}d(a)$, $d(a+b) = d(a) + d(b)$ and $d(ab) = a \cdot db + b \cdot da$ for all $a, b \in A, \lambda \in k$. We claim that we can take Ω^1_A to be N/N' with d which sends a to $\overline{da} \in N/N'$. Indeed one easily checks that $(N/N', d)$ satisfies the universal property. □

Example 6.4.5 For $A = k[x_1, \ldots, x_n]$ we have

$$\Omega^1_A = \{\sum_{i=1}^n f_i dx_i : f_i \in k[x_1, \ldots, x_n]\}.$$

It is a free $k[x_1, \ldots, x_n]$-module with basis $(dx_1, \ldots, dx_n)$. See Exercise 6.7.12.

More generally:

Example 6.4.6 For $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ we have

$$\Omega^1_A = \left(\bigoplus_{i=1}^n A \cdot dx_i \right) / \left(A{\cdot}df_1 + \cdots + A{\cdot}df_r\right)$$

where $df_i = \sum_j (\partial f_i / \partial x_j) dx_j$. Hence Ω^1_A is presented as follows:

$$A^r \xrightarrow{J^t \cdot} A^n \to \Omega^1_A \to 0, \quad \text{where} \quad J = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_r/\partial x_1 & \cdots & \partial f_r/\partial x_n \end{pmatrix}.$$

See Exercise 6.7.13.

Lemma 6.4.7 *Let $\varphi\colon A \to B$ be a morphism of k-algebras, M a B-module, and $D\colon B \to M$ a derivation. Then $D \circ \varphi\colon A \to M$ is a derivation, where M is considered as an A-module via φ.*

Proof See Exercise 6.7.8 □

In particular we have a unique A-linear map $\Omega^1(\varphi)$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle d_A} & \searrow^{d_B \circ \varphi} & \downarrow{\scriptstyle d_B} \\ \Omega^1_A & \xrightarrow[\Omega^1(\varphi)]{} & \Omega^1_B\,. \end{array}$$

The A-linearity of $\Omega^1(\varphi)\colon \Omega^1_A \to \Omega^1_B$ means that for all $a \in A$ and for all $\omega \in \Omega^1_A$ we have

$$(\Omega^1(\varphi))(a\omega) = \varphi(a)(\Omega^1(\varphi))(\omega)\,. \tag{6.4.8}$$

For morphisms of k-algebras $\varphi_1\colon A_1 \to A_2$ and $\varphi_2\colon A_2 \to A_3$ one has

$$\Omega^1(\varphi_2 \circ \varphi_1) = \Omega^1(\varphi_2) \circ \Omega^1(\varphi_1)\,. \tag{6.4.9}$$

So Ω^1 attaches to each k-algebra A an A-module Ω^1_A, functorially in the sense of (6.4.8) and (6.4.9).

6.5 Differential 1-forms on varieties

Definition 6.5.1 Let X be a variety. The *sheaf of* 1*-forms* Ω^1_X on X is the sheafification of the presheaf $U \mapsto \Omega^1_{\mathcal{O}_X(U)}$.

Its main property is the following proposition, which we state without proof.

Proposition 6.5.2 *If $U \subset X$ is open affine, then $\Omega^1_X(U) = \Omega^1_{\mathcal{O}_X(U)}$. Moreover, for affine open subsets $V \subset U \subset X$, we have $\Omega^1_{\mathcal{O}_X(V)} \cong \Omega^1_{\mathcal{O}_X(U)} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ and the restriction $\Omega^1_X(U) \to \Omega^1_X(V)$ is given by $\omega|_V = \omega \otimes 1$.*

From this, one can in principle compute $\Omega^1_X(U)$ for an arbitrary open U. If X is separated, one can proceed as follows. Choose an open affine cover $(U_i)_i$ of U and set $U_{ij} := U_i \cap U_j$. By Proposition 5.4.5, the U_{ij} are themselves open affine. Now the sheaf property implies that

$$\Omega^1_X(U) = \left\{ \left(\omega_i \in \Omega^1_{\mathcal{O}_X(U_i)}\right)_{i \in I} : \forall i, j,\ \omega_j|_{U_{ij}} = \omega_i|_{U_{ij}} \text{ in } \Omega^1_{\mathcal{O}_X(U_{ij})} \right\}.$$

Example 6.5.3 Let $n \in \mathbb{Z}_{\geq 2}$, $X = Z(-y^n + x^{n-1} - 1) \subset \mathbb{A}^2$ and suppose that $n(n-1) \in k^\times$. Let $A := \mathcal{O}_X(X) = k[x, y]/(f)$ where $f = -y^n + x^{n-1} - 1$. Then:

$$\Omega^1_A = (A \cdot dx \oplus A \cdot dy) \,/\, \left(-ny^{n-1}dy + (n-1)x^{n-2}dx\right) .$$

On $D(y) \subset X$ we have: $dy = \frac{n-1}{n} \frac{x^{n-2}}{y^{n-1}} dx$, so $\Omega^1(D(y))$ is free over $\mathcal{O}_X(D(y))$ with basis dx. Similarly, on $D(x) \subset X$ we have $dx = \frac{n}{n-1} \frac{y^{n-1}}{x^{n-2}} dy$, so $\Omega^1(D(x))$ is free over $\mathcal{O}_X(D(x))$ with basis dy.

In fact, we even have that $\Omega^1_X(X)$ is free over $\mathcal{O}_X(X)$, see Exercise 6.7.14.

We end with the definition of rational 1-forms on irreducible varieties.

Definition 6.5.4 Let X be an irreducible variety. Then $K(X)$ is a field, as well as a k-algebra, and so we have the $K(X)$-vector space $\Omega^1_{K(X)}$, whose elements are called rational 1-forms on X. One can show that $\Omega^1_{K(X)}$ is the set of equivalence classes of (U, ω) with $U \subset X$ non-empty open and $\omega \in \Omega^1_X(U)$, where two pairs (U, ω) and (U', ω') are equivalent if and only if $\omega|_{U \cap U'} = \omega'|_{U \cap U'}$.

6.6 Locally free $\mathcal{O}_X$-modules and the Picard group

Definition 6.6.1 Let X be a k-space. A *presheaf of $\mathcal{O}_X$-modules* is a presheaf of abelian groups $\mathcal{F}$ on X, together with, for each open $U \subset X$, a structure of $\mathcal{O}_X(U)$-module on $\mathcal{F}(U)$ (with the given addition), such that the restriction maps are compatible with the module structure. A *sheaf of $\mathcal{O}_X$-modules* is a presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ that is a sheaf. A morphism of presheaves of $\mathcal{O}_X$-modules from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of presheaves of abelian groups $\varphi \colon \mathcal{F} \to \mathcal{G}$ such that, for all open $U \subset X$, $\varphi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. We denote the set of such morphisms by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, which has a natural $\mathcal{O}_X(X)$-module structure.

Example 6.6.2 For $r \geq 0$, the sheaf $\mathcal{O}_X^r$ defined by $\mathcal{O}_X^r(U) := \mathcal{O}_X(U)^r$ is naturally a sheaf of $\mathcal{O}_X$-modules.

Example 6.6.3 An *ideal sheaf* on a k-space X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{I}$ such that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for any open subset $U \subset X$. Suppose $S \subset X$ is a subset, then

$$\mathcal{I}_S(U) := \{ f \in \mathcal{O}_X(U) \,:\, f(p) = 0 \, \forall p \in S \cap U \}$$

defines an ideal sheaf.

Example 6.6.4 If X is an algebraic variety, then Ω^1_X is naturally a sheaf of $\mathcal{O}_X$-modules.

Example 6.6.5 If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $U \subset X$ is open, then $\mathcal{F}|_U$ is naturally a sheaf of $\mathcal{O}_U$-modules.

Definition 6.6.6 Let X be a k-space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *locally free of finite rank* if for each P in X there is an open $U \subset X$ containing P, an integer $r \geq 0$, and an isomorphism of $\mathcal{O}_U$-modules $\mathcal{O}_U^r \to \mathcal{F}|_U$. Note that the number r is constant on each connected component of X and is called the *rank* of $\mathcal{F}$ on that connected component. Locally free sheaves, which have rank 1 on each connected component, are called *invertible sheaves*.

The next proposition is stated without proof.

Proposition 6.6.7 *Let X be an irreducible algebraic variety of dimension d. Assume that X is smooth. Then Ω^1_X is locally free of rank d.*

Our next goal is to introduce the Picard group of a k-space X. For this, we first need to discuss duals and tensor products.

Proposition 6.6.8 *Let X be a k-space and $\mathcal{F}$ and $\mathcal{G}$ sheaves of $\mathcal{O}_X$-modules. Then the presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ defined by*

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \{\varphi\colon \mathcal{F}|_U \to \mathcal{G}|_U \mid \varphi \text{ is a morphism of sheaves of } \mathcal{O}_U\text{-modules}\}$$

(with the obvious restriction maps) is naturally a sheaf of $\mathcal{O}_X$-modules itself.

Proof See Exercise 6.7.18 □

Definition 6.6.9 Now let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. Then we define its *dual sheaf* to be $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Definition 6.6.10 Let X be a k-space and $\mathcal{E}, \mathcal{F}$ sheaves of $\mathcal{O}_X$-modules. Consider the presheaf $\mathcal{E} \otimes^{\mathrm{p}} \mathcal{F}$, which sends an open subset $U \subset X$ to the $\mathcal{O}_X(U)$-module $\mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$. We denote its sheafification by $\mathcal{E} \otimes \mathcal{F}$.

The following lemma will be used several times in this section.

Lemma 6.6.11 *Let $\varphi\colon \mathcal{E} \to \mathcal{F}$ be a morphism of presheaves of abelian groups on a topological space X. Suppose there exists an open cover $\{U_i\}_{i \in I}$ such that $\varphi|_{U_i}$ is an isomorphism for all $i \in I$. Then the induced morphism $\varphi^+\colon \mathcal{E}^+ \to \mathcal{F}^+$ between the sheafifications is an isomorphism. In particular, if $\mathcal{E}$ and $\mathcal{F}$ are sheaves, then φ is an isomorphism.*

Proof See Exercise 6.7.19 □

Duals and tensor products of invertible sheaves have nice properties.

Lemma 6.6.12 *Let $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ be invertible sheaves on a k-space X.*

i. *The sheaf $\mathcal{L}^\vee$ is invertible.*

ii. $\mathcal{L} \otimes (\mathcal{L}' \otimes \mathcal{L}'') \cong (\mathcal{L} \otimes \mathcal{L}') \otimes \mathcal{L}''$.

iii. $\mathcal{L} \otimes \mathcal{L}' \cong \mathcal{L}' \otimes \mathcal{L}$.

iv. $\mathcal{L} \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathcal{L} \cong \mathcal{L}$.

v. $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$.

Proof For any $P \in X$, there exists an open neighbourhood $U \subset X$ of P and an isomorphism of $\mathcal{O}_U$-modules $\varphi\colon \mathcal{O}_U \to \mathcal{L}|_U$. Consider $\mathcal{L}^\vee|_U$. For any open subset $V \subset U$, we have

$$\mathcal{L}^\vee|_U(V) = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{L}|_V, \mathcal{O}_V) \cong \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \mathcal{O}_V) \cong \mathcal{O}_V(V).$$

The first isomorphism is induced by pre-composing with φ. The second isomorphism corresponds to sending $f\colon \mathcal{O}_V \to \mathcal{O}_V$ to $f(V)(1) \in \mathcal{O}_V(V)$ (see Exc. 6.7.17). This gives an isomorphism of sheaves of $\mathcal{O}_U$-modules $\mathcal{L}^\vee|_U \cong \mathcal{O}_U$ (compatibility of restriction maps is checked in Exc. 6.7.20). This proves i.

By Lemma 6.6.11, it suffices to show ii–v at the level of presheaves, i.e. with $\otimes$ replaced by $\otimes^{\mathrm{p}}$. Then ii–iv immediately follow from the corresponding natural isomorphisms for tensor products of modules.

For v, let $U \subset X$ be open and consider the following morphism of $\mathcal{O}_X(U)$-modules

$$\mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U) \to \mathcal{O}_X(U), \quad v \otimes f \mapsto f(U)(v). \tag{6.1}$$

This defines a morphism of pre-sheaves of $\mathcal{O}_X$-modules $\mathcal{L} \otimes^{\mathrm{p}} \mathcal{L}^\vee \to \mathcal{O}_X$. In order to see that it is an isomorphism, it suffices to find an open cover of X on which this morphism restricts to isomorphisms. For any $P \in X$, there exists an open neighbourhood $U \subset X$ of P and an isomorphism $\varphi\colon \mathcal{O}_U \to \mathcal{L}|_U$ together with an induced isomorphism $\mathcal{O}_U \cong \mathcal{L}^\vee|_U$ (part i). Using φ and again Exc. 6.7.17, we obtain isomorphisms

$$\mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U).$$

Combining this isomorphism with (6.1) gives a map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)$, which is the identity map. Hence (6.1), and $(\mathcal{L} \otimes^{\mathrm{p}} \mathcal{L}^\vee)|_U \to \mathcal{O}_X|_U$, are isomorphisms. ☐

Definition 6.6.13 Let X be a k-space. Denote by $\mathrm{Pic}(X)$ the set of isomorphism classes $[\mathcal{L}]$ of invertible sheaves $\mathcal{L}$ on X. Then $\mathrm{Pic}(X)$ forms an abelian group under $\otimes$ by Lemma 6.6.12. The unit is $[\mathcal{O}_X]$ and $[\mathcal{L}]^{-1} = [\mathcal{L}^\vee]$. We refer to $\mathrm{Pic}(X)$ as the *Picard group* of X.

6.7 Exercises

Exercise 6.7.1 If X and Y are smooth varieties, show that $X \times Y$ is smooth.

Exercise 6.7.2 Let X be a variety, $U \subset X$ an open sub-variety, and $P \in U$. Show that X is smooth at P if and only if U is smooth at P.

Exercise 6.7.3 Let k be a field of characteristic different from 2 or 3. Let $a, b \in k$ and $f = y^2 - x^3 - ax - b$. Verify that that f is irreducible and show that $Z(f) \subset \mathbb{A}^2$ is smooth if and only if $4a^3 + 27b^2 \neq 0$.

Exercise 6.7.4 Let n in $\mathbb{Z}_{>1}$ be an integer and k an algebraically closed field. Let $X \subset \mathbb{P}^2$ be the curve given by $x_1^n = x_2 x_0^{n-1} - x_2^n$. Show that X can be covered by two affine curves. (Hint: consider the three affine varieties $X \cap D(x_i)$ for $i = 0, 1, 2$). Is X smooth? (The answer can depend on both n and the characteristic of k.)

Exercise 6.7.5 Give a basis of the tangent space at $(0:0:1)$ along the curve $Z \subset \mathbb{P}^2$ given by the polynomial $y^2 z - x^3$.

Exercise 6.7.6 Prove the claims in Definition 6.3.8.

Exercise 6.7.7 Let k be a field, A a k-algebra and M an A-module. Show that $\mathrm{Der}_k(A, M)$ is an A-module for the addition and multiplication defined by $(D_1 + D_2)g = D_1g + D_2g$, $(fD)g = f(Dg)$.

Exercise 6.7.8 Show that if $\varphi\colon A \to B$ is a morphism of k-algebras and $D \in \mathrm{Der}_k(B, M)$, then $D \circ \varphi$ is in $\mathrm{Der}_k(A, M)$.

Exercise 6.7.9 Let k be a field, A a k-algebra and $\mathfrak{m} \subset A$ a maximal ideal such that the morphism $k \to A \to A/\mathfrak{m} = k$ is an isomorphism.

i. Let $D \in \mathrm{Der}_k(A, A/\mathfrak{m})$. Show that D is zero on $\mathfrak{m}^2$, and hence factors through a derivation $\overline{D}\colon A/\mathfrak{m}^2 \to k$.

ii. Show that the map $\mathrm{Der}_k(A, A/\mathfrak{m}) \to (\mathfrak{m}/\mathfrak{m}^2)^\vee$, $D \mapsto \overline{D}|_{\mathfrak{m}/\mathfrak{m}^2}$ is an isomorphism of A-modules.

Exercise 6.7.10 Let $Q = Z(z-xy)$ in $\mathbb{A}^3$ and let $a \in Q$. Compute $T_Q(a) \subset k^3$. Show that $(a+T_Q(a)) \cap Q$ is a union of two lines, and that all lines on Q are obtained in this way. Try to draw a (real) picture of how Q and $a + T_Q(a)$ intersect.

Exercise 6.7.11 Let X be the union of the three coordinate axes in $\mathbb{A}^3$, and let $Y := Z(xy(x-y))$ in $\mathbb{A}^2$. Are X and Y isomorphic?

Exercise 6.7.12 Let k be a field, $A = k[x_1, \ldots, x_n]$. Show that $(dx_1, \ldots, dx_n)$ is an A-basis of Ω^1_A, and give a formula for df, where $f \in A$.

Exercise 6.7.13 Let k and A be as in the previous exercise. Let $I = (f_1, \ldots, f_r)$ be an ideal in A, and let $q\colon A \to B := A/I$ be the quotient map.

i. Show that, for any B-module M, $q^*\colon \mathrm{Der}_k(B, M) \to \mathrm{Der}_k(A, M)$ is injective and has image the set of those D such that for all i one has $D(f_i) = 0$.

ii. Use the universal property of $d_A\colon A \to \Omega^1_A$ to show that the map $d\colon B \to \Omega^1_A/(I\Omega^1_A + A{\cdot}df_1 + \cdots A{\cdot}df_r)$ which maps $q(a)$ to the class of $d_A(a)$ is well-defined, and satisfies the universal property for $d_B\colon B \to \Omega^1_B$.

iii. Conclude that $\Omega^1_B \cong \Omega^1_A/(I\Omega^1_A + A{\cdot}df_1 + \cdots A{\cdot}df_r)$ as B-modules.

Exercise 6.7.14 Consider the variety X from Example 6.5.3.

i. Show that there exists a unique $\omega \in \Omega^1(X)$ such that

$$\omega|_{D(y)} = \frac{1}{ny^{n-1}}dx \text{ in } \Omega^1(D(y)), \quad \omega|_{D(x)} = \frac{1}{(n-1)x^{n-2}}dy \text{ in } \Omega^1(D(x)).$$

ii. Show that there is an isomorphism $\varphi\colon \mathcal{O}_X \to \Omega^1_X$ which on an open $U \subset X$ is given by

$$\varphi(U)\colon \mathcal{O}_X(U) \to \Omega^1_X(U),\ f \mapsto f\omega|_U.$$

Exercise 6.7.15 Show $\Omega^1_{\mathbb{P}^n}(\mathbb{P}^n) = \{0\}$.

Exercise 6.7.16 Let $A = k[x, y]/(y^2 - x^3)$. Verify that the A-module Ω^1_A is not free of rank one.

Exercise 6.7.17 Let X be a k-space. Prove that there exists a natural isomorphism of $\mathcal{O}_X(X)$-modules $\mathcal{O}_X(X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$, and that any isomorphism $\mathcal{O}_X \to \mathcal{O}_X$ is given by multiplication by an element in $\mathcal{O}_X(X)^\times$.

Exercise 6.7.18 Prove Proposition 6.6.8.

Exercise 6.7.19 Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of presheaves of abelian groups on a topological space X. Suppose there exists an open cover $\{U_i\}_{i \in I}$ of X such that $\varphi|_{U_i}$ is an isomorphism for all $i \in I$.

i. Prove that φ need not be an isomorphism.

ii. If $\mathcal{E}$, $\mathcal{F}$ are sheaves, then φ is an isomorphism.

iii. More generally, the induced morphism $\varphi^+ : \mathcal{E}^+ \to \mathcal{F}^+$ on the sheafifications is an isomorphism.

Exercise 6.7.20 Let X be a k-space and $\mathcal{L}$ an invertible sheaf on X. Let $U \subset X$ be an open subset and $\varphi : \mathcal{O}_U \to \mathcal{L}|_U$ an isomorphism. In Lemma 6.6.12 we constructed an isomorphism $\mathcal{L}^\vee|_U(V) \cong \mathcal{O}_U(V)$ for any open subset $V \subset U$. Prove that these isomorphisms commute with restriction, so we obtain an isomorphism of sheaves of $\mathcal{O}_U$-modules $\mathcal{L}^\vee|_U \cong \mathcal{O}_U$.

Exercise 6.7.21 Let X be a manifold, and $p\colon V \to X$ a vector bundle on X (of rank n). Let $\mathcal{C}_X^\infty$ be the sheaf of smooth functions on X. Define a presheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) = \{s\colon U \to V \mid \ s \text{ is smooth and } p(s(x)) = x \text{ for all } x \in U\ \},$$

for open $U \subset X$.

i. Show that $\mathcal{F}$ is a sheaf, and that it naturally is a sheaf of $\mathcal{C}_X^\infty$-modules.

ii. Show that $\mathcal{F}$ is locally free of rank n.

In fact one can show that every sheaf of $\mathcal{C}_X^\infty$-modules that is locally free of finite rank arises in this way.

Lecture 7

Riemann-Roch and Serre duality

7.1 Uniformizers at smooth points on curves

In this section, we develop the analog of "order of vanishing of the zero/pole of a meromorphic function" for rational functions on smooth curves. The role of a local coordinate will be played by what is called a "uniformizer."

Definition 7.1.1 Let k be an algebraically closed field. A *curve over* k is a quasi-projective algebraic variety over k all of whose irreducible components are of dimension one.

Remark 7.1.2 We will be using the local rings $\mathcal{O}_{X,P}$. By abuse of language, we will often call elements of this ring 'functions' and denote them by $f \in \mathcal{O}_{X,P}$. We think of f as a function defined on some unspecified open neighbourhood U of P, so that f is short-hand for the class of (U, f) in $\mathcal{O}_{X,P}$.

Proposition 7.1.3 *Let X be a curve and $P \in X$ a smooth point. Let $\mathfrak{m} = \mathfrak{m}_{X,P}$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$, and let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then*

i. for every $i \geq 0$, we have $\dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1} = 1$, and $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is generated by the class of t^i;

ii. for every $f \in \mathcal{O}_{X,P}$ and every $s \geq 0$ there exist unique $a_0, \ldots, a_{s-1} \in k$ such that

$$f \equiv a_0 + a_1 t + \cdots + a_{s-1} t^{s-1} \bmod \mathfrak{m}^s$$

iii. $\mathfrak{m} = (t)$;

iv. for every $s \geq 0$, we have $\dim_k \mathcal{O}_{X,P}/\mathfrak{m}^s = s$.

Proof We first show that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is generated by t^i. For $i = 0$ we have $\mathfrak{m}^i/\mathfrak{m}^{i+1} = \mathcal{O}_{X,P}/\mathfrak{m} = k$, generated by $t^0 = 1$. For $i = 1$, note that the smoothness assumption and Proposition 6.2.4 imply that $\dim_k T_X(P) = 1$ so that $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional and generated by t. For $i > 1$, we proceed by induction. The map

$$\mathfrak{m}^{i-1}/\mathfrak{m}^i \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^i/\mathfrak{m}^{i+1},\ (f + \mathfrak{m}^i, g + \mathfrak{m}^2) \mapsto fg + \mathfrak{m}^{i+1}$$

is well-defined, bilinear, and its image generates $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. By the induction hypothesis, we have that $\mathfrak{m}^{i-1}/\mathfrak{m}^i$ is generated by t^{i-1} and $\mathfrak{m}/\mathfrak{m}^2$ is generated by t, so we conclude by induction that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is generated by t^i. In particular, we have $\dim \mathfrak{m}^i/\mathfrak{m}^{i+1} \leq 1$.

Now let $f \in \mathcal{O}_{X,P}$. Then taking $a_0 = f(P)$ we find that $f - a_0 \in \mathfrak{m}$. Since $\mathfrak{m}/\mathfrak{m}^2$ is generated by t, there exists $a_1 \in k$ such that $f - a_0 - a_1 t \in \mathfrak{m}^2$. Similarly, there exists $a_2 \in k$ such that $f - a_0 - a_1 t - a_2 t^2 \in \mathfrak{m}^3$. Repeating this process, we find $a_1, \dots, a_{s-1} \in k$ such that

$$f \equiv a_0 + a_1 t + \cdots + a_{s-1} t^{s-1} \bmod \mathfrak{m}^s.$$

We will use this to show that $\mathfrak{m} = (t)$. Clearly $(t) \subset \mathfrak{m}$. For the converse direction, consider $f_1, \dots, f_r \in \mathfrak{m}$ such that $\mathfrak{m} = (f_1, \dots, f_r)$ (this is possible since $\mathcal{O}_{X,P}$ is Noetherian). Being elements of the stalk $\mathcal{O}_{X,P}$, the functions t and f_i are defined on some open neighbourhood $P \in U \subset X$. We have $t(P) = 0$ (since $t \in \mathfrak{m}$). Since P is a smooth point, it is contained in a unique irreducible component Y. Shrinking U if necessary, we may assume without loss of generality that $U \subset Y$, and hence U is irreducible. Since $t \neq 0$ and since U is an irreducible curve, we have that $Z(t) = \{P, Q_1, \dots, Q_m\}$ for a finite collection of points $Q_i \in U$. Shrinking U again if necessary, we may assume without loss of generality that $Z(t) = \{P\}$ on U, and hence also $Z(f_1, \dots, f_r) = \{P\}$. Without loss of generality, we may take U to be affine. The Nullstellensatz then implies there are $n_i > 0$ such that $f_i^{n_i} \in (t)$ for all i. With $n = \sum n_i$, the pigeon hole principle shows $\mathfrak{m}^n \subset (t)$. Now let $f \in \mathfrak{m}$. Then by the above, we have $a_i \in k$ such that

$$f - (a_1 t + \cdots + a_{n-1} t^{n-1}) \in \mathfrak{m}^n,$$

and since $\mathfrak{m}^n \subset (t)$, we conclude that $f \in (t)$, which establishes the converse inclusion $\mathfrak{m} \subset (t)$.

Now let us prove that $\dim \mathfrak{m}^i/\mathfrak{m}^{i+1} \geq 1$ for all i. Assume for the sake of contradiction that $\mathfrak{m}^i/\mathfrak{m}^{i+1} = 0$, equivalently $\mathfrak{m}^i = \mathfrak{m}^{i+1}$. Then we have $(t^i) = (t^{i+1})$, and so there exists a $u \in \mathcal{O}_{X,P}^\times$ with $t^i = u t^{i+1}$, and hence $t^i(1 - ut) = 0$ in $\mathcal{O}_{X,P}$. Since P is a smooth point, it lies on a unique irreducible component of X, and hence $\mathcal{O}_{X,P}$ is an integral domain (the localization of an integral domain is an integral domain). Moreover $1 - ut \neq 0$, since its value at P is non-zero. We conclude that $t = 0$, which is a contradiction with $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

From the fact that $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is one-dimensional, it follows that the a_i in the second point are unique, and hence that every element of $\mathcal{O}_{X,P}/\mathfrak{m}^s$ can be uniquely represented in the form $\sum_{i=0}^{s-1} a_i t^i$, so that $\dim_k \mathcal{O}_{X,P}/\mathfrak{m}^s = s$. □

Definition 7.1.4 Let X be a curve and $P \in X$ a smooth point. A *uniformizer* of X at P is a $t \in \mathcal{O}_{X,P}$ with $\mathfrak{m} = (t)$, or equivalently $t \in \mathfrak{m} \setminus \mathfrak{m}^2$.

If X is irreducible, then we have seen in Exercise 4.7.17 that $K(X)$ is the fraction field of $\mathcal{O}_{X,P}$, so every $f \in K(X)^\times$ can be written as $f = g/h$ for $g, h \in \mathcal{O}_{X,P}$ with $g, h \neq 0$.

Lemma 7.1.5 *Let P be a smooth point on an irreducible curve X and $t \in \mathcal{O}_{X,P}$ a uniformizer. Let $f \in K(X)^\times$. Then there exist unique $u \in \mathcal{O}_{X,P}^\times$ and $n \in \mathbb{Z}$ such that $f = t^n u$. Moreover, the integer n does not depend on the choice of uniformizer t.*

Proof First assume that $f \in \mathcal{O}_{X,P} \setminus \{0\}$. By Krull's intersection theorem (see [AM, Cor. 10.18]) we have $\cap_i \mathfrak{m}^i = \{0\}$. This means that there is an n such that $f \in \mathfrak{m}^n$ but $f \notin \mathfrak{m}^{n+1}$. We then have $f = t^n u$ (since $f \in \mathfrak{m}^n$) and $u \in \mathcal{O}_{X,P}^\times$ (since $u \notin \mathfrak{m}$). The general case follows from the fact that we can write $f = g/h$ with $g, h \in \mathcal{O}_{X,P}$ and $g, h \neq 0$.

Uniqueness of n and u is clear. For independence of n of the choice of t, it suffices to observe that any other uniformizer t' must be of the form $t' = vt$ for a unit $v \in \mathcal{O}_{X,P}^\times$. □

Definition 7.1.6 Let P be a smooth point on an irreducible curve X. We define the *order of vanishing* at P of a function $f \in K(X)^\times$ as the integer $v_P(f) \in \mathbb{Z}$ so that $f = t^{v_P(f)} u$ for some uniformizer t and unit $u \in \mathcal{O}_{X,P}^\times$.

Proposition 7.1.7 *Let P be a smooth point on an irreducible curve X, let $f, g \in K(X)^\times$. Then* *i.* $v_P(f) \geq 0$ *if and only if* $f \in \mathcal{O}_{X,P}$*;*

ii. $v_P(f) \geq 1$ *if and only if* $f \in \mathfrak{m}_{X,P} \subset \mathcal{O}_{X,P}$*;*

iii. $v_P(f) = 0$ *if and only if* $f \in \mathcal{O}_{X,P}^{\times}$*,*

iv. $v_P(fg) = v_P(f) + v_P(g)$*.*

Proof Exercise 7.7.2 □

Remark 7.1.8 In commutative algebra language, the results of this section can be summarized by saying that $\mathcal{O}_{X,P}$ is a *discrete valuation ring* (with fraction field $K(X)$). This also follows from a general theorem: If R is a noetherian local domain of dimension 1, with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{m}^2$ is a one-dimensional vector space over the residue field $R/\mathfrak{m}$, then R is a discrete valuation ring. See [AM, Prop. 9.2].

7.2 Divisors on curves

Recall from definition 7.1.1 that a *curve* is a quasi-projective variety all whose irreducible components have dimension 1. In particular, in these notes curves are not assumed to be smooth. This will be useful in Chapter 10, where we will need to deal with singular curves when discussing the intersection pairing of divisors on surfaces.

The first goal of this section is to generalize the function v_P from Definition 7.1.6 to non-smooth points. The main difficulty in doing so is that non-smooth points do not admit uniformizers.

Theorem 7.2.1 *Let X be an irreducible curve and let $P \in X$. Then there exists a unique homomorphism*

$$v_P \colon K(X)^{\times} \to \mathbb{Z}$$

such that for all $f \in \mathcal{O}_{X,P}$ with $f \neq 0$ we have $v_P(f) = \dim_k \mathcal{O}_{X,P}/(f)$.

Definition 7.2.2 We call the integer $v_P(f)$ the *order of vanishing of f at P*.

Remark 7.2.3 If P is a smooth point of X, then the function v_P defined in Definition 7.1.6 satisfies the requirements of the above theorem. This follows from Proposition 7.1.3.

The proof of Theorem 7.2.1 is based on the following three lemmas.

Lemma 7.2.4 *Let X be an irreducible curve, let $P \in X$ and let U be an affine open neighbourhood of P in X. Let $f \in \mathcal{O}_X(U)$ and assume that f has no zeroes outside P. Then the natural map*

$$\mathcal{O}_X(U)/(f) \to \mathcal{O}_{X,P}/(f)$$

is an isomorphism.

Proof We first show that the map is injective. So let $g \in \mathcal{O}_X(U)$ such that the image of $\bar{g}$ vanishes. This means that $g = hf$ for some $[(V,h)] \in \mathcal{O}_{X,P}$. Without loss of generality, we may assume $V \subset U$. The function g/f is regular on $U - \{P\}$ and agrees with the regular function h on $V - \{P\}$. Since $U = V \cup (U - \{P\})$, we find that h defines a regular function on all of U, and hence $g \in (f) \subset \mathcal{O}_X(U)$.

For surjectivity, consider an element $\eta \in \mathcal{O}_{X,P}/(f)$. Since the principal affine opens form a basis of the topology of U, there is a g in $\mathcal{O}_X(U)$ such that $P \in D(g) \subset U$ and η is defined on $D(g)$. This means η has the form $h/g^r + (f)$. Since $g(P) \neq 0$, we have $Z(f, g^r) = \emptyset$, and hence $(f, g^r) = \mathcal{O}_X(U)$. In particular, there exist $a, b \in \mathcal{O}_X(U)$ with $h = af + bg^r$. But then $h/g^r + (f)$ is the image of b, and surjectivity follows. □

Lemma 7.2.5 *Let X be an irreducible curve, let $P \in X$ and let $U \subset X$ be an affine open neighbourhood of P. Let $f \in \mathcal{O}_X(U)$ and assume that f has no zeroes outside P. Then the k-vector space $\mathcal{O}_X(U)/(f)$ is finite-dimensional.*

Proof If f is a unit then $\mathcal{O}_X(U)/(f) = \{0\}$ and there is nothing to prove. So we assume that f is not a unit. Then $Z(f) = \{P\}$ and hence $\sqrt{(f)} = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal corresponding to P. Write $\mathfrak{m} = (f_1, \ldots, f_t)$ with $f_i \in \mathcal{O}_X(U)$. It follows that there exist $n_i \in \mathbb{Z}_{\geq 1}$ such that $f_i^{n_i} \in (f)$. As in the proof of Proposition 7.1.3, the pigeon hole principle shows that $\mathfrak{m}^n \subset (f)$ with $n = \sum_i n_i$. And this gives:

$$\dim \mathcal{O}_X(U)/(f) \leq \dim \mathcal{O}_X(U)/\mathfrak{m}^n = \dim \mathcal{O}_X(U)/\mathfrak{m} + \dim \mathfrak{m}/\mathfrak{m}^2 + \cdots + \dim \mathfrak{m}^{n-1}/\mathfrak{m}^n.$$

Notice that $\mathcal{O}_X(U)/\mathfrak{m} = k$. It is enough to show that $\dim \mathfrak{m}^b/\mathfrak{m}^{b+1} < \infty$ (for any $b \in \mathbb{Z}_{\geq 1}$). First observe that $\mathfrak{m}^b/\mathfrak{m}^{b+1}$ is a finitely generated $\mathcal{O}_X(U)$-module. Now $\mathfrak{m} \subset \mathcal{O}_X(U)$ annihilates $\mathfrak{m}^b/\mathfrak{m}^{b+1}$ (indeed, if $x \in \mathfrak{m}^b$ and $y \in \mathfrak{m}$, then $xy \in \mathfrak{m}^{b+1}$). So $\mathfrak{m}^b/\mathfrak{m}^{b+1}$ is even a finitely generated $\mathcal{O}_X(U)/\mathfrak{m}$-module, hence a finite-dimensional k-vector space. So $\dim_k \mathcal{O}_X(U)/(f) < \infty$. □

Lemma 7.2.6 *Let X be an irreducible curve, let $P \in X$ and let $U \subset X$ be an affine open neighbourhood of P. Let $f, g \in \mathcal{O}_X(U)$ and assume that f and g have no zeroes outside P. Then*

$$\dim_k \mathcal{O}_X(U)/(fg) = \dim_k \mathcal{O}_X(U)/(f) + \dim_k \mathcal{O}_X(U)/(g).$$

Proof This follows from the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(U)/(g) \xrightarrow{f\cdot} \mathcal{O}_X(U)/(fg) \longrightarrow \mathcal{O}_X(U)/(f) \longrightarrow 0,$$

where $f\cdot$ is multiplication by f. □

We now prove Theorem 7.2.1:

Proof Let $f \in K(X)^\times$. Then by Exercise 4.7.17 there exists an affine open neighbourhood U of P and functions $g, h \in \mathcal{O}_X(U)$ such that $f = g/h$. Since neither g nor h is zero, they have only finitely many zeroes, and shrinking U if necessary we may assume that g and h have no zeroes outside P. In this case, we must have

$$v_P(f) = v_P(g) - v_P(h) = \dim_k \mathcal{O}_X(U)/(g) - \dim_k \mathcal{O}_X(U)/(h),$$

which shows uniqueness.

For existence, we can use the above formula as a definition of $v_P(f)$. Lemma 7.2.5 guarantees that the right-hand side makes sense, and Lemmas 7.2.6 and 7.2.4 guarantee that it is independent of the choice of U, and of the choice of the representation of f as a quotient of regular functions. Moreover, Lemma 7.2.6 shows that v_P is indeed a homomorphism. □

We now introduce the concept of divisors, which will play a fundamental role in our study of curves.

Definition 7.2.7 Let X be a curve. A *divisor* on X is a $\mathbb{Z}$-valued function D on X such that for at most finitely many P in X, $D(P) \neq 0$. In other words, it is a function $D\colon X \to \mathbb{Z}$ with finite *support*, which we denote by $\operatorname{Supp} D$. The $\mathbb{Z}$-module of divisors is $\operatorname{Div}(X) := \mathbb{Z}^{(X)}$, the free $\mathbb{Z}$-module with basis X. Often a divisor D is written as a formal finite sum $D = \sum_{P \in X} D(P) \cdot P$. The *degree* of a divisor D is defined as $\deg(D) = \sum_P D(P)$.

Example 7.2.8 An example of an element of $\mathbb{Z}^{(X)}$ is $2P + 3Q - R$, where $P, Q, R \in X$. The degree of this divisor is 4.

Lemma 7.2.9 *Let X be an irreducible curve and f in $K(X)^\times$. Then the set of P in X with $v_P(f) \neq 0$ is finite.*

Proof Removing from X the finite set X_{sing}, we may without loss of generality assume that X is smooth. Now since X is quasi-projective (by our definition of curve), it can be covered by finitely many nonempty open affines U_i, such that for each of them, $f|_{U_i} = g_i/h_i$ with g_i and h_i in $\mathcal{O}_X(U_i)$, both non-zero. For each i, U_i is irreducible and affine and of dimension one. The subsets $Z(g_i)$ and $Z(h_i)$ are proper and closed, hence finite. □

Theorem 7.2.10 *Let X be an irreducible curve. The map*

$$K(X)^\times \to \mathrm{Div}(X),\ f \mapsto \mathrm{div}(f) := \sum_{P \in X} v_P(f)P$$

is a homomorphism.

Proof This is a direct consequence of the definition. □

Definition 7.2.11 Let X be an irreducible curve, and $f \in K(X)^\times$. We call the divisor $\mathrm{div}(f)$ defined in Theorem 7.2.10 the *divisor of f*. A divisor of the form $\mathrm{div}(f)$ is called a *principal divisor*. By Theorem 7.2.10, the principal divisors form a subgroup of $\mathrm{Div}(X)$, which we denote by $P(X)$.

Definition 7.2.12 Let X be an irreducible curve and D, D' divisors on X. We say that D and D' are *linearly equivalent* when $D' - D \in P(X)$, in which case we write $D \sim_{\mathrm{lin}} D'$. We denote the group of linear equivalence classes of divisors on X by $\mathrm{Cl}(X) := \mathrm{Div}(X)/P(X)$. The group $\mathrm{Cl}(X)$ is called the *divisor class group* of X.

Definition 7.2.13 Let X be an irreducible curve, and D and D' divisors on X. Then we say that $D \leq D'$ if for all $P \in X$, $D(P) \leq D'(P)$. This relation "$\leq$" is a partial ordering. A divisor D is called *effective* when $0 \leq D$.

Example 7.2.14 Let P, Q and R be distinct points on X. Then $P - 3Q + R \leq 2P - 2Q + R$. Note however that $P + Q \not\leq 2Q$ and that $2Q \not\leq P + Q$, so the partial ordering is not a total ordering.

7.3 Invertible sheaves and divisors on smooth curves

For the remainder of this chapter, we work with smooth curves.

Definition 7.3.1 Let X be an irreducible smooth curve, and D a divisor on X. Then we define a sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ as follows. For $U = \emptyset$ we set $\mathcal{O}_X(D)(U) = \{0\}$. For $U \subset X$ open and non-empty we define

$$\mathcal{O}_X(D)(U) = \{f \in K(X) \colon v_P(f) + D(P) \geq 0 \text{ for all } P \in U\},$$

where we set $v_P(0) = +\infty$.

Equivalently, for $U \subset X$ non-empty open we have

$$\mathcal{O}_X(D)(U) = \{f \in K(X)^\times : \mathrm{div}(f)|_U + D|_U \geq 0\} \cup \{0\}.$$

Example 7.3.2 Let X be an irreducible smooth curve, $U \subset X$ open and non-empty, and P in X. If P is not in U then $\mathcal{O}_X(P)(U)$ is the set of f in $K(X)$ with no pole in U. If P is in U, then $\mathcal{O}_X(P)(U)$ is the set of f in $K(X)$ with a pole of order at most 1 at P and no other poles in U.

Proposition 7.3.3 *We have $\mathcal{O}_X(0) = \mathcal{O}_X$ as sheaves of $\mathcal{O}_X$-modules.*

Proof We show that for every non-empty open $U \subset X$ we have $\mathcal{O}_X(U) = \mathcal{O}_X(0)(U)$ as sub-$\mathcal{O}_X(U)$-modules of $K(X)$. The inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_X(0)(U)$ follows from the fact that $v_P(f) \geq 0$ for any $f \in \mathcal{O}_{X,P}$.

For the other inclusion, let $f \in \mathcal{O}_X(0)(U)$ be non-zero, so that $f \in K(X)^\times$ with $v_P(f) \geq 0$ for all $P \in U$. By definition of $K(X)$, we can represent f by a regular function $f \in \mathcal{O}_X(W)$ for some non-empty open $W \subset X$. We need to show that f can be represented by a regular function $f \in \mathcal{O}_X(U)$.

For every $P \in U$ we have $v_P(f) \geq 0$, so that $f \in \mathcal{O}_{X,P}$. In other words, there exists an open neighbourhood $P \in V_P$ and an $f_P \in \mathcal{O}_X(V_P)$ such that (V_P, f_P) represents f. Given points $P, Q \in U$, we have that f_P and f_Q agree on some non-empty open subset of $V_P \cap V_Q$, so that they agree on $V_P \cap V_Q$. But this means that there exists a regular function $f \in \mathcal{O}_X(U)$ with $f|_{V_P} = f_P$ for all P. □

Example 7.3.4 If X fails to be smooth, then we may have $\mathcal{O}_X(0) \neq \mathcal{O}_X$. For example, if X is the affine curve $Z(y^2 - x^3)$ in $\mathbb{A}^2$, then we have $\mathcal{O}_X(X) \subset \mathcal{O}_X(0)(X)$ and the inclusion is strict. In fact, we may identify $\mathcal{O}_X(X) = k[x,y]/((y^2 - x^3)$ with the sub-k-algebra $k[t^2, t^3]$ of $k[t]$ by sending x to t^2 and y to t^3. In Exercise 7.7.19 we will see that we may identify $\mathcal{O}_X(0)$ with the larger k-algebra $k[t]$.

Proposition 7.3.5 *Let X be an irreducible smooth projective curve, and let D be a divisor on X. Then $\mathcal{O}_X(D)(X)$ is a k-vector space of finite dimension.*

Proof See Exercise 7.7.10. □

In Definition 6.6.13 we introduced the Picard group $\operatorname{Pic}(X)$ of a k-space X, which consists of isomorphism classes of invertible sheaves on X. We first observe that the sheaves $\mathcal{O}_X(D)$ are invertible.

Lemma 7.3.6 *Let X be a smooth irreducible curve, let D be a divisor on X, let $g \in K(X)^\times$, and set $D' = D + \operatorname{div}(g)$. Then there is an isomorphism of sheaves of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{O}_X(D) \tilde{\to} \mathcal{O}_X(D')$ satisfying $\varphi(U)(f) = f/g$ for every non-empty open $U \subset X$.*

Proof See Exercise 7.7.8. □

Proposition 7.3.7 *Let D be a divisor on an irreducible smooth curve X. Then $\mathcal{O}_X(D)$ is an invertible sheaf on X.*

Proof Let $P \in X$. We claim that there exists an open neighbourhood $U \subset X$ of P and an element $g \in K(X)^\times$ such that $D|_U = \operatorname{div}(g)|_U$. Indeed, one can find an open neighbourhood $P \in U \subset X$, and a $t \in \mathcal{O}_X(U)$ such that t is a uniformizer at P, and such that $v_Q(t) = 0$ and $D(Q) = 0$ for all $Q \in U \setminus \{P\}$. The claim then follows by taking $g = t^{D(P)}$.

For such U, Lemma 7.3.6 then gives an isomorphism $\mathcal{O}_X(0)|_U \to \mathcal{O}_X(D)|_U$, and by Proposition 7.3.3 we have $\mathcal{O}_X(0)|_U \cong \mathcal{O}_U$, so we are done. □

We end this section with the following result.

Theorem 7.3.8 *Let X be an irreducible smooth curve. Then the map*

$$\operatorname{Cl}(X) \to \operatorname{Pic}(X), \quad [D] \mapsto [\mathcal{O}_X(D)]$$

is a group isomorphism.

Proof We already established that $\mathcal{O}_X(D)$ is invertible (Proposition 7.3.7), and by Lemma 7.3.6 the map is well-defined.

In order to see that it is a group homomorphism, we must show that

$$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1 + D_2).$$

For this, we constuct a morphism of presheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(D_1) \otimes^{\mathrm{p}} \mathcal{O}_X(D_2) \to \mathcal{O}_X(D_1 + D_2)$ as follows. Over an open subset $U \subset X$ we send $f_1 \otimes f_2$, where $f_i \in K(X)^\times$ satisfy $\operatorname{div} f_i|_U + D_i|_U \geq 0$, to $f_1 f_2$. Then $\operatorname{div} f_1 f_2|_U + D_1|_U + D_2|_U = \operatorname{div} f_1|_U + D_1|_U + \operatorname{div} f_2|_U + D_2|_U \geq 0$ (Theorem 7.2.10). By Lemma 6.6.11, it suffices to show that there exists an open cover of X on which this morphism restricts to isomorphisms. For this, we use the trivialization constructed in Proposition 7.3.7. Let $P \in X$ and let $U \subset X$ be an open neighbourhood of P such that $D_i|_U = \operatorname{div} g_i|_U$, where $g_1, g_2 \in K(X)^\times$. As shown in Proposition 7.3.7, this gives isomorphisms of $\mathcal{O}_U$-modules $\mathcal{O}_U \cong \mathcal{O}_X(D_i)|_U$ (by mapping f to f/g_i). Then the composition

$$\mathcal{O}_U \to \mathcal{O}_X(D_1)|_U \otimes^{\mathrm{p}} \mathcal{O}_X(D_2)|_U \to \mathcal{O}_X(D_1 + D_2)|_U$$

sends $f \mapsto f/g_1 \otimes 1/g_2 \mapsto f/g_1 g_2$, which is the trivialization of $\mathcal{O}_X(D_1 + D_2)|_U$, and therefore an isomorphism. We conclude that the second map is an isomorphism as well.

For the rest of the proof, we only provide a sketch. We show how to construct from an invertible sheaf $\mathcal{L}$ on X a divisor D and claim that this defines a map $\operatorname{Pic}(X) \to \operatorname{Div}(X)$, $[\mathcal{L}] \mapsto [D]$ which is an inverse to $\operatorname{Div}(X) \to \operatorname{Pic}(X)$.

Suppose $\mathcal{L}$ is an invertible sheaf on X. By definition, this means there exists an open cover $(U_i)_i$ of X and a collection of isomorphisms $s_i \colon \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$. Consider the isomorphism

$$s_j^{-1}|_{U_{ij}} \circ s_i|_{U_{ij}} \colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}},$$

where $U_{ij} := U_i \cap U_j$. By Exercise 6.7.17 this is given by multiplication by an element

$$h_{ij} \in \mathcal{O}_X(U_{ij})^\times \subset K(X)^\times.$$

Chasing the isomorphisms, one verifies that for all indices i, j, k one has $h_{ij} h_{jk} = h_{ik}$ in $K(X)^\times$.

Now fix an index $i = i_0$, and consider for all indices j the divisor

$$D_j = \operatorname{div}_{U_j}(h_{ij}) \in \operatorname{Div}(U_j).$$

Note that $D_i = 0$. Moreover, for all j, k we have that D_j and D_k agree on U_{jk}, since h_{jk} is invertible on U_{jk}. This implies that there exists a divisor D on X such that $D|_{U_j} = D_j$ for all j. □

7.4 The genus of a smooth projective curve

For a sheaf of abelian groups $\mathcal{F}$ on a topological space X we define $\mathrm{H}^0(X, \mathcal{F})$ to be the abelian group $\mathcal{F}(X)$ of global sections of $\mathcal{F}$.

Lemma 7.4.1 *Let X be a topological space, and let $X = U_1 \cup U_2$ with U_1 and U_2 open. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then the sequence*

$$0 \to \mathcal{F}(X) \xrightarrow{\iota} \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \xrightarrow{\delta} \mathcal{F}(U_1 \cap U_2),$$

with $\iota(f) = (f|_{U_1}, f|_{U_2})$ and $\delta(f_1, f_2) = f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2}$, is exact.

Proof This follows from the sheaf property for the open cover $\{U_1, U_2\}$. See also Exercise 4.7.21. □

For a *curve* X there exist affine open subsets U_1 and U_2 of X such that $X = U_1 \cup U_2$ (see Exercise 7.7.17). By Proposition 5.4.5 the intersection $U_1 \cap U_2$ is also affine.

Definition 7.4.2 Let X be a smooth irreducible curve and D a divisor on X. Let $\{U_1, U_2\}$ be an affine open cover of X, and let δ be as in Lemma 7.4.1, with $\mathcal{F} = \mathcal{O}_X(D)$. We define k-vector spaces

$$\mathrm{H}^0(X, D) := \mathrm{H}^0(X, \mathcal{O}_X(D)) := \ker \delta$$

and

$$\mathrm{H}^1(X, D) := \mathrm{H}^1(X, \mathcal{O}_X(D)) := \operatorname{coker} \delta.$$

Note that by Lemma 7.4.1 we have $\mathrm{H}^0(X, D) \cong \mathcal{O}_X(D)(X)$. We now give a number of properties of these vector spaces, mostly without proofs.

Proposition 7.4.3 *The vector spaces* $\mathrm{H}^0(X, D)$ *and* $\mathrm{H}^1(X, D)$ *are independent of the choice of affine open cover* $\{U_1, U_2\}$.

We omit the proof, but note that for $\mathrm{H}^0(X, D)$ this follows from Lemma 7.4.1.

Proposition 7.4.4 *If* X *is affine, then* $\mathrm{H}^1(X, D) = 0$.

Proof By Proposition 7.4.3 we may compute $\mathrm{H}^1(X, D)$ using the affine open cover $\{X, \emptyset\}$. One then sees immediately that $\mathrm{H}^1(X, D) = 0$. □

Proposition 7.4.5 *If* X *is projective, then* $\mathrm{H}^0(X, D)$ *and* $\mathrm{H}^1(X, D)$ *are finite-dimensional.*

Proof The space $\mathrm{H}^0(X, D) = \mathcal{O}_X(D)(X)$ is finite-dimensional by Proposition 7.3.5.

For $\mathrm{H}^1(X, D)$ we only give a sketch. An explicit computation with the standard open cover shows that $\mathrm{H}^1(\mathbb{P}^1, D)$ is finite-dimensional, see Exercise 7.7.12. For a general curve, one can use a non-constant morphism $f\colon X \to \mathbb{P}^1$ to deduce the finite-dimensionality from the case of the projective line. □

Definition 7.4.6 Let X be a smooth irreducible projective curve. Then the integer $\dim_k \mathrm{H}^1(X, 0)$ is called the *genus* of X.

Example 7.4.7 The genus of $\mathbb{P}^1$ is 0, see Exercise 7.7.12. The genus of the curve $X \subset \mathbb{P}^2$ given by $x_1^n = x_2 x_0^{n-1} - x_2^n$ of Exercise 6.7.4 is $(n-1)(n-2)/2$, see Exercise 7.7.13.

7.5 The Riemann-Roch theorem

In this section, we state and prove the famous Riemann-Roch theorem. In the proof we will make use of the Chinese remainder theorem from elementary algebra, which we recall.

Proposition 7.5.1 (Chinese remainder theorem) *Let* R *be a ring and* $I_1, \ldots, I_r$ *ideals in* R *such that* $I_i + I_j = R$ *for all* $i \neq j$. *Then the natural map*

$$R/\bigcap_{i=1}^r I_i \longrightarrow \prod_{i=1}^r R/I_i$$

is an isomorphism.

In our application, the hypotheses of the Chinese remainder theorem will be verified by the following lemma.

Lemma 7.5.2 *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be distinct maximal ideals in a ring R, and let n_1 and n_2 be positive integers. Then $\mathfrak{m}_1^{n_1} + \mathfrak{m}_2^{n_2} = R$.*

Proof See Exercise 7.7.20. □

We will also use the following lemma on exact sequences of finite-dimensional vector spaces.

Lemma 7.5.3 *Let*

$$0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} V_n \xrightarrow{\alpha_n} 0$$

be an exact sequence of finite-dimensional vector spaces. Then

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

Proof For all i the map α_i induces an isomorphism

$$V_i/(\ker \alpha_i) \to \operatorname{im} \alpha_i = \ker \alpha_{i+1}.$$

In particular, $\dim V_i = \dim \ker \alpha_i + \dim \ker \alpha_{i+1}$, from which the lemma follows immediately. □

Theorem 7.5.4 (Riemann–Roch) *Let X be a smooth, irreducible projective curve. Let g be the genus of X and D a divisor on X. Then $\dim \mathrm{H}^0(X, D) - \dim \mathrm{H}^1(X, D) = 1 - g + \deg(D)$.*

Proof Note that the statement is true for $D = 0$, as $\dim \mathrm{H}^0(X, 0) = 1$ and $\dim \mathrm{H}^1(X, 0) = g$. It now suffices to show that for all D and all $P \in X$, the statement is true for D if and only if it is true for $D' := D + P$.

To ease the notation, for a non-empty open $U \subset X$ we will write $\mathrm{H}^0(U, D)$ for $\mathrm{H}^0(U, D|_U)$. Note that we have $\mathrm{H}^0(U, D|_U) = \mathcal{O}_X(D)(U)$. We then have the following exact sequences:

$$0 \to \mathrm{H}^0(X, D) \to \mathrm{H}^0(U_1, D) \oplus \mathrm{H}^0(U_2, D) \to \mathrm{H}^0(U_1 \cap U_2, D) \to \mathrm{H}^1(X, D) \to 0$$

$$0 \to \mathrm{H}^0(X, D') \to \mathrm{H}^0(U_1, D') \oplus \mathrm{H}^0(U_2, D') \to \mathrm{H}^0(U_1 \cap U_2, D') \to \mathrm{H}^1(X, D') \to 0$$

Let $\{U_1, U_2\}$ be an affine open cover of X. Note that we have the following inclusions:

$$\begin{aligned} \alpha : \mathrm{H}^0(U_1, D) \oplus \mathrm{H}^0(U_2, D) &\to \mathrm{H}^0(U_1, D') \oplus \mathrm{H}^0(U_2, D') \\ \beta : \mathrm{H}^0(U_1 \cap U_2, D) &\to \mathrm{H}^0(U_1 \cap U_2, D') \end{aligned}$$

We can form a large diagram (with exact rows and columns):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{H}^0(X, D) & & \mathrm{H}^0(X, D') & & A' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{H}^0(U_1, D) \oplus \mathrm{H}^0(U_2, D) & \xrightarrow{\alpha} & \mathrm{H}^0(U_1, D') \oplus \mathrm{H}^0(U_2, D') & \to & A & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \gamma & & \\
0 & \to & \mathrm{H}^0(U_1 \cap U_2, D) & \xrightarrow{\beta} & \mathrm{H}^0(U_1 \cap U_2, D') & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{H}^1(X, D) & & \mathrm{H}^1(X, D') & & B' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In this diagram A and B are the cokernels of α respectively β; γ is the map induced by the δ's above it and A' and B' are the kernel and cokernel of γ, respectively.

We can now apply the snake lemma [AM, Prop. 2.10], and we obtain the following exact sequence:

$$0 \to \mathrm{H}^0(X, D) \to \mathrm{H}^0(X, D') \to A' \to H^1(X, D) \to H^1(X, D') \to B' \to 0.$$

We apply Lemma 7.5.3 a few times. If A and B are finite-dimensional then we see from the last column of the large diagram that:

$$\dim B' - \dim A' = \dim B - \dim A.$$

From the exact sequence obtained from the snake lemma and from the previous line we get:

$$\begin{aligned}(\dim \mathrm{H}^0(X, D) - \dim H^1(X, D)) - (\dim \mathrm{H}^0(X, D') - \dim H^1(X, D'))\\ = \dim B' - \dim A'\\ = \dim B - \dim A.\end{aligned}$$

So it suffices to show that $\dim A$ and $\dim B$ are finite and that $\dim A - \dim B = 1$.

We claim that for $U \subset X$ open affine and non-empty:

$$\dim \operatorname{coker}\left(\mathrm{H}^0(U, D) \to \mathrm{H}^0(U, D')\right) = \begin{cases} 0 & \text{if } P \notin U \\ 1 & \text{if } P \in U \end{cases}$$

If $P \notin U$, the claim is obvious as $D|_U = D'|_U$. Suppose that $P \in U$. Let us first argue that the cokernel of $\mathrm{H}^0(U, D) \to \mathrm{H}^0(U, D')$ has dimension at most one. Let $t \in \mathcal{O}_{X,P}$ be a uniformizer at P. Let $n := -D'(P)$. Each element f in $\mathrm{H}^0(U, D')$ can be written uniquely as $f = a_n(f)t^n + t^{n+1}h$ with $a_n(f)$ in k and h in $K(X)$ regular at P. Such an f is in $\mathrm{H}^0(U, D)$ if and only if $a_n(f) = 0$. Hence $\mathrm{H}^0(U, D)$ is the kernel of the map $\mathrm{H}^0(U, D') \to k$, $f \mapsto a_n(f)$. Hence the cokernel has dimension at most one. To prove that it is one, it suffices to show that there is an f in $\mathrm{H}^0(U, D')$ that is not in $\mathrm{H}^0(U, D)$. We put $g := t^n$. Then g is in $K(X)^\times$, and $v_P(g) = n = -D'(P)$. We claim that there exists an h in $\mathcal{O}_X(U)$ such that $h(P) = 1$ and $f := h{\cdot}g$ is in $\mathrm{H}^0(U, D')$. A element $h \neq 0$ in $\mathcal{O}_X(U)$ has this property if and only if $h(P) = 1$ and for all Q in U, $v_Q(h) \geq -v_Q(g) - D'(Q)$. This means that $h(P) = 1$ and at a finite number of distinct points $Q_1, \ldots, Q_r$, and elements n_i in $\mathbb{Z}_{\geq 0}$, we must have $v_{Q_i}(h) \geq n_i$. This is a consequence of the Chinese remainder theorem, which says that the morphism of k-algebras $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{m}_P \times \prod_{i=1}^r \mathcal{O}_X(U)/\mathfrak{m}_{Q_i}^{n_i}$ is surjective. (Thanks to Lemma 7.5.2 it holds that $\mathfrak{m}_P + \mathfrak{m}_{Q_i}^{n_i} = R$ and $\mathfrak{m}_{Q_i}^{n_i} + \mathfrak{m}_{Q_j}^{n_j} = R$ for all $i \neq j$.) This finishes the proof of the claim.

Using the claim, we can now finish the proof. From the claim we get:

	$\dim A$	$\dim B$
$P \in U_1 \cap U_2$	2	1
$P \notin U_1 \cap U_2$	1	0

So indeed $\dim B - \dim A = -1$, and we are done with the proof. □

Corollary 7.5.5 *Let X be an irreducible projective smooth curve over k. Then for every effective divisor D on X, we have $\dim \mathrm{H}^1(X, D) \leq g$.*

Proof This follows from Exercise 7.7.10 and the Riemann–Roch theorem. □

7.6 Serre duality

We start by summarizing the main features of one-forms on smooth curves.

Proposition 7.6.1 *Let X be a smooth and irreducible curve.*

i. For every $t \in K(X)$ with $t \notin k$ we have $\Omega^1_{K(X)} = K(X)dt$

ii. For every $P \in X$ with uniformizer t we have $\Omega^1_{\mathcal{O}_{X,P}} = \mathcal{O}_{X,P}dt$

Proof We omit the proof. □

Definition 7.6.2 A *rational* 1*-form* is an element of $\Omega^1_{K(X)}$. Let ω be a rational 1-form on X, let $P \in X$ and let t be a uniformizer at P. Assume that $\omega \neq 0$. Then $\omega = f dt$ for some $f \in K(X)^\times$. We define the *order of vanishing* of ω at P to be $v_P(\omega) := v_P(f) \in \mathbb{Z}$.

Lemma 7.6.3 *$v_P(\omega)$ is independent of the choice of uniformizer t.*

Proof If t' is another uniformizer, then $t' = ut$ for some $u \in \mathcal{O}^\times_{X,P}$. We have $dt' = udt + tdu$. By Proposition 7.6.1 we have $du = hdt$ for some $h \in \mathcal{O}_{X,P}$, so we find $dt' = (u + ht)dt$. Note that $u + ht$ is non-zero at P, and hence $v_P(u + ht) = 0$. Now if we have $f dt = f' dt'$, then $f = (u + ht)f'$ and hence $v_P(f) = v_P(f')$. □

Let X be a smooth and irreducible curve. Then by Proposition 6.6.7 the sheaf of differentials Ω^1_X is invertible.

Definition 7.6.4 The *divisor* of a non-zero rational 1-form ω is defined to be

$$\operatorname{div}(\omega) = \sum_P v_P(\omega)P \in \operatorname{Div}(X).$$

Note that this sum is a finite sum.

We now introduce sheaves $\Omega^1_X(D)$ analogous to $\mathcal{O}_X(D)$.

Definition 7.6.5 Let X be a smooth and irreducible curve. Let D be a divisor on X. Then the sheaf of $\mathcal{O}_X$-modules $\Omega^1_X(D)$ is defined by $\Omega^1_X(D)(\emptyset) = \{0\}$ and

$$\Omega^1_X(D)(U) = \{0 \neq \theta \in \Omega^1_{K(X)} : \operatorname{div}(\theta)|_U + D|_U \geq 0\} \cup \{0\}$$

for non-empty open $U \subset X$.

Proposition 7.6.6 *$\Omega^1_X(D) \cong \Omega^1_X \otimes \mathcal{O}_X(D)$ as sheaves of $\mathcal{O}_X$-modules.*

Proof See Exercise 7.7.25. □

In particular, $\Omega^1_X(D)$ is *invertible*, and $\Omega^1_X(0) \cong \Omega^1_X$.

Definition 7.6.7 Let X be a smooth and irreducible curve. By Theorem 7.3.8, there exists a divisor K_X on X such that $\Omega^1_X \cong \mathcal{O}_X(K_X)$. Any such divisor is called a *canonical divisor* of X. Its class in $\operatorname{Cl}(X)$ is called the *canonical divisor class*.

By abuse of language, one often refers to K_X as 'the canonical divisor', even though K_X is only defined up to linear equivalence.

Proposition 7.6.8 *Let X be a smooth irreducible curve and $\theta \in \Omega^1_{K(X)}$ non-zero. Then there exists an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(\operatorname{div}(\theta)) \cong \Omega^1_X$. In particular, $\operatorname{div}(\theta)$ is a canonical divisor.*

Proof By Proposition 7.6.1, for any two non-zero $\theta, \theta' \in \Omega^1_{K(X)}$, there exists a (unique) $f \in K(X)^\times$ such that $\theta' = f\theta$. Consider the morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(\operatorname{div}\theta) \to \Omega^1_X(0),$$

which, over $U \subset X$ open, maps $f \in K(X)^\times$ satisfying $\operatorname{div} f|_U + \operatorname{div}\theta|_U \geq 0$ to $f\theta \in \Omega^1_X(U)$. This morphism has an inverse mapping $\theta' = f\theta$ to f. We conclude $\mathcal{O}_X(\operatorname{div}\theta) \cong \Omega^1_X(0) \cong \Omega^1_X$. □

Corollary 7.6.9 *Let X be a smooth and irreducible curve, and let K_X be a canonical divisor on X. Then $\Omega^1_X(D) \cong \mathcal{O}_X(K_X + D)$.*

Proof This follows from Proposition 7.6.6 and Proposition 7.6.8. □

Corollary 7.6.10 *If X is a smooth irreducible projective curve, and D a divisor on X then the k-vector space $\mathrm{H}^0(X, \Omega^1_X(D)) := \Omega^1_X(D)(X)$ is finite-dimensional.*

Proof This follows from Corollary 7.6.9 and Proposition 7.4.5. □

Lemma 7.6.11 *Let X be an irreducible curve and $P \in X$ a smooth point and let t be a uniformizer at P. Let $f \in K(X)^\times$ with $v_P(f) = -n < 0$. Then there exist $a_{-n}, a_{-n+1}, \ldots, a_{-1} \in k$ and $h \in \mathcal{O}_{X,P}$ such that*

$$f = a_{-n}t^{-n} + a_{-n+1}t^{-n+1} + \cdots + a_{-1}t^{-1} + h.$$

Moreover, the a_i and h are uniquely determined.

Proof See Exercise 7.7.7. □

Definition 7.6.12 Let $\omega \in \Omega^1_{K(X)}$ be a rational 1-form, and $P \in X$ a point with uniformizer t. We define $\operatorname{res}_P(\omega) \in k$, the *residue* of ω at P, as follows. By Proposition 7.6.1 we have $\omega = g{\cdot}dt$ for some $g \in K(X)$. If $g = 0$ or $v_P(g) \geq 0$, then $\operatorname{res}_P(\omega) := 0$. If $v_P(g) = -n$ with $n \geq 1$, then by Lemma 7.6.11 we have

$$g = a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} + h$$

with $a_i \in k$ and $h \in \mathcal{O}_{X,P}$. We define $\operatorname{res}_P(\omega) := a_{-1}$.

Lemma 7.6.13 *$\operatorname{res}_P(\omega)$ is independent of the choice of uniformizer t.*

Proof We omit the proof. See Exercise 7.7.21 for the case where the field k is of characteristic 0. □

Theorem 7.6.14 (Residue theorem) *Let X be an irreducible smooth projective curve. Then for all rational 1-forms $\theta \in \Omega^1_{K(X)}$ we have $\sum_{P \in X} \operatorname{res}_P(\theta) = 0$.*

Proof We omit the proof. For the case of $X = \mathbb{P}^1$, see Exercise 7.7.24. □

Remark 7.6.15 It it is crucial for the residue theorem that X is projective. Indeed it is easy to produce counter-examples: it suffices to remove from a projective curve X a point P with $\operatorname{res}_P(\theta) \neq 0$.

We now have all the ingredients to come to the formulation of Serre duality.

Recall that we have the following map for $X = U_1 \cup U_2$, U_1, U_2 affine open:

$$\delta \colon \mathrm{H}^0(U_1, D) \oplus \mathrm{H}^0(U_2, D) \to \mathrm{H}^0(U_1 \cap U_2, D), \quad (f_1, f_2) \mapsto f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2}.$$

The cokernel of this map was defined to be $\mathrm{H}^1(X, D)$. We define the following pairing:

$$\langle \cdot, \cdot \rangle : \mathrm{H}^0(U_1 \cap U_2, D) \times \mathrm{H}^0(X, \Omega^1_X(-D)) \to k, \quad (g, \omega) \mapsto \sum_{P \in X - U_2} \operatorname{res}_P(g{\cdot}\omega). \tag{7.6.16}$$

Theorem 7.6.17 (Serre duality) *Let X be a smooth irreducible projective curve. The pairing in (7.6.16) induces a perfect pairing*

$$\mathrm{H}^1(X, D) \times \mathrm{H}^0(X, \Omega^1_X(-D)) \to k.$$

It does not depend on the choice of the pair (U_1, U_2).

Remark 7.6.18 We have chosen to sum over residues at the points missing U_2. One can also decide to do it for the points missing U_1, and this would change the pairing by a factor -1. To see this, first notice that $U_1^c \cap U_2^c = \emptyset$ and by definition $g{\cdot}\omega$ is regular at the points of $U_1 \cap U_2$. Using the residue theorem, we find:

$$\begin{aligned}
0 &= \sum_{P\in X} \mathrm{res}_P(g{\cdot}\omega) \\
&= \sum_{P\in X-U_1} \mathrm{res}_P(g{\cdot}\omega) + \sum_{P\in X-U_2} \mathrm{res}_P(g{\cdot}\omega) + \sum_{P\in U_1\cap U_2} \mathrm{res}_P(g{\cdot}\omega) \\
&= \sum_{P\in X-U_1} \mathrm{res}_P(g{\cdot}\omega) + \sum_{P\in X-U_2} \mathrm{res}_P(g{\cdot}\omega).
\end{aligned}$$

The proof of the Serre duality theorem falls outside the scope of these lecture notes. Instead, we explore a number of interesting consequences arising from combining Riemann-Roch and Serre duality. For the remainder of this section, we focus on some first corollaries. In the exercises, you explore further consequences for irreducible smooth projective curves of genus 0 and 1.

Corollary 7.6.19 *Let X be a smooth irreducible projective curve. Then $\Omega^1_X(X)$ and $\mathrm{H}^1(X, \mathcal{O}_X)$ are both of dimension g, the genus of X.*

Proof By Theorem 7.6.17 the finite-dimensional k-vector spaces $\Omega^1_X(X) = \mathrm{H}^0(X, \Omega^1_X(0))$ and $\mathrm{H}^1(X, \mathcal{O}_X)$ are isomorphic to each other's dual, hence they have the same dimension. □

Combining Riemann-Roch and Serre duality, we obtain the following formula.

Corollary 7.6.20 *Let X be a smooth irreducible projective curve, and D a divisor on X. Then*

$$\dim \mathrm{H}^0(X, D) - \dim \mathrm{H}^0(X, K_X - D) = \deg(D) + 1 - g.$$

Proposition 7.6.21 *Let X be a smooth irreducible projective curve.*

i. Let $\theta \in \Omega^1_{K(X)}$ be non-zero. Then $\deg(\mathrm{div}(\theta)) = 2g - 2$. In other words, every canonical divisor on X has degree $2g - 2$.

ii. Let f be in $K(X)^\times$. Then $\deg(\mathrm{div}(f)) = 0$. In other words, every principal divisor has degree zero.

iii. Let D be a divisor on X. Then $\mathrm{H}^0(X, D) = \{0\}$ if $\deg D < 0$, and $\dim \mathrm{H}^0(X, D) = \deg D + 1 - g$ if $\deg D > 2g - 2$.

Proof i. We use Theorem 7.6.20 with $D = \mathrm{div}(\theta)$ in combination with the above lemma and Corollary 7.6.19. We get:

$$\begin{aligned}
\deg(D) &= g - 1 + \dim \mathrm{H}^0(X, D) - \dim \mathrm{H}^0(X, \Omega^1(-D)) \\
&= g - 1 + \dim \mathrm{H}^0(X, \Omega^1(0)) - \dim \mathrm{H}^0(X, 0) \\
&= g - 1 + g - 1 \\
&= 2g - 2.
\end{aligned}$$

ii. Take $\omega \in \Omega^1_{K(X)}$ nonzero. Then $\mathrm{div}(f{\cdot}\theta) = \mathrm{div}(f) + \mathrm{div}(\theta)$. Hence $\mathrm{div}(f)$ is the difference of two canonical divisors and therefore it has degree zero. Another proof of ii is given in Exercise 7.7.15.

iii. The first case follows directly from ii. For the second case, notice that $\mathrm{H}^0(X, \Omega^1_X(-D)) = \{0\}$ and apply Theorem 7.6.20. □

Let X be an irreducible smooth projective curve. By Proposition 7.6.21(ii), the degree map of Definition 7.2.7 factors over the divisor class group

$$\deg : \mathrm{Cl}(X) \longrightarrow \mathbb{Z}. \tag{7.1}$$

Since the canonical divisor K_X is actually a divisor class, we can rewrite Proposition 7.6.21(i) as

$$\deg K_X = 2g - 2.$$

The kernel of (7.1) is denoted by $J(X)$ and is called the *Jacobian* of X. Besides its group structure, $J(X)$ also turns out to have the structure of a variety (we will not use this). Pick a point $O \in X$. The map

$$X \to J(X), \quad P \mapsto [P] - [O]$$

is known as the *Abel-Jacobi map*. In Exc. 7.7.27, you prove that the Abel-Jacobi map is bijective when $g(X) = 1$. Irreducible smooth projective curves of genus 1 are called *elliptic curves*. Hence, for any choice of $O \in X$, an elliptic curve inherits a group structure from its Jacobian.

Remark 7.6.22 We note that $2 - 2g$ is the Euler characteristic of the sphere with g handles attached to it.

7.7 Exercises

Exercise 7.7.1 Let X be an irreducible curve, $P \in X$ and $f \in \mathcal{O}_{X,P}$ with $f \neq 0$. Show that there exists an open neighbourhood $P \in U$ such that f can be represented by a pair (U, f) with $f(Q) \neq 0$ for all $Q \in U \setminus \{P\}$.

Exercise 7.7.2 Show that the function v_P of Definition 7.1.6, satisfies the properties claimed in Proposition 7.1.7.

Exercise 7.7.3 Let $a \in k$, and $f, g \in k[x] = \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. Write $f = (x-a)^n u$ and $g = (x-a)^m v$ with $u, v \in k[x]$ and $u(a) \neq 0$ and $v(a) \neq 0$. Show that $v_a(f/g) = n - m$.

Exercise 7.7.4 Consider the standard affine $\mathbb{A}^1 \subset \mathbb{P}^1$, and denote by ∞ the point $(1 : 0)$, so that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Let g and h be nonzero elements of $k[x] = \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. Show that $v_\infty(g/h) = \deg(h) - \deg(g)$.

Exercise 7.7.5 Let P be a smooth point on an irreducible curve X, and let n be an integer. Let $f, g \in K(X)$ with $v_P(f) \geq n$ and $v_P(g) \geq n$. Show that $v_P(f+g) \geq n$. Here we use the convention that $v_P(0) = +\infty$.

Exercise 7.7.6 Let f be in $k[x,y]$, $X := Z(f) \subset \mathbb{A}^2$ and $P \in X$, such that $(\partial f/\partial x)(P) \neq 0$. In this exercise we will prove that $y - y(P)$ is a uniformizer of X at P by showing that $y - y(P)$ is not in $\mathfrak{m}_P^2$.

i. Show that by applying a translation in $\mathbb{A}^2$ we may assume that $P = (0,0)$. We assume this for the rest of this exercise.

ii. Write $f = ax + by + g$ with $g \in (x^2, xy, y^2)$ and a and b in k, with $a \neq 0$. Show that:

$$\mathcal{O}_X(X)/\mathfrak{m}_P^2 = k[x,y]/(f, x^2, xy, y^2) = k[x,y]/(ax+by, x^2, xy, y^2) = k[y]/(y^2)$$

and conclude that the image of y in $\mathcal{O}_X(X)$ is not in $\mathfrak{m}_P^2$.

Exercise 7.7.7 Prove Lemma 7.6.11.

Exercise 7.7.8 Prove Lemma 7.3.6.

Exercise 7.7.9 Verify that $\mathcal{O}_X(D)$, as defined in Definition 7.3.1 is indeed a sheaf of $\mathcal{O}_X$-modules.

Exercise 7.7.10 Let X be a smooth and irreducible curve.

i. Let $D' \geq D$ be divisors on X. Show that there is an exact sequence

$$0 \to \mathcal{O}_X(D)(X) \to \mathcal{O}_X(D')(X) \to \bigoplus_{P \in X} \mathcal{O}_{X,P}/\mathfrak{m}^{D'(P)-D(P)}$$

of k-vector spaces. (Hint: first show that if $f \in \mathcal{O}_X(D')(X)$ and t is a uniformizer at $P \in X$, then $t^{D'(P)} f \in \mathcal{O}_{X,P}$.)

ii. Let D be an effective divisor on X, and assume that X is projective. Show that $\dim_k \mathcal{O}_X(D)(X) \leq \deg(D)+1$.

iii. Let D be an arbitrary divisor on X, and assume that X is projective. Show that $\mathcal{O}_X(D)(X)$ is finite-dimensional.

Exercise 7.7.11 Show (without using the results of Section 7.6):

i. $\deg \operatorname{div}(f) = 0$ for all $f \in K(\mathbb{P}^1)^\times$.

ii. For every $P \in \mathbb{P}^1$ there exists an $f \in K(\mathbb{P}^1)^\times$ with $\operatorname{div}(f) = P - \infty$

iii. The map $\mathrm{Cl}(\mathbb{P}^1) \to \mathbb{Z}$, $[D] \mapsto \deg(D)$ is a well-defined isomorphism.

Hint: use Exercise 7.7.4.

Exercise 7.7.12 Show that the dimensions of $\mathrm{H}^0(\mathbb{P}^1, D)$ and $\mathrm{H}^1(\mathbb{P}^1, D)$ depend only on the degree of D. Give formulas for these dimensions, and verify that the genus of $\mathbb{P}^1$ is 0. (Hint: compute $\mathrm{H}^0(\mathbb{P}^1, n\infty)$ and $\mathrm{H}^1(\mathbb{P}^1, n\infty)$ using the standard open cover).

Exercise 7.7.13 Consider the curve $X \subset \mathbb{P}^2$ given by $x_1^n = x_2 x_0^{n-1} - x_2^n$, see also Exercise 6.7.4. Assume that X is smooth. Show that the genus of X is $(n-1)(n-2)/2$.

Hint: compute the cokernel of δ using bases for the infinite-dimensional vector spaces $\mathcal{O}_X(X_1)$, $\mathcal{O}_X(X_2)$ and $\mathcal{O}_X(X_1 \cap X_2)$ that are as simple as possible. As a sanity check on your computation: the kernel of δ should be one-dimensional!

Exercise 7.7.14 Let X be a smooth and irreducible curve. Show (without using Serre duality) that the dimension of $\mathrm{H}^0(X, D)$ and $\mathrm{H}^1(X, D)$ depend only on the class of D in $\mathrm{Cl}(D)$.

Exercise 7.7.15 Let X be smooth, projective and irreducible. Use Riemann-Roch and Exercise 7.7.14 to show that $\deg(\operatorname{div}(f)) = 0$ for every $f \in K(X)^\times$.

Exercise 7.7.16 Let X be a smooth projective and irreducible curve and P a point of X. Use the Riemann-Roch theorem to show that $\mathcal{O}_X(X - \{P\})$ is infinite-dimensional.

Exercise 7.7.17 Let $X \subset \mathbb{P}^n$ be a closed curve. Show that there exists hyperplanes H_1 and H_2 in $\mathbb{P}^n$ such that $H_1 \cap H_2 \cap X = \emptyset$. Now generalise this as follows (quite a lot harder): for $X \subset \mathbb{P}^n$ a quasi-projective curve there exist hypersurfaces $Z(f_1)$ and $Z(f_2)$ in $\mathbb{P}^n$ such that $Z(f_1) \cap Z(f_2) \cap X = \emptyset$ and $X \cap D(f_i)$ is closed in $D(f_i)$ for both i.

Deduce that X is the union of two open affine subsets.

Exercise 7.7.18 Let X be a smooth, projective and irreducible curve. Let $f\colon X \to \mathbb{P}^1$ be a morphism of varieties. Recall from Section 3.3 that the group $\mathrm{PGL}_2(k)$ is the group of projective transformations of $\mathbb{P}^1$.

i. Show that f is either constant or surjective (hint: use that all morphisms from X to $\mathbb{A}^1$ are constant).

ii. Let U be the complement of $f^{-1}\{(1:0)\}$ and assume that U is non-empty. Show that $f|_U$, seen as a map to $\mathbb{A}^1 = k$ defines an element $\tilde{f}$ of $K(X)$.

iii. Show that $f \mapsto \tilde{f}$ defines a bijection between the set of morphisms $X \to \mathbb{P}^1$ whose image is not $\{(1:0)\}$ and $K(X)$. Hint: let $\tilde{f}$ be in $K(X)$, consider its divisor and cover X by opens U_1 and U_2 such that $\tilde{f} \in \mathcal{O}_X(U_1)$ and $1/\tilde{f} \in \mathcal{O}_X(U_2)$.

iv. From now on let $X = \mathbb{P}^1$ and let $f\colon X \to \mathbb{P}^1$ be an isomorphism. We denote the coordinate ring of $\mathbb{A}^1$ by $k[x]$. Then $K(\mathbb{P}^1)$ is the field of fractions of $k[x] = \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1)$. Show that the map $f^*\colon K(\mathbb{P}^1) \to K(X) = K(\mathbb{P}^1)$ induced by f sends x to $\tilde{f}$.

v. Show that there exist $a, b, c, d \in k$ such that $\tilde{f} = (ax+b)/(cx+d)$. Deduce that $\mathrm{PGL}_2(k)$ is the group of automorphisms of the variety $\mathbb{P}^1$. Hint: $\mathrm{div}(x) = 0 - \infty$, what does this imply for $\mathrm{div}(\tilde{f})$?

Exercise 7.7.19 Let $X \subset \mathbb{A}^2$ be the curve defined by $x^3 - y^2$.

i. Show that X is irreducible.

ii. Show that X is not smooth.

iii. Let P be the point $(0,0)$. Show that there is no pair (U, f) with $P \in U \subset X$ open affine, $f \in \mathcal{O}_X(U)$ and $v_P(f) = 1$. (Hint: consider $k[x,y]/\mathfrak{m}^2$ with $\mathfrak{m} = (x,y)$.)

iv. Use the morphism $\varphi\colon \mathbb{A}^1 \to X$, $a \mapsto (a^2, a^3)$ to prove that φ^* induces an isomorphism of k-algebras from $k[x,y]/(y^2 - x^3) = \mathcal{O}_X(X)$ to the sub-k-algebra $k[t^2, t^3]$ of $k[t]$, and that $\mathcal{O}_X(0)(X) = k[t]$, as claimed in Example 7.3.4.

Exercise 7.7.20 Prove Lemma 7.5.2.

Exercise 7.7.21 In this exercise you prove Lemma 7.6.13 in characteristic 0. Let X be a smooth and irreducible curve, let $P \in X$.

i. Show that every $\omega \in \Omega^1_{K(X)}$ can be written as $\omega = a\frac{dt}{t} + df + \theta$ with $a \in k$, $f \in K(X)$ and θ a rational 1-form with $v_P(\theta) \geq 0$. (Hint: what is the integral of $t^{-n}dt$ for $n > 1$?).

ii. Show that $\mathrm{res}_P\, df = 0$ for every $f \in K(X)^\times$.

iii. Let t and t' be uniformizers at P. Show that the residue of dt'/t', computed with respect to the uniformizer t, equals 1.

iv. Show that the residue of a rational 1-form at a point P does not depend on the choice of uniformizer t at P.

Exercise 7.7.22

i. Prove that $g(\mathbb{P}^1) = 0$.

ii. Let X be an irreducible smooth projective curve satisfying $g(X) = 0$. Use Riemann-Roch to show that there exists a $f \in K(X)^\times$ with a pole of order 1 and a zero of order 1 and no further poles and zeroes.

By Exc. 7.7.18 we obtain a surjective morphism $f : X \to \mathbb{P}^1$. This map can be shown to be an isomorphism. As a consequence, any irreducible smooth projective curve of genus 0 is isomorphic to $\mathbb{P}^1$. In the language of Ch. 9 (and working over $k = \mathbb{C}$), the morphism $f : X \to \mathbb{P}^1$ induces a holomorphic map between the corresponding Riemann surfaces. This map has degree 1 and is therefore an isomorphism.

Exercise 7.7.23 Consider the rational 1-form $x^{-1}dx$ on $\mathbb{P}^1$. Compute its order and residue at all $P \in \mathbb{P}^1$.

Exercise 7.7.24 Prove that for all rational 1-forms ω on $\mathbb{P}^1$ we have $\sum_P \operatorname{res}_P(\omega) = 0$, where the sum is over all $P \in \mathbb{P}^1$. Hint: write $\omega = f{\cdot}dx$, with $f \in k(x)$, and use a suitable k-basis of $k(x)$. (See also Theorem 7.6.14).

Exercise 7.7.25 Prove Proposition 7.6.6.

Exercise 7.7.26 Let $n \in \mathbb{Z}_{\geq 2}$, $X = Z(-x_1^n + x_0^{n-1}x_2 - x_2^n) \subset \mathbb{P}^2$. Assume that $n(n-1)$ is in $k^\times$. We have already seen in Exercise 6.7.4 that X is smooth. You may now use without proof that X is irreducible. Let $U := X \cap \mathbb{A}^2$. Then $U = Z(f)$ with $f = -y^n + x^{n-1} - 1$.

i. Show that in $\Omega^1(U)$ we have $(n-1)x^{n-2}dx = ny^{n-1}dy$.

ii. We define a rational 1-form ω_0 by:

$$\omega_0 = \frac{dx}{ny^{n-1}} = \frac{dy}{(n-1)x^{n-2}}.$$

Show that ω_0 has no poles on U. Hint: $U = (U \cap D(x)) \cup (U \cap D(y))$.

iii. Show that for all $P \in U \cap D(x)$, $y - y(P)$ is a uniformizer at P.

iv. Show that for all $P \in U \cap D(y)$, $x - x(P)$ is a uniformizer at P.

v. Show that for all $P \in U$, ω_0 has order 0 at P. Hence (you do not need to prove this) $\Omega^1(U)$ is a free $\mathcal{O}(U)$-module, with basis ω_0.

vi. Let $Q = X \cap Z(x_2)$ be the point at infinity of X. Give a uniformizer at Q. Hint: see Example 2.4.1 and choose the appropriate coordinates.

vii. Compute $v_Q(x)$ and $v_Q(y)$.

viii. Compute $v_Q(\omega_0)$.

ix. Show that the x^iy^j with $0 \leq i$ and $0 \leq j < n$ form a k-basis of $\mathcal{O}(U)$, and that the $v_Q(x^iy^j)$ are all distinct.

x. For $n \in \{2, 3, 4\}$, give a basis (and hence the dimension) of $\Omega^1(X)$.

Exercise 7.7.27 Let X be an elliptic curve, i.e. an irreducible smooth projective curve X with $g(X) = 1$.

i. Show that $\Omega^1_X \cong \mathcal{O}_X$, which implies that X has a nowhere vanishing regular 1-form. *Hint: Show that* $\mathrm{H}^0(X, \Omega^1_X) \neq 0$ *and conclude that the canonical divisor can be represented by an effective divisor. What is the degree of this divisor?*

ii. Pick a point $O \in X$. Prove that the Abel-Jacobi map is a bijection. *Hint: Let* $[D] \in J(X)$ *and show that* $\dim \mathrm{H}^0(X, D + O) = 1$. *For surjectivity of the Abel-Jacobi: show that* $D + O \sim_{\mathrm{lin}} P$ *for some* $P \in X$. *For injectivity: show that if* $D + O = P \sim_{\mathrm{lin}} Q$*, for some* $P \neq Q \in X$*, then* $\dim \mathrm{H}^0(X, D + O) > 1$.

Hence for any elliptic curve X and $O \in X$, the elliptic curve X inherits a group structure from its Jacobian.

Lecture 8

Complex varieties and analytification

8.1 Holomorphic functions in several variables

There is a rich theory of holomorphic functions in several complex variables. We will only touch on a small part of it. In this section, we work with the standard Euclidean topology on $\mathbb{C}^n$, which is not the same as the Zariski topology unless $n = 0$.

Definition 8.1.1 Let $U \subset \mathbb{C}^n$ be an open subset, and $f \colon U \to \mathbb{C}$. Let $u = (u_1, \dots, u_n) \in U$. We say f *is holomorphic at* u if there exist $\epsilon \in \mathbb{R}_{>0}$ and complex numbers $c_{\underline{i}} : \underline{i} \in \mathbb{N}^n$ such that on the ball $B_\epsilon(u)$ we have an equality of functions

$$f(z_1, \dots, z_n) = \sum_{\underline{i} \in \mathbb{N}^n} c_{\underline{i}} \prod_{j=1}^{n} (z_j - u_j)^{i_j} .$$

Implicitly we mean that the right hand side converges absolutely at every point in $B_\epsilon(u)$.

We say f *is holomorphic on* U if f is holomorphic at u for every $u \in U$.

If $g \colon U \to \mathbb{C}^m$ is another function and $u \in U$, we say g is *holomorphic at* u if each of the m components of g is holomorphic at u (i.e. if for each of the m coordinate projections $\mathbb{C}^m \to \mathbb{C}$, the composite with g is holomorphic). Similarly, we say g is *holomorphic on* U if it is holomorphic at u for every $u \in U$.

We list some basic properties of holomorphic functions.

Lemma 8.1.2 *Holomorphic functions are C^∞ (hence continuous).*

Examples:

i. Any polynomial function, or power series which converges on U gives a holomorphic function.

ii. If f and g are polynomials and g has no zeros on U then the rational function f/g is holomorpic on U. For example, if $U = \mathbb{C} \setminus \{0\}$, $f = 1$ and $g = z$ then we see that not every holomorphic function can be globally defined by a power series.

iii. Not every holomorphic function can be written as a ratio of polynomials, even locally. For example, the exponential function.

Lemma 8.1.3 *i. Let $f \colon U \to \mathbb{C}$ be a holomorphic function which does not vanish anywhere. Then $1/f$ is also holomorphic.*

ii. Let $f \colon U \to V \subset \mathbb{C}^n$ and $g \colon V \to \mathbb{C}^m$ be holomorphic. Then $g \circ f$ is holomorphic.

Lemma 8.1.4 *Let $f\colon U \to \mathbb{C}^n$ be holomorphic. Then $\{u \in U : f(u) = 0\}$ is a closed subset (in the Euclidean topology).*

Proof By Lemma 8.1.2, f is continuous, so the result follows. □

8.2 Complex manifolds

Definition 8.2.1 Let $U \subset \mathbb{C}^n$ be Euclidean open. Define a $\mathbb{C}$-space $(U, \mathrm{hol}(U, \mathbb{C}))$ where U has the Euclidean topology, and $\mathrm{hol}(U, \mathbb{C})$ is the subsheaf of complex valued functions which are holomorphic.

These $\mathbb{C}$-spaces will play the role of 'affine varieties' in defining complex manifolds. Note that they are always *open* in $\mathbb{C}^n$, in contrast to affine varieties.

Definition 8.2.2 A *complex manifold* is a $\mathbb{C}$-space which is everywhere locally isomorphic to $(U, \mathrm{hol}(U, \mathbb{C}))$ for some n and some open subset $U \subset \mathbb{C}^n$.

A morphism of complex manifolds is just a morphism as $\mathbb{C}$-spaces (so the complex manifolds form a full subcategory of $\mathbb{C}$-spaces, just like $\mathbb{C}$-varieties).

There is an obvious notion of the dimension of a complex manifold. If you have seen real manifolds, note that the underlying topological space of a complex manifold of dimension n is a real manifold of dimension $2n$ - we will come back to this in the next lecture.

Example 8.2.3

i. Any union of open subsets of $\mathbb{C}^n$ gives a complex manifold, these are never compact unless empty or $n = 0$.

ii. Glueing complex manifolds works in exactly the same way as glueing varieties, cf. Exercise 5.5.14. Let $X_1 = X_2 = \mathbb{C}$ with its sheaf of holomorphic functions. Let $X_{12} = \{z \in X_1 : z \neq 0\}$ and similarly $X_{21} = \{z \in X_2 : z \neq 0\}$, these are open submanifolds. Define $\varphi_{1,2}\colon X_{12} \to X_{21}$ by $\varphi(z) = 1/z$ (with the obvious map on sheaves, cf 8.1.3). Then the complex manifold obtained from this glueing data is called $\mathbb{CP}^1$, i.e. the complex projective line. As a ringed space, this is not isomorphic to the variety $\mathbb{P}^1_{\mathbb{C}}$. For example, on the level of topological spaces, $\mathbb{CP}^1$ is Hausdorff but $\mathbb{P}^1_{\mathbb{C}}$ is not. Note that the constructions of $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{CP}^1$ look rather similar, though they are carried out in different categories. This will be generalised when we talk about 'analytification' of complex smooth varieties — it will turn out that $\mathbb{CP}^1$ is the analytification of $\mathbb{P}^1_{\mathbb{C}}$.

Though they are both special kinds of $\mathbb{C}$-spaces, $\mathbb{C}$-varieties and complex manifolds are very different - this is illustrated in the exercises.

8.3 Analytification

Let $\mathrm{SmVar}_{\mathbb{C}}$ be the full subcategory of $\mathrm{Var}_{\mathbb{C}}$ consisting of varieties that are smooth. The *analytification functor* takes as input a smooth complex variety (or morphism of such) and outputs a complex manifold (or map of such). From now until the end of this section, fix a smooth complex variety X. We will define a complex manifold X^{an}, called the 'analytification of X'.

The set

We define the underlying set of X^{an} to be the same as the underlying set of X.

The topology

The easiest way to define the analytic topology on X is to consider *all* isomorphisms of $\mathbb{C}$-spaces

$$\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} Z(I_\alpha) \subset \mathbb{A}^{n_\alpha}\}_\alpha,$$

where $U_\alpha \subset X$ is an open subset. We define $U \subset X$ to be *open in the analytic topology* if and only if $\varphi_\alpha(U \cap U_\alpha) \subset Z(I_\alpha) \subset \mathbb{C}^n$ is an open subset of $Z(I_\alpha) \subset \mathbb{C}^n$ with its Euclidean subspace topology. When we consider X with its analytic topology, we write X^{an}.

We now discuss two desirable properties. The first states that any Euclidean open subset of a chart $(U_\alpha, \varphi_\alpha)$ gives an analytic open subset. The second states that we can check openness in the analytic topology for any cover by affine opens (not just the maximal affine open cover considered above).

Proposition 8.3.1 *Let X be a complex variety and consider X^{an}.*

i. Suppose $\varphi_\alpha : U_\alpha \xrightarrow{\cong} Z(I_\alpha) \subset \mathbb{A}^{n_\alpha}$ is an isomorphism of $\mathbb{C}$-spaces with $U_\alpha \subset X$ open. Let $V \subset Z(I_\alpha) \subset \mathbb{C}^{n_\alpha}$ be any open subset of $Z(I_\alpha) \subset \mathbb{C}^{n_\alpha}$ in the Euclidean topology. Then $\varphi_\alpha^{-1}(V) \subset X^{\mathrm{an}}$ is open.

ii. Suppose $\{\varphi_i : U_i \xrightarrow{\cong} Z(I_i) \subset \mathbb{A}^{n_i}\}_i$ is any collection of morphisms of $\mathbb{C}$-spaces with $\{U_i\}_i$ an open cover of X (i.e. not necessarily the maximal such cover considered above). Suppose $U \subset X$ and $\varphi_i(U \cap U_i) \subset Z(I_i) \subset \mathbb{C}^{n_i}$ is an open subset of $Z(I_i) \subset \mathbb{C}^{n_i}$, with its Euclidean topology, for all i. Then $U \subset X^{\mathrm{an}}$ is open.

Proof Both statements follow from the lemma below. □

Lemma 8.3.2 *Let $I, J \subset \mathbb{C}[x_1, \ldots, x_n]$ be ideals and $V \subset Z(I)$, $W \subset Z(J)$ open subsets in the Zariski topology. Then $V \subset Z(I)$, $W \subset Z(J)$ are open subsets in the Euclidean topology and any isomorphism $f : V \to W$ of $\mathbb{C}$-spaces is also a homeomorphism between V, W with their Euclidean topologies.*

Proof Regular functions are continuous in the Euclidean topology. □

The $\mathbb{C}$-space structure

Up to now we have not used the smoothness of X, but at this point it will be crucial. We repeat definition 6.1.2 for the convenience of the reader:

Definition 8.3.3 Let X be a variety. For P in X, X is *smooth* at P if there is an open subvariety U of X containing P and an isomorphism $\varphi\colon U \tilde{\to} Z(f_1, \ldots, f_{n-d}) \subset \mathbb{A}^n$ for some $d \leq n$ and $f_1, \ldots, f_{n-d}$, such that the rank of the $n-d$ by n matrix over k:

$$\left(\frac{\partial f_i}{\partial x_j}(\varphi P)\right)_{i,j}$$

equals $n-d$, i.e. is maximal. The variety X is *smooth* if it is smooth at all its points. Points of X where X is not smooth are called *singular* and we denote the subset of singular points of X by X_{sing}.

The key to the construction is the *holomorphic implicit function theorem*, which we state without proof:

Theorem 8.3.4 (Holomorphic implicit function theorem) *Let $U \subset \mathbb{C}^n$ be Euclidean open and $f_1, \ldots, f_{n-d}$ be holomorphic functions on U. Let $p \in U$ be such that the $n-d$ by $n-d$ matrix*

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i,j \leq n-d}$$

is invertible. Then there exist

- *an open neighbourhood U' of p contained in U*
- *a open subset $W \subset \mathbb{C}^d$;*
- *holomorphic functions $w_1, \ldots, w_{n-d} \colon W \to \mathbb{C}$;*

such that for all $(z_1, \cdots, z_n) \in U'$ we have that

$$(f_i(z_1, \cdots, z_n) = 0 \text{ for all } 1 \leq i \leq n-d) \iff (z_i = w_i(z_{n-d+1}, \cdots, z_n) \text{ for all } 1 \leq i \leq n-d)\,.$$

Proof Omitted, see for example [KK, Section 0.8]. □

Now let X be a *smooth* complex variety and consider any isomorphism of $\mathbb{C}$-spaces

$$\varphi : U \to Z(f_1, \ldots, f_{n-d}) \subset \mathbb{A}^n,$$

where $U \subset X$ is open and

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq n-d}$$

has rank $n-d$ on a Zariski open subset $U' \subset \varphi(U)$ (recall that matrices of polynomials attain their maximal rank on a Zariski open subset). The holomorphic IFT gives an open subset $U' \subset Z(f_1, \ldots, f_{n-d})$ (in the Euclidean topology) and a map

$$\begin{aligned} &h : W \subset \mathbb{C}^d \to U' \subset Z(f_1, \ldots, f_{n-d}) \subset \mathbb{C}^n \\ &(z_{n-d+1}, \ldots, z_n) \mapsto \big(w_1(z_{n-d+1}, \ldots, z_n), \ldots, w_{n-d}(z_{n-d+1}, \ldots, z_n)\big) \end{aligned}$$

is bijective and holomorphic with inverse given by projection to the last d coordinates. In fact, h is a biholomorphic map between complex manifolds. Note that $\varphi^{-1}(U') \subset X^{\mathrm{an}}$ is open by Proposition 8.3.1(i). We therefore get a bijective map

$$\varphi^{-1}(U') \longrightarrow U' \longrightarrow W.$$

We consider *all* analytic open subsets $\varphi^{-1}(U')$ arising this way, together with the maps φ, h.

Let $O \subset X^{\mathrm{an}}$ be open. We define a $\mathbb{C}$-algebra of $\mathcal{O}_X^{\mathrm{an}}(O)$ of analytic functions on O as follows. A $\mathbb{C}$-valued function $f : O \to \mathbb{C}$ is defined to be an element of $\mathcal{O}_X^{\mathrm{an}}(O)$ if and only if for all U, φ, U', h, W as above, the composition

$$h^{-1}(\varphi(O \cap U) \cap U') \subset W \to \mathbb{C}$$

defined by $f \circ \varphi^{-1} \circ h$ is a holomorphic function. Note that $h^{-1}(\varphi(O \cap U) \cap U') \subset \mathbb{C}^d$ is open in the Euclidean topology. Then $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ defines a $\mathbb{C}$-space (Definition 4.3.1). Using the above notation, $(U, \mathcal{O}_X^{\mathrm{an}}|_U)$ is isomorphic (as a $\mathbb{C}$-space) to $(W, \mathrm{hol}(W))$, therefore $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ is a complex manifold. We refer to $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ as *the analytification of X*.

Analytification of morphisms

If $f \colon X \to Y$ is a morphism of $\mathbb{C}$-varieties, we want to get a morphism of complex manifolds from X^{an} to Y^{an}. This is straightforward because regular functions are holomorphic; we omit the details.

8.4 Examples

We can now give a large number of examples of complex manifolds - any smooth complex variety gives one after analytification.

Projective line

We have already seen $\mathbb{CP}^1$, but now you can check that $\mathbb{CP}^1 = \left(\mathbb{P}^1_{\mathbb{C}}\right)^{\mathrm{an}}$. Note that the latter is compact and Hausdorff (it is a sphere). We will come back to this in the next lecture.

Affine space

The analytification of $\mathbb{A}^n$ is $\mathbb{C}^n$ with the usual sheaf of holomorphic functions. It works similarly for any open subvariety of $\mathbb{A}^n$.

Note that (with the Zariski topology) any open subset X of $\mathbb{A}^n$ is compact (Exc. 8.5.1). On the other hand, the analytification X^{an} of such a subset is never compact unless $n = 0$ or it is empty.

With the Zariski topology, an open subset X of $\mathbb{A}^n$ is Hausdorff if and only if it is empty or $n = 0$ (Exc. 8.5.1). On the other hand, the analytification X^{an} of such a subset is always Hausdorff, since $\mathbb{C}^n$ is.

This suggests that studying X^{an} may not be a good way to gain information on X, but in fact this is far from true, and in the next lecture we will begin to develop a dictionary between them.

8.5 Exercises

Exercise 8.5.1 Let $U \subset \mathbb{C}^n$ be a non-empty open subset in the Zariski topology and $n > 0$

i. Prove that U is compact (see also Exc. 1.6.12).

ii. Prove that U is not Hausdorff

Exercise 8.5.2 Give an example of a holomorphic function on $\mathbb{C}$ whose zero set is not closed in the Zariski topology.

Exercise 8.5.3 Here we check some basic facts about rational and holomorphic functions, in the 1-variable case for simplicity. Let $f \in \mathbb{C}[x]$ be a polynomial with $f(0) \neq 0$.

i. Show that the image of f in the ring $\mathbb{C}[[x]]$ of formal power series is a unit.

ii. Show that the formal inverse of f that you found above has a positive radius of convergence.

iii. If you are following the commutative algebra course, show that $\mathbb{C}[[x]]$ is a *local ring*.

Exercise 8.5.4 Show that the underlying topological space of $\mathbb{CP}^1$ is a sphere. If you get stuck, think of 'stereographic projection'.

Exercise 8.5.5 Let X be a complex variety.

i. Assume X is separated. Show that X^{an} is Hausdorff.

ii. Assume X^{an} is connected. Show that X is connected.

In fact the converses also hold, but this is harder and is omitted.

Exercise 8.5.6 Let X be the complement of the origin in $\mathbb{A}^1_{\mathbb{C}}$. Pick any basepoint in X.

i. Compute the fundamental group of X^{an} with the Euclidean topology.

ii. Compute the fundamental group of X with the Zariski topology.

It turns out that there is a good notion of the fundamental group of an algebraic variety, even for varieties not over $\mathbb{C}$ (the 'étale fundamental group').

Lecture 9

Riemann surfaces and Riemann-Hurwitz

The main aim of this lecture is to discuss the analytification functor in more detail in the case of varieties of dimension 1. We will see that the topological space underlying a complex manifold is much nicer to work with than the topological space underlying a variety. We start by giving a 'dictionary' relating properties of varieties and properties of manifolds (with no restrictions on dimension yet). Then we focus on the case of (complex) dimension 1, i.e. Riemann surfaces. After recalling/introducing some facts from topology, we discuss the Riemann-Hurwitz theorem.

9.1 Dictionary between varieties and manifolds

Projective varieties and compactness

The complex variety $\mathbb{P}^n_{\mathbb{C}}$ is smooth, so you can analytify it. This is a very important example:

Theorem 9.1.1 *The (underlying topological space of the) complex manifold $\mathbb{CP}^n$ obtained by analytifying $\mathbb{P}^n_{\mathbb{C}}$ is compact.*

Proof Recall that the continuous image of a compact space is compact, so it is enough to construct a continuous surjection from a compact space. In fact, we will construct a continuous surjection from the $(2n+1)$-sphere S^{2n+1}.

Recall that we can think of $\mathbb{P}^n_{\mathbb{C}}$ as the space of (complex) lines through the origin in $\mathbb{C}^{n+1}$. Note that $\{z \in \mathbb{C}^m : |z| = 1\}$ is naturally S^{2m-1}. Given a point $z \in \mathbb{C}^{n+1}$ with $|z| = 1$, there is a unique (complex) line through 0 and z, and every complex line arises in this way. This gives a surjection from $S^{2(n+1)-1}$ to $\mathbb{CP}^n$. We leave the (straightforward, but somewhat tedious) verification of continuity to the reader. □

Exercise 9.6.1 gives an alternative proof.

Corollary 9.1.2 *Let X be a smooth projective complex variety. Then X^{an} is compact.*

Proof There exists a closed embedding $X \to \mathbb{P}^n$ for some n. Then X^{an} is a subspace of $\mathbb{CP}^n$ given as the zero set of some homogeneous polynomials and so is a closed subset of $\mathbb{CP}^n$. Hence it is compact by the above theorem. □

Further properties

Above, we saw that if X is projective then X^{an} is compact. The converse fails, though it is true up to dimension 1. What about Hausdorff?

Theorem 9.1.3 *X is separated if and only if X^{an} is Hausdorff.*

Proof The direction 'X separated implies X^{an} Hausdorff was shown in Exc. 8.5.5. The converse is a lot harder and is omitted. □

Theorem 9.1.4 *X is connected if and only if X^{an} is connected.*

Proof One direction was in Exc. 8.5.5. Again, the converse is harder, and is omitted. □

We summarise this (and a few other properties) in a table:

X		X^{an}
projective	$\Longrightarrow$	compact
separated	$\Longleftrightarrow$	Hausdorff
connected	$\Longleftrightarrow$	connected
dimension n	$\Longleftrightarrow$	dimension n

We also mention a rather wonderful theorem:

Theorem 9.1.5 *The analytification functor is fully faithful on smooth projective varieties.*

Proof See [Artin, Algebrization of formal moduli II, theorem 7.3]. □

9.2 Riemann surfaces

We now look more closely at the case of complex varieties of dimension 1, with particular attention to the projective case.

Definition 9.2.1 A *Riemann surface* is a Hausdorff complex manifold of dimension 1.

Example 9.2.2

- We have already seen $\mathbb{CP}^1$;
- If X is a separated complex variety of dimension 1 then X^{an} is a Riemann surface. If X is projective then X^{an} is compact. Moreover, X is connected if and only if X^{an} is connected.

If we restrict to compact Riemann surfaces, it turns out that this is the only source of examples. We noted above that the analytification functor is fully faithful on projective varieties. In the case of dimension 1 it is also essentially surjective:

Theorem 9.2.3 *The analytification gives an equivalence of categories between*

- *smooth projective complex varieties of dimension 1;*
- *compact Riemann surfaces.*

Proof Omitted, see e.g. [Algebraic Curves and Riemann Surfaces by Rick Miranda]. □

9.3 Triangulations, Euler characteristics, and coverings

In this section, we discuss some facts (without proof) from topology.

Definition 9.3.1 A *topological n-manifold* is a Hausdorff topological space X such that for every $x \in X$ there exists an open neighbourhood U of x, an open subset $V \subset \mathbb{R}^n$, and a homeomorphism $U \to V$. We refer to topological 2-manifolds as *topological surfaces*.

Note that being a topological n-manifold is a *property* of a topological space, there is no extra data attached.

An easy exercise shows that if M is a complex manifold, say of dimension m, then the underlying topological space is a topological $2m$-manifold. In particular, a Riemann surface (1-dimensional complex manifold) has a topological surface as its underlying topological space, which explains the terminology.

Just as studying complex manifolds provides information about complex varieties, studying topological manifolds can tell us about complex manifolds. Be warned that topological manifolds are much less 'rigid' than complex manifolds - for example, many non-isomorphic compact Riemann surfaces can have isomorphic underlying topological manifolds.

We want to talk about triangulations of Riemann surfaces. Let Δ be the closed subset

$$\Delta = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \text{ and } x + y \leq 1\}.$$

The edges and vertices are the obvious closed subsets (more technically speaking, we consider the subset $\{(0,0), (1,0), (0,1)\}$ of $\mathbb{R}^2$, and the convex hulls of the 7 non-empty subsets).

Definition 9.3.2 Let X be a topological surface and let I be a finite set. A *triangulation* $\mathcal{T} = \{(X_i, \tau_i)\}_{i \in I}$ of X consists of:

- a closed subset $X_i \subset X$, for each $i \in I$, such that $X = \bigcup_{i \in I} X_i$,
- a homeomorphism $\tau_i \colon X_i \to \Delta$ for each $i \in I$,

such that for every $i, j \in I$ with $i \neq j$ we have that $\tau_i(X_i \cap X_j)$ is empty, or an edge or a vertex of Δ.

The following non-trivial result is due to to Radó. A nice exposition can be found in [*The Jordan-Schoenflies theorem and the classification of surfaces* by Carsten Thomassen, `http://www.maths.ed.ac.uk/~aar/jordan/thomass.pdf`]. Interestingly, the analogous statement in dimension 4 or higher is false.

Theorem 9.3.3 *Every compact topological surface admits a triangulation.*

Definition 9.3.4 Let X be a topological surface and let $\mathcal{T} = \{(X_i, \tau_i)\}_{i \in I}$ be a triangulation of X. Then we define $F(X, \mathcal{T}) = \#I$, i.e. the number of faces of $\mathcal{T}$, and analogously $E(X, \mathcal{T})$ the number of edges of $\mathcal{T}$ and $V(X, \mathcal{T})$ the number of vertices of $\mathcal{T}$. We define the *Euler characteristic* of $(X, \mathcal{T})$ as

$$\chi(X, \mathcal{T}) = V(X, \mathcal{T}) - E(X, \mathcal{T}) + F(X, \mathcal{T}).$$

Clearly $V(X, \mathcal{T})$, $E(X, \mathcal{T})$ and $F(X, \mathcal{T})$ depend on $\mathcal{T}$ as well as X. However, we have the following theorem:

Theorem 9.3.5 *If $\mathcal{T}$ and $\mathcal{T}'$ are two triangulations of a compact topological surface X, then we have $\chi(X, \mathcal{T}) = \chi(X, \mathcal{T}')$. We denote this number by $\chi(X)$ and refer to it as the* Euler characteristic *of X.*

The Euler characteristics of the following surfaces can be found by drawing an appropriate triangulations (Exc. 9.6.2).

Example 9.3.6 Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere with induced Euclidean topology.

i. Then $\chi(S^2) = 2$.

Let $R = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be the unit square with its Euclidean topology.

ii. Identifying $\{(0, y)\}$ with $\{(1, y)\}$ and $\{(x, 0)\}$ with $\{(x, 1)\}$, for all $0 \leq x, y \leq 1$, gives the torus T which satisfies $\chi(T) = 0$.

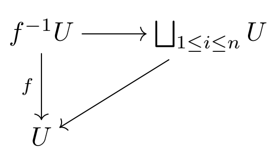
iii. Identifying $\{(0, y)\}$ with $\{(1, y)\}$ and $\{(x, 0)\}$ with $\{(1 - x, 1)\}$, for all $0 \leq x, y \leq 1$, gives the Klein bottle K which satisfies $\chi(K) = 0$.

iv. Identifying $\{(0, y)\}$ with $\{(1, 1 - y)\}$ and $\{(x, 0)\}$ with $\{(1 - x, 1)\}$, for all $0 \leq x, y \leq 1$, gives the real projective plane $\mathbb{RP}^2$ which satisfies $\chi(\mathbb{RP}^2) = 1$.

Definition 9.3.7 Let $f : X \to S$ be a surjective continuous map between topological spaces. We say that f is a *covering of degree* n if for every $s \in S$ there exists an open neighbourhood U of s and a homeomorphism

$$f^{-1}U \to \bigsqcup_{1 \leq i \leq n} U,$$

which fits in the commutative diagram



where, for all i, the diagonal arrow maps the ith copy U of $\bigsqcup_{1 \leq i \leq n} U$ identically to U.

Theorem 9.3.8 *Let X and S be compact topological surfaces and let $f : X \to S$ be a covering of degree n. Then*

$$\chi(X) = n\chi(S).$$

Proof We choose a sufficiently fine triangulation $\mathcal{T}$ of Y in the following sense. Any triangle of $\mathcal{T}$ lies entirely inside one of the open subsets U on which f trivializes, i.e. one of the open subsets appearing in Definition 9.3.7. We do not prove the existence of such a triangulation.

We can give X a triangulation by just pulling back the triangulation from Y to X in the obvious manner. Then it is clear that there are n faces of X over every face of S, and the same for edges and vertices. □

We can now connect back to complex manifolds (and hence varieties):

Theorem 9.3.9 *Let X be a smooth connected projective complex curve of genus g. Then $\chi(X^{\mathrm{an}}) = 2 - 2g$, where g denotes the genus of X.*

Let us consider the example of $\mathbb{P}^1_{\mathbb{C}}$. You checked that its analytic topological space is a sphere S^2, so $\chi(\mathbb{CP}^1) = 2$ by Ex. 9.3.6 above. On the other hand, the genus is 0 of $\mathbb{P}^1_{\mathbb{C}}$ Exc. 7.7.22(i), so we have $2 - 2g = 2$.

9.4 The Riemann-Hurwitz formula

In this section, we discuss the Riemann-Hurwitz formula. We start with two facts from the theory of Riemann surfaces. Let $f : X \to Y$ be a surjective holomorphic map between compact Riemann surfaces.

Fact 1. There exists a finite set $R \subset Y$, such that $f : X \setminus f^{-1}(R) \to Y \setminus R$ is a covering of degree d for some $d \geq 1$. We refer to R as the *ramification locus* of f and to $B := f^{-1}(R)$ as the *branch locus* of f.

Fact 2. For any branch point $P \in B$, there exists an open neighbourhood U of P and an open neighbourhood V of $Q = f(P)$ such that $f(U) \subset V$ and with the following properties. There are isomorphisms $U \cong U' \subset \mathbb{C}$ and $V \cong V' \subset \mathbb{C}$, where U', V' are open neighbourhoods of 0, fitting in a commutative diagram

$$\begin{array}{ccc} U \subset X & \xrightarrow{f} & V \subset Y \\ \downarrow \cong & & \downarrow \cong \\ U' \subset \mathbb{C} & \xrightarrow{z^{d'}} & V' \subset \mathbb{C} \end{array}$$

for some $2 \leq d' \leq d$, i.e. the map f near P is $z \mapsto z^{d'}$. We refer to $e_P := d'$ as the *ramification index* of P. We say that P has *simple ramification* when $e_P = 2$.

We can now present the Riemann-Hurwitz formula.

Theorem 9.4.1 (Riemann-Hurwitz) *Let $f : X \to Y$ be a surjective holomorphic map of degree d between compact Riemann surfaces. Then*

$$\chi(X) = d\chi(Y) - \sum_{P \in B} (e_P - 1).$$

Proof We choose a sufficiently fine triangulation $\mathcal{T}$ of Y in the following sense. (1) Every point of R corresponds to a vertex of $\mathcal{T}$. (2) Except for the vertices corresponding to points of R, every triangle of $\mathcal{T}$ lies entirely inside an open subset U of the trivialization of the covering $f\colon X \setminus f^{-1}(R) \to Y \setminus R$ (as in the proof of Theorem 9.3.8). We do not prove the existence of such a triangulation.

Granted such a triangulation exists, the rest of the proof goes as follows. We obtain a triangulation $f^*\mathcal{T}$ of X defined by pulling back the triangulation $\mathcal{T}$ of Y. Since f has degree d, we find

$$F(X, f^*\mathcal{T}) = dF(Y, \mathcal{T}), \quad E(X, f^*\mathcal{T}) = dE(Y, \mathcal{T}).$$

If there are no ramification points, then we have $V(X, f^*\mathcal{T}) = dV(Y, \mathcal{T})$. However, any ramification point $Q \in R$ has only $d - \sum_{P \in f^{-1}(Q)} (e_P - 1)$ distinct branch points mapping to it, so

$$V(X, f^*\mathcal{T}) = dV(Y, \mathcal{T}) - \sum_{P \in B} (e_P - 1).$$

We conclude (Theorem 9.3.5)

$$\begin{aligned} \chi(X) &= V(X, f^*\mathcal{T}) - E(X, f^*\mathcal{T}) + F(X, f^*\mathcal{T}) \\ &= dV(Y, \mathcal{T}) - \sum_{P \in B} (e_P - 1) - dE(Y, \mathcal{T}) + dF(Y, \mathcal{T}) \\ &= d\chi(Y) - \sum_{P \in B} (e_P - 1). \end{aligned}$$

□

We now have all the main tools of the theory of curves at our disposal:

- $\dim H^0(X,D) - \dim \mathrm{H}^1(X,D) = \deg D + 1 - g$ for any irreducible smooth projective curve X of genus g and any divisor D on X (Riemann-Roch).
- $\dim \mathrm{H}^1(X,D) = \dim H^0(X, K_X - D)$ for any irreducible smooth projective curve X and any divisor D on X (Serre duality).
- $\chi(X^{\mathrm{an}}) = 2 - 2g$ for any irreducible smooth projective *complex* curve X of genus g.
- Given a surjective morphism of irreducible smooth projective *complex* curves $f\colon X \to Y$, we have $\chi(X^{\mathrm{an}}) = d\chi(Y^{\mathrm{an}}) - \sum_{P\in B}(e_P - 1)$, where d is the degree of f (Riemann-Hurwitz).

9.5 Application: the genus of a Fermat curve

Consider the following projective curve in $\mathbb{P}^2_{\mathbb{C}}$ known as a *Fermat curve*

$$F_d : x^d + y^d = z^d,$$

where $d \geq 1$ and x, y, z are homogeneous coordinates on $\mathbb{P}^2$. Fermat curves are irreducible and smooth (Exc. 9.6.3). Hence F_d^{an} is a Riemann surface (compact, Hausdorff, connected by Sect. 9.1).

Consider the following surjective morphism (Exc. 9.6.3)

$$f : F_d \to \mathbb{P}^1, \quad (x : y : z) \mapsto (x : z).$$

We study the fibres of f. Let $P = (x : z) \in \mathbb{P}^1$.

Case 1: $z \neq 0$**.** After scaling, we study the fibre over $(x : 1)$. When $x^d \neq 1$, the equation $y^d = 1 - x^d \neq 0$ has d solutions and $\#f^{-1}(P) = d$. We deduce that f has degree d. When $x^d = 1$, the equation $y^d = 0$ has one solution, so $\{(\zeta^k : 1)\}$, where ζ is a primitive dth root of unity and $k = 0, \ldots, d-1$, are ramification points with (maximal) ramification index d.

Case 2: $z = 0$**.** After scaling, we study the fibre over $(1 : 0)$. This fibre has d elements, so $(1 : 0)$ is not a ramification point.

Using Riemann-Hurwitz and $\chi(\mathbb{CP}^1) = 2$ (Sect. 9.3), we find

$$\chi(F_d) = d\chi(\mathbb{CP}^1) - d(d-1) = 2d - d(d-1) = 2 - 2g,$$

where we write g for the genus of the Fermat curve. We conclude that

$$g = \frac{(d-1)(d-2)}{2}.$$

We mention as a fact that any projective curve in $\mathbb{P}^2$, defined by a homogeneous polynomial of degree d without multiple factors, has genus $g = (d-1)(d-2)/2$.

9.6 Exercises

Exercise 9.6.1 The closed unit polydisk in $\mathbb{C}^n$ is defined to be the set of $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ such that for each $i = 1, \ldots, n$ we have $|z_i| \leq 1$. Take the open cover (in the complex topology) of $\mathbb{CP}^n$ by standard affine opens $U_0, \ldots, U_n$. Show that $\mathbb{CP}^n$ is covered by the closed unit polydisks in the U_i. Show that $\mathbb{CP}^n$ is compact.

Exercise 9.6.2

i. Prove that $\chi(S^2) = 2$. *Hint: Inscribe a tetrahedron inside a sphere and radially project it (from the origin) onto the surface of the sphere. Extra: it is instructive to do the same calculation using the triangulation of S^2 obtained by radially projecting an octahedron or icosahedron onto the circumscribed sphere.*

ii. Prove that $\chi(T) = \chi(K) = 0$ and $\chi(\mathbb{RP}^2) = 1$ (Example 9.3.6). *Hint: Draw appropriate triangulations on the unit rectangle and ensure they remain triangulations after identification.*

iii. (More work) Let S be a "doughnut with two holes". Prove that $\chi(S) = -2$. *Hint: Define S by taking an octogon in $\mathbb{R}^2$ and identiying opposite sides in an appropriate way. Then draw a triangulation on the octogon, which remains a triangulation after identifications.*

Exercise 9.6.3

i. Prove that the Fermat curve $F_d \subset \mathbb{P}^2$ is an irreducible smooth projective curve.

ii. Prove that $f : F_d \to \mathbb{P}^1$, $(x : y : z) \mapsto (x : z)$ is a well-defined morphism of varieties.

Exercise 9.6.4 Suppose $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a map of complex varieties such that $f^{\mathrm{an}}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a topological cover of degree d for some d. What are the possible values of d?

Exercise 9.6.5 Let a, b, c be distinct complex numbers. Define a curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ by the equation

$$y^2 = (x - a)(x - b)(x - c).$$

Define $\pi\colon C \to \mathbb{P}^1_{\mathbb{C}}$ by sending $(X : Y : Z)$ to $(X : Z)$ if $(X : Y : Z)$ is not equal to $(0 : 1 : 0)$, and sending $(0 : 1 : 0)$ to $(1 : 0)$.

i. Verify that π is a map of complex varieties.

ii. Verify that π^{an} satisfies the hypotheses of theorem 9.4.1 for some d (what is that d?). Take care with the point at $Z = 0$.

iii. Compute the genus of C.

Exercise 9.6.6 Give an example of a map of topological spaces from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ which does not arise as the analytification of any map of varieties $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. You can describe your map in words, but you must show that it does not come from a map of varieties.

Exercise 9.6.7 In this exercise $k = \mathbb{C}$. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero set of the bi-homogeneous polynomial $f(x_0, x_1, y_0, y_1) = (x_0^3 + x_1^3)y_0^2 + (x_0^3 - x_1^3)y_1^2$. Consider the morphism $f\colon X \to \mathbb{P}^1$, which sends $((a_0 : a_1), (b_0 : b_1))$ to $(a_0 : a_1)$.

i. Prove that X is a smooth curve.

ii. Determine the degree of f and use Riemann-Hurwitz to show that the genus of X is 2.

iii. Let $P \in X$. Show that any divisor D on X satisfies $\deg D \leq \dim H^0(X, D + P)$.

Lecture 10

Curves on surfaces

In this lecture, for closed curves Z_1 and Z_2 on a smooth irreducible projective surface X, we will define their intersection number $Z_1 \cdot Z_2$.

10.1 Divisors

Let X be a smooth connected quasi-projective variety of dimension d. So in particular X is irreducible.

Definition 10.1.1 A *prime divisor* on X is a closed irreducible subset $Z \subset X$ of dimension $d-1$.

Definition 10.1.2 A *divisor* is an element of the free abelian group generated by the prime divisors. We denote this group by $\mathrm{Div}(X)$.

So divisors are formal expressions of the form $\sum_i n_i Z_i$ with Z_i a finite collection of prime divisors, and the n_i integers.

Proposition 10.1.3 *Let X be a smooth, connected, quasi-projective variety. Let $Z \subset X$ be a prime divisor. Then there is a finite open affine cover $\{U_i\}_i$ of X, such that there are nonzero $f_i \in \mathcal{O}_X(U_i)$ with the property that $I(Z \cap U_i) = (f_i)$ as ideals in $\mathcal{O}_X(U_i)$.*

Proof We omit the proof. For $d = 1$, this proposition follows from the proof of Proposition 7.3.7. For $d > 2$, it uses the fact that the local ring $\mathcal{O}_{X,P}$ of a variety X at a *smooth* point P is a unique factorization domain. □

Now we want to associate a valuation to a prime divisor. Let $Z \subset X$ be a prime divisor. Use an affine cover $\{U_i : i \in I\}$ as in the above proposition. Then choose an i with $Z \cap U_i \neq \emptyset$. For $0 \neq f \in \mathcal{O}_X(U_i)$ we define:

$$v_Z(f) := \text{the largest integer } n \text{ such that } f \in (f_i^n).$$

Such a largest integer exists and it does not depend on the chosen cover $\{U_i : i \in I\}$ or the particular choice of i. This v_Z has the property $v_Z(fg) = v_Z(f) + v_Z(g)$. As usual we extend this to a morphism $v_Z \colon K(X)^\times \to \mathbb{Z}$.

Definition 10.1.4 Let X be a smooth, connected, quasi-projective variety. Then we define the divisor map

$$\mathrm{div} \colon K(X)^\times \longrightarrow \mathrm{Div}(X), \quad f \mapsto \mathrm{div}(f) := \sum_{Z \text{ prime}} v_Z(f) Z.$$

Note that div is a group homomorphism. To see that the sum occurring in $\mathrm{div}(f)$ is finite, first reduce to the case that X is affine (it has a cover by finitely many), then write f as g/h and note that nonzero coefficients only occur at Z that are irreducible components of $Z(g)$ or $Z(h)$. A divisor of the form $\mathrm{div}(f)$ is called a *principal divisor*.

Definition 10.1.5 For X a smooth, connected, quasi-projective variety we call two divisors D_1, D_2 on X *linearly equivalent* when $D_1 - D_2 = \mathrm{div}(f)$ for some $f \in K(X)^\times$. If this is the case, then we write $D_1 \sim_{\mathrm{lin}} D_2$. We define the *divisor class group* of X by $\mathrm{Cl}(X) := \mathrm{Div}(X)/\sim_{\mathrm{lin}}$ or, in other words, the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors.

Example 10.1.6 By Proposition 4.6.9 every prime divisor of $\mathbb{A}^d$ is of the form $Z(f)$ for $f \in k[x_1, \ldots, x_d]$ irreducible. But then every prime divisor is principal, hence $\mathrm{Cl}(\mathbb{A}^d) = 0$.

Proposition 10.1.7 *Let $X = \mathbb{P}^d$ with $d \in \mathbb{Z}_{\geq 1}$. Then $\mathrm{Cl}(\mathbb{P}^d) \cong \mathbb{Z}$, generated by the class of a hyperplane.*

Proof By Proposition 4.6.11, the prime divisors of $\mathbb{P}^n$ are of the form $Z(f)$ where $f \in k[x_0, \ldots, x_d]$ is homogeneous and irreducible. We now define $\deg(Z(f)) := \deg(f)$. We extend this to a morphism of groups and obtain a map deg as follows:

$$\deg\colon \mathrm{Div}(X) \longrightarrow \mathbb{Z}, \quad \sum_i n_i Z_i \mapsto \sum_i n_i \deg(Z_i).$$

We now claim that $D = \sum_i n_i Z_i$ is principal if and only if $\deg(D) = 0$. Indeed, consider a divisor $\mathrm{div}(f)$ for some $f \in K(X)^\times$ and write $f = g/h$ with g and h in $k[x_0, \ldots, x_d]$ homogeneous of the same degree. Decompose g and h into irreducibles, $g = \prod_i g_i^{n_i}$ and $h = \prod_i h_i^{m_i}$, then

$$\deg(\mathrm{div}(f)) = \sum_i n_i \deg(g_i) - \sum_i m_i \deg(h_i) = \deg(g) - \deg(h) = 0.$$

On the other hand, if $\deg(\sum n_i Z_i) = 0$, then let $Z_i = Z(f_i)$ and consider $f := \prod f_i^{n_i}$. By construction $\deg(f) = 0$ and so $f \in K(X)^\times$ and $\mathrm{div}(f) = \sum n_i Z_i$.

So the degree factors through an injective map $\mathrm{Cl}(X) \to \mathbb{Z}$. The map is also surjective, since for example $\deg Z(x_0) = 1$. □

Remark 10.1.8 Note that for $d = 1$, the degree of a divisor on $\mathbb{P}^d$ constructed in the proof above coincides with the degree of a divisor on a curve, as defined in Definition 7.2.7.

10.2 The intersection pairing on surfaces

Let X be a smooth connected projective surface (a smooth connected projective variety of dimension 2). In this section we define the intersection pairing on $\mathrm{Div}(X)$, show that it factors through $\mathrm{Pic}(X)$, and derive Bézout's theorem for $\mathbb{P}^2$ as a simple consequence.

For prime divisors Z_1 and Z_2 on X the intersection number $Z_1 \cdot Z_2$ in $\mathbb{Z}$ is defined as the degree on Z_1 of a locally free $\mathcal{O}_{Z_1}$-module of rank one, $\mathcal{O}_X(Z_2)|_{Z_1}$. As we have not defined these notions (lack of time) we give the procedure that produces $Z_1 \cdot Z_2$ in terms of concepts that we have defined, and that one would use even if one had the notions that we did not define. This definition of $Z_1 \cdot Z_2$ does not assume that Z_1 and Z_2 are distinct.

Definition 10.2.1 Let Z_1 and Z_2 be prime divisors on X.

i. Choose open subsets $(U_i)_{i \in I}$ ($I = \{1, \ldots, r\}$ for some r) in X and f_i in $\mathcal{O}_X(U_i)$ such that the U_i cover Z_2, each U_i meets Z_2, and such that $\mathrm{div}(f_i) = Z_1 \cap U_i$ on U_i. In particular, $I(Z_1 \cap U_i) = (f_i)$ (as in Proposition 10.1.3).

ii. Since f_i and f_j generate the same ideal of $\mathcal{O}_X(U_{ij})$, where $U_{ij} := U_i \cap U_j$, there are unique f_{ij} in $\mathcal{O}_X(U_{ij})^\times$ such that $f_i = f_{ij}f_j$ in $\mathcal{O}_X(U_{ij})$. Note that $f_{ij}{\cdot}f_{jk} = f_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$.

iii. Define $g_i := f_{i1} \in \mathcal{O}_{Z_2}(Z_2 \cap U_{i1})^\times$. Remark that $g_1 = 1$ and that $g_i = f_{ij}g_j$ in $\mathcal{O}_{Z_2}(Z_2 \cap U_{ij1})$. This shows that $g_i \neq 0$ in $\mathcal{O}_{Z_2}(Z_2 \cap U_{i1})$. For $P \in Z_2$ and i such that $P \in U_i$, the number $v_P(g_i)$ is independent of i. We finally define:

$$Z_1{\cdot}Z_2 := \sum_{P \in Z_2} v_P(g_{i_P}) = \sum_{P \in Z_2 - Z_2 \cap U_1} v_P(g_{i_P}), \quad \text{where } i_P \in I \text{ such that } P \in U_{i_P}.$$

Remark 10.2.2 The ideas behind Definition 10.2.1 may be understood, very briefly, as follows. On U_i, the $\mathcal{O}_X$-module $\mathcal{O}_X(Z_1)$ is generated by $1/f_i$. In particular, $1/f_1$ provides a non-zero section

$$s_{Z_1} := 1/f_1 : \mathcal{O}_X \to \mathcal{O}_X(Z_1).$$

See Exc. 10.3.6 for the definition of "section of an invertible sheaf". When $Z_1 \neq Z_2$, restriction to Z_2 gives a non-zero section

$$s_{Z_1}|_{Z_2} : \mathcal{O}_{Z_2} \to \mathcal{O}_X(Z_1)|_{Z_2}.$$

Then the above definition can be rewritten as $Z_1 \cdot Z_2 = \deg(\mathrm{div}(s_{Z_1}|_{Z_2}))$. There exists a notion of "degree of an invertible sheaf on an irreducible projective curve" (what would it be for *smooth* irreducible projective curves?), which can be used to show that $Z_1 \cdot Z_2 = \deg(\mathcal{O}_X(Z_1)|_{Z_2})$. Interestingly, the last equality also holds when $Z_1 = Z_2$ and can therefore be taken as the definition of $Z_1 \cdot Z_2$.

As promised, we will show that this really is a good definition. We will make frequent use of the following fact, which we will not prove. We refer to Exercise 7.7.15 for a proof in the smooth case.

Proposition 10.2.3 *If f is a non-zero rational function on an irreducible projective curve X then* $\deg \mathrm{div}(f) = 0$.

Lemma 10.2.4 *The integer $Z_1{\cdot}Z_2$ does not depend on the choice of the f_i.*

Proof Assume that f_i' for i in I satisfy the same conditions as the f_i. Then $f_i' = u_i f_i$ with $u_i \in \mathcal{O}_X(U_i)^\times$, and $f_{ij}' := f_i'/f_j' = (u_i/u_j)f_{ij}$ and $g_i' = (u_i/u_1)g_i$. This then gives (we use that $v_P(u_i) = 0$ for all $P \in U_i$ and that the degree of a principal divisor is 0):

$$(Z_1{\cdot}Z_2)' = Z_1{\cdot}Z_2 + \sum_P v_P(u_{i_P}/u_1) = Z_1{\cdot}Z_2 + \deg(\mathrm{div}(1/u_1)) = Z_1{\cdot}Z_2.$$

□

Lemma 10.2.5 *The integer $Z_1{\cdot}Z_2$ does not depend on the choice of 1 in $\{1, \ldots, r\}$ in step iii.*

Proof Assume that we use U_2 instead. Then $g_i' = f_{i2} = f_{i1}f_{12} = g_i f_{12}$. Hence:

$$(Z_1{\cdot}Z_2)' = Z_1{\cdot}Z_2 + \deg(\mathrm{div}(f_{12})) = Z_1{\cdot}Z_2$$

□

Lemma 10.2.6 *The integer $Z_1{\cdot}Z_2$ does not depend on the choice of the open cover $\{U_i : i \in I\}$.*

Proof Given two covers $\{U_i : i \in I\}$ and $\{U_j' : j \in J\}$, one can consider a common refinement (given by for example the open $\{U_i \cap U_j' : i \in I, j \in J\}$). So it is enough to show that the lemma holds for a refinement, and this is just a calculation which we leave to the reader. □

As $\mathrm{Div}(X)$ is the free $\mathbb{Z}$-module with basis the set of prime divisors on X, the map "$\cdot$" extends bilinearly and we obtain a bilinear map:

$$\cdot\colon\ \mathrm{Div}(X)\times\mathrm{Div}(X)\longrightarrow\mathbb{Z},\quad (Z_1,Z_2)\mapsto Z_1{\cdot}Z_2.$$

Bilinearity means that $(aZ_1{+}bZ_2){\cdot}Z_3=a(Z_1{\cdot}Z_3){+}b(Z_2{\cdot}Z_3)$ and $Z_1{\cdot}(aZ_2{+}bZ_3)=a(Z_1{\cdot}Z_2){+}b(Z_1{\cdot}Z_3)$ for all $a,b\in\mathbb{Z}$ and for all divisors Z_1,Z_2,Z_3.

Proposition 10.2.7 *Let $Z_1\neq Z_2$ be prime divisors. Then $Z_1\cap Z_2$ is finite. For all P in $Z_1\cap Z_2$ there is an open affine $U_P\subset X$ with $P\in U_P$ such that $U_P\cap Z_1\cap Z_2=\{P\}$ and $f_{1,P}$ and $f_{2,P}\in\mathcal{O}_X(U_P)$ such that $I(Z_1\cap U_P)=(f_{1,P})$ and $I(Z_2\cap U_P)=(f_{2,P})$. For such a collection of U_P we have:*

$$Z_1{\cdot}Z_2=\sum_{P\in Z_1\cap Z_2}\dim\mathcal{O}_X(U_P)/(f_{1,P},f_{2,P}).$$

For each $P\in Z_1\cap Z_2$ the integer $\dim\mathcal{O}_X(U_P)/(f_{1,P},f_{2,P})$ is independent of the choice of U_P, and is called the local intersection multiplicity of Z_1 and Z_2 at P.

Proof As $Z_1\cap Z_2$ is closed in the projective curve Z_1, and not equal to Z_1, it is finite. The existence of a collection of $(U_P,f_{1,P},f_{2,P})$ as in the proposition follows from the fact that the set of open affines in X is a basis for the topology, together with Proposition 10.1.3. But note that $Z_1\cap Z_2$ may be empty. We extend this collection of $(U_P,f_{1,P},f_{2,P})$ to one $(U_i,f_{1,i},f_{2,i})$, $i\in I$, such that the $U_i\cap Z_1\cap Z_2$ have at most one element and are disjoint, and the conditions in step i of Definition 10.2.1 are met: the U_i cover Z_2 and all meet Z_2. For P in Z_2, let i_P be an $i\in I$ such that U_i contains P; this i_P is unique if P is in $Z_1\cap Z_2$.

As Z_1 and Z_2 are distinct all $f_{1,i}\in\mathcal{O}_X(U_i)$ are not identically zero on $Z_2\cap U_i$, and give nonzero rational functions on Z_2, regular on $U_i\cap Z_2$, that we still denote by $f_{1,i}$. Definition 10.2.1 gives

$$Z_1{\cdot}Z_2=\sum_{P\in Z_2}v_P(f_{1,i_P}/f_{1,1}).$$

As for every i we have $\mathcal{O}_{Z_2}(U_i{\cap}Z_2)=\mathcal{O}_X(U_i)/(f_{2,i})$, and the degree of a principal divisor on a projective curve is zero, and for $i\in I$ such that $U_i\cap Z_1\cap Z_2$ is empty, $\mathcal{O}_X(U_i)/(f_{1,i},f_{2,i})=0$, we get:

$$\begin{aligned}Z_1{\cdot}Z_2&=\sum_{P\in Z_2}v_P(f_{1,i_P})-\sum_{P\in Z_2}v_P(f_{1,1})=\sum_{P\in Z_2}\dim\mathcal{O}_{Z_2}(Z_2\cap U_{i_P})/(f_{1,i_P})\\&=\sum_{P\in Z_2}\dim\mathcal{O}_X(U_{i_P})/(f_{2,i_P},f_{1,i_P})\\&=\sum_{P\in Z_1\cap Z_2}\dim\mathcal{O}_X(U_{i_P})/(f_{1,i_P},f_{2,i_P}).\end{aligned}$$

□

Using local rings, the above definition of the local intersection multiplicity may be made more intrinsic (and hence visibly independent of choices) as follows. Let $Z_1\neq Z_2$ be prime divisors, and $P\in Z_1\cap Z_2$. Inside the local ring $\mathcal{O}_{X,P}$ we have the ideal $I(Z_1)$ given by classes of (U,f) such that f vanishes along $Z_1\cap U$. The ideal $I(Z_1)$ is principal, say $I(Z_1)=(f_1)$. Similarly we have an ideal $I(Z_2)\subset\mathcal{O}_{X,P}$ associated to Z_2 and an element $f_2\in\mathcal{O}_{X,P}$ such that $I(Z_2)=(f_2)$. Then the local intersection multiplicity of Z_1,Z_2 at P is equal to $\dim\mathcal{O}_{X,P}/(f_1,f_2)$. We leave the verification of this as an exercise for the interested reader.

Corollary 10.2.8 *If $Z_1\neq Z_2$ are distinct then $Z_1{\cdot}Z_2\geq 0$.*

Remark 10.2.9 If $Z_1=Z_2$, then $Z_1{\cdot}Z_2$ can be negative, as can be seen in Exercise 10.3.1.

Corollary 10.2.10 *The intersection pairing* $\cdot\colon \operatorname{Div}(X)\times\operatorname{Div}(X)\to\mathbb{Z}$, $(Z_1,Z_2)\mapsto Z_1{\cdot}Z_2$ *is symmetric.*

Proof In view of Proposition 10.2.7 this is now obvious. □

Proposition 10.2.11 *Situation as in Proposition 10.2.7. If* $Z_1\neq Z_2$ *and for all* P *in* $Z_1\cap Z_2$ *the tangent spaces* $T_{Z_1}P$ *and* $T_{Z_2}P$ *have a trivial intersection (inside* T_XP*), then* $Z_1{\cdot}Z_2=\#(Z_1\cap Z_2)$*. In this case we say that* Z_1 *and* Z_2 *intersect transversally.*

Theorem 10.2.12 *The intersection pairing* $\cdot\colon \operatorname{Div}(X)\times\operatorname{Div}(X)\to\mathbb{Z}$ *factors through* $\operatorname{Cl}(X)\times\operatorname{Cl}(X)$.

Proof It suffices (by symmetry) to verify that $Z_1{\cdot}Z_2=0$ for $Z_1=\operatorname{div}(f)$ for some $f\in K(X)^\times$ and Z_2 a prime divisor. Write $Z_1=\sum_Z Z_1(Z)Z$ with Z ranging over the set of prime divisors on X. We take an open cover $\{U_i : i\in I\}$ such that for all the Z with $Z_1(Z)\neq 0$ and for each i in I we have an $f_{i,Z}\in\mathcal{O}_X(U_i)$ such that $\mathcal{O}_X(U_i){\cdot}f_{i,Z}$ is the ideal of $Z\cap U_i$ (take a common refinement if necessary). Then for each i there is a u_i in $\mathcal{O}_X(U_i)^\times$ such that $\prod_Z f_{i,Z}^{Z_1(Z)}=u_if$. Linearity in Z_1, Definition 10.2.1, the fact that $v_P(u_{i_P})=0$, and Theorem 7.6.21 ii, give

$$\begin{aligned} Z_1{\cdot}Z_2 &= \sum_Z Z_1(Z)\sum_{P\in Z_2} v_P(f_{i_P,Z}/f_{1,Z}) = \sum_{P\in Z_2} v_P\left(\frac{\prod_Z f_{i_P,Z}^{Z_1(Z)}}{\prod_Z f_{1,Z}^{Z_1(Z)}}\right) = \sum_{P\in Z_2} v_P\left(\frac{u_{i_P}f}{u_1f}\right)\\ &= \sum_{P\in Z_2} v_P(u_{i_P}/u_1) = -\sum_{P\in Z_2} v_P(u_1) = -\deg(\operatorname{div}(u_1|_{Z_2}))=0. \end{aligned}$$

□

Corollary 10.2.13 (Bézout) *Let* Z_1 *and* Z_2 *be prime divisors in* $\mathbb{P}^2$*, then* $Z_1{\cdot}Z_2=\deg(Z_1){\cdot}\deg(Z_2)$.

Proof By Theorem 10.2.12 the intersection pairing is given by $\cdot\colon\operatorname{Cl}(\mathbb{P}^2)\times\operatorname{Cl}(\mathbb{P}^2)\to\mathbb{Z}$. The degree map $\deg\colon\operatorname{Cl}(\mathbb{P}^2)\to\mathbb{Z}$ is an isomorphism by Lemma 10.1.7. The induced bilinear map $\cdot\colon\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is determined by the value of $(1,1)$. So it suffices to prove that there are two lines Z_1 and Z_2 in $\mathbb{P}^2$ such that $Z_1{\cdot}Z_2=1$. Take the lines $Z_1=Z(x_1)$ and $Z_2=Z(x_2)$, and apply Proposition 10.2.7. □

10.3 Exercises

Exercise 10.3.1 Assume that the characteristic of k is not 3. Let $X\subset\mathbb{P}^3$ be the surface given by $x_0^3-x_1^3+x_2^3-x_3^3=0$. Verify that X is smooth. Let $Z\subset X$ be the line consisting of the points $(s:s:t:t)$ with $(s,t)\in k^2-\{0\}$. Show that $Z{\cdot}Z=-1$.

Exercise 10.3.2 Show that any morphism $f\colon\mathbb{P}^2\to\mathbb{P}^1$ is constant. Hint: if not show that f is surjective and that $f^{-1}(0:1)$ and $f^{-1}(1:0)$ are curves. Use Bézout to obtain a contradiction.

Exercise 10.3.3 (cf. Exercise 5.5.7) Give closed subsets $X\subset\mathbb{P}^m$ and $Y\subset\mathbb{P}^n$ for some m,n and a morphism $f\colon X\to Y$ of projective varieties such that f does not extend to a morphism of varieties $\overline{f}\colon\mathbb{P}^m\to\mathbb{P}^n$.

Exercise 10.3.4 Assume that 6 is invertible in k. Let $C\subset\mathbb{P}^2$ be a smooth curve given by a homogeneous polynomial $f\in k[x_0,x_1,x_2]$ of degree 3. Given a point $P\in C$ denote by $L_P\subset\mathbb{P}^2$ the tangent line in P to C.

i. Show that L_P intersects C in only the point P if and only if the local intersection multiplicity of L_P and C at P is 3. If this is the case P is called a flex-point of C.

ii. Show that $P = (p_0 : p_1 : p_2)$ is a flex point if and only if the determinant of the matrix $(\partial^2 f/\partial x_i \partial x_j)$ is zero at $(x, y, z) = (p_0, p_1, p_2)$.

iii. Show that C has 9 flex-points.

Exercise 10.3.5 Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ and use coordinates $x : y$ on the first factor and $u : v$ on the second factor.

If $f \in k[x, y, u, v]$ is polynomial which is homogeneous of degree d in x, y and homogeneous of degree e in u, v then we say that f is bihomogeneous and has bidegree (d, e). For example, $x^3 u + xy^2 v - y^3 v$ is bihomogeneous of bidegree $(3, 1)$.

Denote the prime divisors $\{(0 : 1)\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{(0 : 1)\}$ by V and H, respectively.

i. Show that V is equivalent with $V' = \{(1 : 1)\} \times \mathbb{P}^1$ and deduce that $V \cdot V = 0$. Same for $H \cdot H$.

ii. Show that $H \cdot V = 1$.

iii. If f is irreducible and bihomogeneous of bidegree (d, e) show that

$$Z(f) = \{((a_0 : a_1), (b_0 : b_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 : f(a_0, a_1, b_0, b_1) = 0\}$$

is a prime divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ which is equivalent with $dV + eH$.

iv. Show that the prime divisor $\Delta = \{(P, P) \colon P \in \mathbb{P}^1\}$ satisfies $\Delta \cdot \Delta = 2$.

Exercise 10.3.6 Let X be an irreducible k-variety and let $\mathcal{L}$ be an invertible sheaf on X. We refer to a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{O}_X \to \mathcal{L}$ as a *section* of $\mathcal{L}$.

i. Prove that there exists a natural isomorphism of $\mathcal{O}_X(X)$-modules $\mathcal{L}(X) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{L})$ (compare to Exc. 6.7.17).

ii. Prove that sections of $\mathcal{L}$ are either zero or injective.

Lecture 11

Cubic surfaces

Note: this chapter is currently in preliminary form. It will be finished and polished in the coming weeks.

In this chapter, we apply the techniques developed in these lecture notes to *cubic surfaces* X, which are defined as smooth 2-dimensional projective varieties of the form

$$X = Z_{\mathrm{proj}}(f) \subset \mathbb{P}^3,$$

where $f \in k[x_0, x_1, x_2, x_3]$ is a homogeneous polynomial of degree 3. Throughout most of chapter, the ground field is an algebraically closed field k of characteristic zero.

The first two sections cover the classical fact that X contains precisely 27 lines. A *line* in $\mathbb{P}^3$ is a closed subset $L \subset \mathbb{P}^3$ of the form

$$L = \{(a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3 \,:\, (a_0, a_1, a_2, a_3) \in V\}$$

for some 2-dimensional k-linear subspace $V \subset k^4$. In the third section, we study the divisor class group $\mathrm{Cl}(X)$ and its intersection product. The material in this chapter partially draws from the excellent exposition on the 27 lines in [Beau], [Reid] and [Reid2].

11.1 Existence of a line

Let $X = Z_{\mathrm{proj}}(f) \subset \mathbb{P}^3$ be a cubic surface. Before we construct the 27 lines on X, we need to show the existence of a line on X. For this, we use a dimension argument involving an incidence locus. This type of reasoning is used often in algebraic geometry. The proof also involves the crucial fact that any morphism between projective varieties is closed (see Exercise 5.5.18).

We first introduce Grassmannians. For any positive integers $m \leq n$, we denote by $\mathrm{Gr}(m, n)$ the set of m-dimensional k-linear subspaces $V \subset k^n$. This set can be given the structure of a smooth variety of dimension $m(n-m)$, known as the *Grassmannian* of m-dimensional subspaces of k^n, as we will now discuss. As the reader will notice, for $m = 1$, the construction recovers $\mathbb{P}^{n-1}$ glued from its standard open subsets $U_i \cong \mathbb{A}^{n-1}$.

An m-dimensional linear subspace $V \subset k^n$ can be represented by an $n \times m$ matrix A of rank m, where we view the column vectors as a basis of V. Recall that A has rank m if and only if A has an $m \times m$ invertible submatrix. The choice of basis of V is not unique and two such matrices A, B define the same V if and only if $A = Bg$ for some $g \in \mathrm{GL}_m(k)$. For any $n \times m$ matrix A and index set $I = \{i_1, \ldots, i_\ell\}$ with $1 \leq i_1 < \ldots < i_\ell \leq n$, we denote by A_I the $\ell \times m$ submatrix of A defined by rows $i_1, \ldots, i_\ell$ of A.

Let $I = \{i_1, \ldots, i_m\}$ with $1 \leq i_1 < \ldots < i_m \leq n$. We denote by

$$U_I \subset \mathrm{Gr}(m, n)$$

the subset of $V \in \mathrm{Gr}(m,n)$ such that V has a basis for which the corresponding $m \times n$ matrix A has the property that A_I is invertible. Note that this property does not depend on the choice of basis, so U_I is well-defined. Define the map

$$\varphi_I : U_I \to \mathbb{A}^{m(n-m)}, \quad A \mapsto (AA_I^{-1})_{\{1,\dots,n\}\setminus I}$$

i.e. we send A to the $(n-m) \times m$ submatrix of AA_I^{-1} defined by the rows indexed by $\{1,\dots,n\} \setminus I$. Note that $(AA_I^{-1})_I$ is the $m \times m$ identity matrix. Also observe that this map is well-defined, *i.e.* it is independent of the choice basis of V. It is easy to verify that φ_I is bijective and $\{U_I\}_I$ forms a cover of $\mathrm{Gr}(m,n)$. In Exercise 11.4.1, for $m = 2$ and $n = 4$, you verify that $\varphi_I(U_I \cap U_J) \subset \mathbb{A}^4$ is an open subset and the components of the transition functions

$$\varphi_J \circ \varphi_I^{-1} : \varphi_I(U_I \cap U_J) \subset \mathbb{A}^4 \to \varphi_J(U_I \cap U_J) \subset \mathbb{A}^4$$

are rational functions. This holds for any m, n. It follows from Exercise 5.5.14 that $\mathrm{Gr}(m,n)$ is a smooth variety of dimension $m(n-m)$. In fact, $\mathrm{Gr}(m,n)$ is projective by the Plücker embedding

$$\mathrm{Gr}(m,n) \hookrightarrow \mathbb{P}^N,$$

where $N := \dim \wedge^m k^n - 1$ (Exercise 11.4.2). The case of interest to cubic surfaces is $\mathrm{Gr}(2,4)$, which is a 4-dimensional smooth projective variety whose points correspond to the lines $\ell \subset \mathbb{P}^3$.

Proposition 11.1.1 *Suppose a cubic surface containing finitely many lines exists. Then every cubic surface contains a line.*

Proof A homogeneous polynomial of degree 3 in 4 variables has $\binom{3+3}{3} = 20$ coefficients. Two such polynomials define the same zero set in $\mathbb{P}^3$ when they differ by a non-zero scalar. Therefore, we can view such polynomials, up to multiplication by a non-zero scalar, as elements of $\mathbb{P}^{19}$.

We view X as the parameter space of all surfaces of the form $Z_{\mathrm{proj}}(f) \subset \mathbb{P}^3$ with f homogeneous of degree 3. Consider the subset (incidence locus)

$$\Sigma := \{(X,L) : \ L \subset X\} \subset \mathbb{P}^{19} \times \mathrm{Gr}(2,4).$$

Then Σ is a closed subset of $\mathbb{P}^{19} \times \mathrm{Gr}(2,4)$ whose irreducible components Σ_i have codimension at most 4. This can be seen by using the standard affine open cover of $\mathrm{Gr}(2,4)$. E.g. for any $(a,b,c,d) \in U_{12} \cong \mathbb{A}^4$ and $[f] \in \mathbb{P}^{19}$, we see that the line defined by $\mathrm{span}_k((1,0,a,c),(0,1,b,d))$ lies on $Z_{\mathrm{proj}}(f)$ if and only if

$$f(\lambda, \mu, \lambda a + \mu b, \lambda c + \mu d) = 0, \quad \forall \lambda, \mu \in k.$$

Since $f(\lambda, \mu, \lambda a + \mu b, \lambda c + \mu d)$ is homogeneous of degree 3 in λ, μ, this imposes 4 polynomial equations on a, b, c, d and the coefficients of f. Therefore, $\dim(\Sigma_i) \geq \dim(\mathbb{P}^{19} \times \mathrm{Gr}(2,4)) - 4 = 19$.

Next, we consider the morphism $\pi \colon \Sigma \to \mathbb{P}^{19}$ defined by inclusion followed by projection

$$\pi \colon \Sigma \subset \mathbb{P}^{19} \times \mathrm{Gr}(2,4) \to \mathbb{P}^{19}.$$

We need to prove that π is surjective. Since $\mathrm{Gr}(2,4)$ is projective, we find that $\mathbb{P}^{19} \times \mathrm{Gr}(2,4)$, and hence also Σ is projective. By Exercise 5.5.18, the image $\pi(\Sigma) \subset \mathbb{P}^{19}$ is closed. If $\dim \pi(\Sigma_i) < 19$ for all components Σ_i, then for every i the morphism $\pi \colon \Sigma_i \to f(\Sigma_i)$ satisfies $\dim(\Sigma_i) > \dim(f(\Sigma_i))$, so all the fibres of π have positive dimension. This contradicts the assumption that a cubic surface with finitely many lines exists. We conclude that there is an i with $\dim \pi(\Sigma_i) = 19$. Since $\pi(\Sigma_i)$ is closed, we see that $\pi(\Sigma_i) = \mathbb{P}^{19}$, hence π is surjective. $\square$

Corollary 11.1.2 *Every cubic surface contains a line.*

Proof See Exercise 11.4.4, where it is shown that the Fermat cubic surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

has finitely many lines. □

Remark 11.1.3 A classical (much lenghtier) "coordinate proof" of Corollary 11.1.2 can be found in [Reid].

11.2 27 lines

In this section, we use Corollary 11.1.2 to construct the 27 lines on a cubic surface.

We start with an elementary observation. Suppose $X = Z_{\mathrm{proj}}(f) \subset \mathbb{P}^3$ is a cubic surface and $H \subset \mathbb{P}^3$ is a *plane*, i.e. a closed subset defined by a homogeneous polynomial of degree 1. Then $X \cap H \subset H \cong \mathbb{P}^2$ is a closed subset defined by a homogeneous polynomial of degree 3. Then $X \cap H$ may be of the following form:

(1) $X \cap H \subset H \cong \mathbb{P}^2$ is a smooth cubic,[1]

(2) $X \cap H \subset H \cong \mathbb{P}^2$ is the union of a smooth conic and a line,

(3) $X \cap H \subset H \cong \mathbb{P}^2$ is the union of three distinct lines, which may or may not pass through one point.

Lemma 11.2.1 *Let $X \subset \mathbb{P}^3$ be a cubic surface and $H \subset \mathbb{P}^3$ a plane. Then $X \cap H$ is of the form (1)–(3). In other words, $X \cap H$ is not the union of one or two lines.*

Proof The proof relies on smoothness of X. We use the following general fact (proved in Exercise 11.4.3). Given $X = Z_{\mathrm{proj}}(f) \subset \mathbb{P}^n$ such that f is not divisible by g^2 for some homogeneous polynomial $g \in k[x_0, \ldots, x_n]$ with $\deg(g) > 0$. Then $(a_0 : \ldots : a_n) \in \mathbb{P}^n$ is a singular point of X if and only if $\frac{\partial f}{\partial x_i}|_{(a_0,\ldots,a_n)} = 0$ for all i.

Suppose $X \cap H$ is the union of one or two lines and let $L \subset X \cap H$ be such a line. Choose homogeneous coordinates such that $H = Z_{\mathrm{proj}}(x_3)$ and $L = Z_{\mathrm{proj}}(x_2, x_3)$ (Theorem 3.3.2). Then

$$f(x_0, x_1, x_2, x_3) = x_3 g(x_0, x_1, x_2, x_3) + x_2^2 h(x_0, x_1, x_2, x_3),$$

where $g, h \in k[x_0, x_1, x_2, x_3]$ are homogeneous of degree two and one respectively. But then any point of the form $(\lambda : \mu : 0 : 0)$ such that $g(\lambda, \mu, 0, 0) = 0$ is a singularity of X. □

Suppose L is a line on a cubic surface X. Let $\mathbb{P}^1 \subset \mathbb{P}^3$ be an auxiliary line not contained in X. There exists a natural map

$$\varphi \colon X \setminus L \to \mathbb{P}^1$$

which sends a point $P \in X \setminus L$ to the point on $\mathbb{P}^1$ obtained by intersecting $\mathbb{P}^1$ with the plane spanned by L and P. This map is a morphism as we now show. Choose homogeneous coordinates such that $L = Z_{\mathrm{proj}}(x_2, x_3)$ and $\mathbb{P}^1 = Z_{\mathrm{proj}}(x_0, x_1)$ (Theorem 3.3.2). Then

$$f = x_2 g(x_0, x_1, x_2, x_3) + x_3 h(x_0, x_1, x_2, x_3),$$

where $g, h \in k[x_0, x_1, x_2, x_3]$ are homogeneous of degree 2 and

$$\varphi \colon X \setminus L \to \mathbb{P}^1, \quad (a_0 : a_1 : a_2 : a_3) \mapsto (a_2 : a_3) = (-h(a_0, a_1, a_2, a_3) : g(a_0, a_1, a_2, a_3)).$$

[1] We refer to smooth curves in $\mathbb{P}^2$ defined by homogeneous polynomials of degree 1,2,3 as *lines*, *smooth conics*, and *smooth cubics* respectively.

Although the projection $\mathbb{P}^3 \setminus L \to \mathbb{P}^1$ does not extend to a morphism $\mathbb{P}^3 \to \mathbb{P}^1$, φ *does* extend to a morphism $\varphi\colon X \to \mathbb{P}^1$. For this, it suffices to note that g, h cannot simultaneously vanish on any point $(a_0 : a_1 : 0 : 0)$, which follows from smoothness of X by the same type of argument given in the proof of Lemma 11.2.1.

The fibres of $\varphi : X \to \mathbb{P}^1$ are interesting. For any plane H containing L, the intersection $X \cap H$ consists of L and (i) a smooth conic or (ii) the union of two distinct lines (different from L), which we will refer to as a *singular conic* (Lemma 11.2.1). Therefore, the fibres of φ are smooth or singular conics.

Proposition 11.2.2 *There exist 5 distinct points $P \in \mathbb{P}^1$ such that $\varphi^{-1}(P)$ is a singular conic.*

Proof We first recall a fact about conics. A conic is a projective curve of the form $X = Z_{\text{proj}}(f) \subset \mathbb{P}^2$ such that $f \in k[x_0, x_1, x_2]$ is homogeneous of degree 2 and not the square of a linear form. Then X is smooth if and only if X is irreducible. Write $f(x_0, x_1, x_2) = \sum_{i,j} a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$. Then X is smooth if and only if $\det\{a_{ij}\}_{i,j} \neq 0$ (we essentially already saw this in the proof of Pascal's theorem 3.5.2 but it also follows from Exercise 11.4.3).

Back to a cubic surface X and a line $L \subset X$. As above, we use homogeneous coordinates such that $L = Z_{\text{proj}}(x_2, x_3)$. Then f does not contain monomials in x_0, x_1 of degree 3, so

$$f(x_0, x_1, x_2, x_3) = a_{00}x_0^2 + 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{02}x_0 + 2a_{12}x_1 + a_{22},$$

where $a_{00}, a_{01}, a_{11}, a_{02}, a_{12}, a_{22} \in k[x_2, x_3]$ are homogeneous of degrees $1, 1, 1, 2, 2, 3$ respectively. The factors 2 and subscripts are for notational convenience. The planes containing L are of the form

$$H_{(\lambda:\mu)} := Z_{\text{proj}}(\lambda x_2 + \mu x_3), \quad (\lambda : \mu) \in \mathbb{P}^1.$$

There is at least one plane $H_{(\lambda:\mu)}$ such that $X \cap H_{(\lambda:\mu)}$ is the union of L with a smooth conic, or else X contains infinitely many lines, which we showed is impossible in the previous section. We can choose coordinates such that $H_{(1:0)}$ is one of these planes (Theorem 3.3.2), so we can restrict attention to $H_{(\lambda:1)}$. Then $X \cap H_{(\lambda:1)}$ is defined by the homogeneous polynomial

$$\begin{aligned} f(x_0, x_1, x_2, -\lambda x_2) = x_2 \Big\{ & a_{00}(1, -\lambda)x_0^2 + 2a_{01}(1, -\lambda)x_0x_1 + a_{11}(1, -\lambda)x_1^2 \\ & + 2a_{02}(1, -\lambda)x_0x_2 + 2a_{12}(1, -\lambda)x_1x_2 + a_{22}(1, -\lambda)x_2^2 \Big\}, \end{aligned}$$

where we used homogeneity of the a_{ij}. Consequently

$$\begin{aligned} q_\lambda(x_0, x_1, x_2) := {} & a_{00}(1, -\lambda)x_0^2 + 2a_{01}(1, -\lambda)x_0x_1 + a_{11}(1, -\lambda)x_1^2 \\ & + 2a_{02}(1, -\lambda)x_0x_2 + 2a_{12}(1, -\lambda)x_1x_2 + a_{22}(1, -\lambda)x_2^2 \end{aligned}$$

defines a conic in $H_{(\lambda:1)}$ and we claim that it is singular for precisely 5 values of λ.

As we discussed above, the conic $Z_{\text{proj}}(q_\lambda) \subset \mathbb{P}^2$ is singular if and only if

$$\det\{a_{ij}(1, -\lambda)\}_{ij} = 0.$$

This is a degree 5 equation in λ, so it remains to show that its roots are distinct. Suppose λ_1 is one of these roots, then $X \cap H_{(\lambda_1:1)}$ consists of three lines L, L_1, L_1', which either pass through one point or not. Suppose they do not pass through one point (the other case is Exercise 11.4.5). Again by Theorem 3.3.2, there exist homogeneous coordinates such that $\lambda_1 = 0$ and

$$L = Z_{\text{proj}}(x_2, x_3), \quad L_1 = Z_{\text{proj}}(x_1, x_3), \quad L_1' = Z_{\text{proj}}(x_0, x_3).$$

This implies $f(x_0, x_1, x_2, x_3) = x_3 F(x_0, x_1, x_2, x_3) + x_0x_1x_2$, for some $F \in k[x_0, x_1, x_2, x_3]$ homogeneous of degree 2. In particular, x_3 divides $a_{00}, a_{11}, a_{02}, a_{12}, a_{22}$ and $a_{01} = \frac{1}{2}x_2 + cx_3$ for some $c \in k$. This implies

$$\det\{a_{ij}\}_{ij} = \tfrac{1}{4}x_2^2 a_{22}(x_2, x_3) + O(x_3^2),$$

where a_{22} is divisible by x_3 and $O(x_3^2)$ denotes the terms of degree ≥ 2 in x_3. Finally, we observe that a_{22} is not divisible by x_3^2 because $(0:0:1:0) \in X$ and X is smooth. Therefore $\det\{a_{ij}(1,-\lambda)\}_{ij}$ is divisible by λ but not λ^2. □

Given a cubic surface X and a line $L \subset X$, from the previous proposition we obtain 10 more lines $\{(L_i, L_i')\}_{i=1,\dots,5}$. Observe that

$$(L_i \cup L_i') \cap (L_j \cup L_j') = \varnothing, \quad \forall i \neq j.$$

This follows from the fact that $L_i \cup L_i'$ and $L_j \cup L_j'$ can only intersect on the line L, but when three lines on X pass through a point they are coplanar (by smoothness of X). We also see that there exist at least two disjoint lines on X.

In order to find the remaining lines, we will use the solution to a famous problem raised by H. Schubert in the 19th century: "how many lines intersect *four* mutually disjoint lines in $\mathbb{P}^3$? In order to address Schubert's question, we first discuss a nice property of *three* mutually disjoint lines in $\mathbb{P}^3$. A *quadric surface* is a smooth 2-dimensional projective variety of the form $Q = Z_{\text{proj}}(f) \subset \mathbb{P}^3$, where $f \in k[x_0, x_1, x_2, x_3]$ is a homogeneous polynomial of degree 2. In Exercise 11.4.6 you show that any three mutually disjoint lines $L_1, L_2, L_3 \subset \mathbb{P}^3$ always lie on a unique quadric surface Q, and Q is precisely the union of all lines intersecting L_1, L_2, L_3.

Proposition 11.2.3 *Suppose $L_1, \dots, L_4 \subset \mathbb{P}^3$ are mutually disjoint. Then*

(1) If L_4 lies on the quadric surface determine by L_1, L_2, L_3, then there are infinitely many *lines intersecting L_1, L_2, L_3, L_4.*

(2) If L_4 does not lie on the quadric surface determined by L_1, L_2, L_3 and is tangent to it, then there is one *line intersecting L_1, L_2, L_3, L_4.*

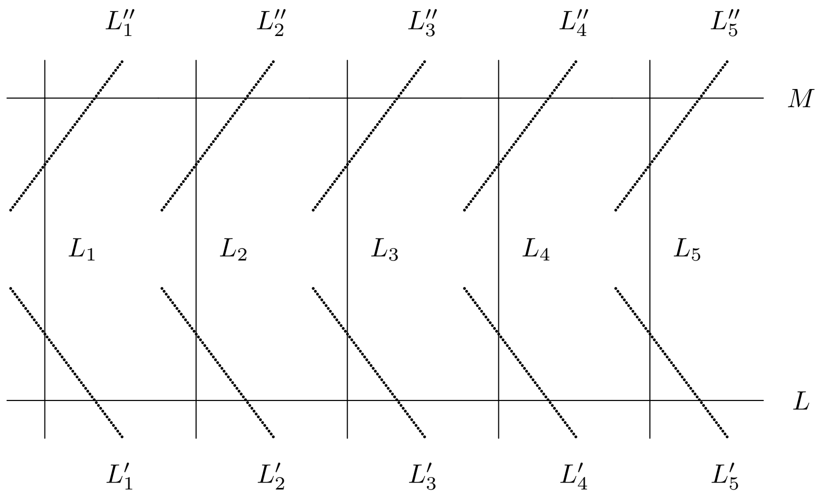
(3) If L_4 does not lie on the quadric surface determined by L_1, L_2, L_3 and is not tangent to it, then there are two *lines intersecting L_1, L_2, L_3, L_4.*

Proof Using the homogeneous coordinates of Exercise 11.4.6, we have $L_1 = Z_{\text{proj}}(x_0, x_1)$, $L_2 = Z_{\text{proj}}(x_2, x_3)$, and $L_3 = Z_{\text{proj}}(x_0 - x_2, x_1 - x_3)$. The quadric determined by L_1, L_2, L_3 is $Q = Z_{\text{proj}}(x_0x_3 - x_1x_2)$, which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ (see Exercise 3.6.7). Therefore L_1, L_2, L_3 correspond to the lines $\{(0:1)\} \times \mathbb{P}^1$, $\{(1:0)\} \times \mathbb{P}^1$, $\{(1:1)\} \times \mathbb{P}^1$. If $L_4 \subset Q$, then it must correspond to $\{P\} \times \mathbb{P}^1$ with $P \notin \{(0:1), (1:0), (1:1)\}$. Therefore all lines corresponding to $\mathbb{P}^1 \times \{R\}$, for arbitrary $R \in \mathbb{P}^1$, intersect L_1, L_2, L_3, L_4.

When $L_4 \not\subset Q$, then L_4 intersects Q in one or two points (depending on whether it is tangent or not). From this, the remaining cases follow (use the Segre embedding to visualize $L_4 \cap Q$, L_1, L_2, L_3 and the cases that can occur). □

Remark 11.2.4 For "generic" L_1, L_2, L_3, L_4, we see that the answer to Schubert's question is 2. This is an example of a problem in the area of *enumerative geometry*.

Now let us go back to a cubic surface X. Take two disjoint lines $L, M \subset X$. By Proposition 11.2.2 applied to X, L, we obtain 10 lines $\{L_i, L_i'\}_{i=1,\dots5}$ such that $(L_i \cup L_i') \cap (L_j \cup L_j') = \varnothing$ for all $i \neq j$. Observe that, for each $i = 1, \dots, 5$, M intersects L_i or L_i', but not both. Indeed, consider the plane $H_i \subset \mathbb{P}^3$ containing L, L_i, L_i', then M intersects this plane. But H_i intersects X precisely in L, L_i, L_i' (and not more) and M cannot intersect both L_i, L_i' (because X is smooth). We label the lines which M intersects by L_1, L_2, L_3, L_4, L_5. Applying Proposition 11.2.2 to X, M, we obtain 5 pairs of lines $\{L_i, L_i''\}_{i=1,\dots,5}$. We now have 17 lines which intersect as follows:



There are two warnings regarding this figure. Firstly, we should bear in mind that L, L_i, L_i' may intersect in one single point (and likewise for M, L_i, L_i''). Secondly, not all intersections are shown. Indeed

$$L_i' \cap L_j'' \neq \varnothing, \quad \forall i \neq j. \tag{11.1}$$

As above, this is because L_j'' intersects the plane H_i containing L, L_i, L_i' and since L_j'' does not intersect L, L_i, it must intersect L_i'.

We end by constructing the 10 remaining lines:

Theorem 11.2.5 *A cubic surface contains 27 lines.*

Proof For any 4 disjoint lines on X, we note that there are always one or two lines intersecting all four, i.e. possibilities (2) or (3) in Proposition 11.2.3. This follows, for instance, by using the Segre embedding to see what would happen on $\mathbb{P}^1 \times \mathbb{P}^1$ when a quadric surface and cubic surface would intersect in four mutually disjoint lines.

Continuing the notation above, we so far have the lines $L, M, \{L_i, L_i', L_i''\}_{i=1,\dots,5}$. Suppose N is any *other* line on X. We claim N intersects precisely 3 out of $\{L_i\}_{i=1,\dots,5}$. This can be seen as follows. Firstly, since L, M already intersect all L_i, it is not possible for N to intersect 4 or 5 of the L_i. Secondly, suppose N meets at most 2 of the lines L_i. After relabelling, this means N meets $L_1', L_2', L_3', L_4', L_5'$ or $L_1, L_2', L_3', L_4', L_5'$ or $L_1, L_2, L_3', L_4', L_5'$. In each case, L and L_1'' already meet four of the given lines, so we reach a contradiction. This proves the claim, which also implies that there are *at most* $\binom{5}{3} = 10$ more lines on X (why?).

By Proposition 11.2.2, there are 10 lines intersecting L_1. So besides L, M, L_1', L_1'', there exist 6 more such lines. Each such line intersects 3 out of $\{L_i\}_{i=2,\dots,5}$. Therefore, we can label these lines by $\{L_{1jk}\}_{2 \leq j < k \leq 5}$. Continuing in this way for the lines $L_2, \dots, L_5$, we see that for each $1 \leq i < j < k \leq 5$, there exists a line L_{ijk} intersecting L_i, L_j, L_k. This produces the final 10 lines on X. □

Remark 11.2.6 Given a cubic surface X, we constructed its 27 lines

$$L, M, \{L_i, L_i', L_i''\}_{i=1,\dots,5}, \{L_{ijk}\}_{1 \leq i < j < k \leq 5}.$$

We already discussed all possible intersections among $L, M, \{L_i, L_i', L_i''\}_{i=1,\dots,5}$. In addition to L_1, L_2, L_3, it can be shown that L_{123} intersects $L_{145}, L_{245}, L_{345}, L_4', L_5', L_4'', L_5''$ (and similarly for the other L_{ijk}). This describes all intersections among the 27 lines.

11.3 Divisor class group

In this section, we study the divisor class group $\mathrm{Cl}(X)$ (Definition 10.1.5) and its intersection product (Theorem 10.2.12) for X a cubic surface.

Consider the subgroup generated by classes of lines on X, i.e.

$$\Lambda(X) := \langle [L] \,:\, L \subset X \text{ a line} \rangle \subset \mathrm{Cl}(X).$$

We give a complete description of $\Lambda(X)$ and the restriction of the intersection product $\mathrm{Cl}(X)\times\mathrm{Cl}(X) \to \mathbb{Z}$ to $\Lambda(X) \times \Lambda(X)$. As we briefly sketch at the end of this section, we in fact have $\Lambda(X) = \mathrm{Cl}(X)$. Throughout the proof of the next two results, we repeatedly use that the intersection product is bilinear and well-defined on $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$ (Theorem 10.2.12) , i.e. independent of choices of representative of linear equivalence classes.

Lemma 11.3.1 *For any line L on a cubic surface X, we have $[L] \cdot [L] = -1$.*

Proof Take a line $L \subset X$. Let $H_1 \subset \mathbb{P}^3$ be a plane not containing L and such that $X \cap H_1 = M \cup M' \cup M''$ is the union of three lines. Then L intersects precisely one of M, M', M'', so

$$[L] \cdot ([M] + [M'] + [M'']) = 1.$$

Next, we take a plane $H_2 \subset \mathbb{P}^3$ such that $X \cap H_2 = L \cup L' \cup L''$ is the union of three lines, including L. Such a plane exists by Proposition 11.2.2. There exist linear forms $a_1, a_2 \in k[x_0, x_1, x_2, x_3]$ such that $H_i = Z_{\mathrm{proj}}(a_i)$ and therefore

$$\mathrm{div}\left(\frac{a_1}{a_2}\Big|_X\right) = M + M' + M'' - (L + L' + L''),$$

i.e. $M + M' + M'' \sim_{\mathrm{lin}} L + L' + L''$ (Definition 10.1.5). Hence

$$1 = [L] \cdot ([M] + [M'] + [M'']) = [L] \cdot ([L] + [L'] + [L'']) = [L] \cdot [L] + 2$$

and the result follows. □

Proposition 11.3.2 *Let X be a cubic surface and use the notation from the previous section for its 27 lines. Define*

$$e_0 := [L_5] + [L_5'] + [L_5''], \quad e_1 := [L_1], \ldots, e_4 := [L_4], \quad e_5 := [L_5'], \quad e_6 := [L_5''].$$

Then $e_0 \cdot e_0 = 1$, $e_i \cdot e_i = -1$ for all $i = 1, 2, 3, 4$, and $e_i \cdot e_j = 0$ for all $i \neq j$. In particular, $(e_0, \cdots, e_6)$ forms a basis for $\Lambda(X) \cong \mathbb{Z}^7$ with respect to which the intersection product $\Lambda(X) \times \Lambda(X) \to \mathbb{Z}$ is given by the matrix $\mathrm{diag}(1, -1, -1, -1, -1, -1, -1)$.

Proof In the previous section, we gave a complete description of how the 27 lines intersection. From this it follows at once that $e_i \cdot e_j = 0$ for all $i, j \in \{1, \ldots, 6\}$ with $i \neq j$ (simply because the corresponding lines involved are disjoint). Moreover, $e_0 \cdot e_i = 0$ for all $i = 1, 2, 3, 4$ and

$$e_0 \cdot e_5 = [L_5] \cdot [L_5'] + [L_5'] \cdot [L_5'] = 1 - 1 = 0,$$

because L_5, L_5' intersect in a point and $[L_5'] \cdot [L_5'] = -1$ by Lemma 11.3.1. Similarly $e_0 \cdot e_6 = 0$. Furthermore, we have $e_i \cdot e_i = -1$ for all $i \in \{1, \ldots, 6\}$ by Lemma 11.3.1. Finally, using that L_5, L_5' intersect in a point, L_5, L_5'' intersect in a point, and L_5', L_5'' are disjoint, we find

$$\begin{aligned} e_0 \cdot e_0 &= [L_5] \cdot [L_5] + [L_5'] \cdot [L_5'] + [L_5''] \cdot [L_5''] + 2[L_5] \cdot [L_5'] + 2[L_5] \cdot [L_5''] + 2[L_5'] \cdot [L_5''] \\ &= -1 - 1 - 1 + 2 + 2 + 0 = 1. \end{aligned}$$

□

The remainder of this section can be safely skipped and is included to give a flavour of the powerful interplay between algebraic geometry and algebraic topology.

There are two ways to establish $\Lambda(X) = \mathrm{Cl}(X)$. One way is by showing that a cubic surface is isomorphic to $\mathbb{P}^2$ blown-up in 6 points and using general facts about the behaviour of the divisor class group under blow-ups. In these lecture notes, we do not discuss blow-ups. Another way, reduced the problem to $k = \mathbb{C}$, where it is amenable to techniques from algebraic topology. We give a *rough sketch* of the argument for the interested reader. By Section 8.3, any smooth projective surface X over $\mathbb{C}$ determines a complex manifold of (complex) dimension 2 and therefore a (compact, oriented) differentiable manifold of (real) dimension 4. There exists a group homomorphism

$$c : \mathrm{Cl}(X) \to \mathrm{H}^2(X, \mathbb{Z}),$$

where $\mathrm{H}^2(X, \mathbb{Z})$ denotes the second singular homology group of the topological space X^{an}. This map sends $[D]$, where $D \subset X$ is a prime divisor, to the Poincaré dual of the class of the 2-cycle $[D] \in \mathrm{H}_2(X, \mathbb{Z})$. Moreover, this map is compatible with the cup product, i.e.

$$c([C] \cdot [D]) = c([C]) \cup c([D]), \quad \forall [C], [D] \in \mathrm{Cl}(X).$$

In general, c is neither injective nor surjective. However, if $\mathrm{H}_1(X, \mathbb{Z}) = 0$, then the map c is injective. For X a cubic surface, or any smooth projective surface in $\mathbb{P}^3$, we have $\mathrm{H}_1(X, \mathbb{Z}) = 0$ (by the so-called Lefschetz hyperplane theorem). So for X a cubic surface, we have

$$\mathbb{Z}^7 \cong \Lambda(X) \hookrightarrow \mathrm{Cl}(X) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z}).$$

It therefore suffices to show that $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free and $\mathrm{rk}(\mathrm{H}^2(X, \mathbb{Z})) = 7$. As far as torsion is concerned, we have $\mathrm{H}^2(X, \mathbb{Z})_{\mathrm{tor}} \cong \mathrm{H}_1(X, \mathbb{Z})_{\mathrm{tor}}$ (this holds for any compact oriented 4-dimensional manifold and follows from combining Poincaré duality and the universal coefficient theorem). Hence $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free for X a cubic surface, or any smooth projective surface in $\mathbb{P}^3$. The Euler characteristic of X^{an}, defined via triangulations in Section 9.3, is given by

$$\chi(X) = \sum_{i=0}^{4} (-1)^i \mathrm{rk}(\mathrm{H}^i(X, \mathbb{Z})) = 2 + \mathrm{rk}(\mathrm{H}^2(X, \mathbb{Z})).$$

We want to show that $\chi(X) = 9$ when X is a cubic surface. This follows nicely by considering the morphism $\varphi : X \to \mathbb{P}^1$ of Proposition 11.2.2. The general fibre of this map is a smooth conic C, which has Euler characteristic $\chi(C) = 2 - 2g(C) = 2$. However, φ has 5 fibres corresponding to singular conics C_{sing} with Euler characteristic $\chi(C_{\mathrm{sing}}) = 3$.[2] Using properties of Euler characteristics, we obtain

$$\chi(X) = (\chi(\mathbb{P}^1) - 5)\chi(C) + 5\chi(C_{\mathrm{sing}}) = -3 \cdot 2 + 5 \cdot 3 = 9$$

as desired.

11.4 Exercises

Exercise 11.4.1 Consider the set $\mathrm{Gr}(2, 4)$. Use the notation of Section 11.1 and let $I = \{1, 2\}$. For $J = \{3, 4\}$ and $J = \{1, 3\}$, show that $\varphi_I(U_I \cap U_J) \subset \mathbb{A}^4$ and $\varphi_J(U_I \cap U_J) \subset \mathbb{A}^4$ are open. Show in both cases that $\varphi_J \circ \varphi_I^{-1} : \varphi_I(U_I \cap U_J) \subset \mathbb{A}^4 \to \varphi_J(U_I \cap U_J) \subset \mathbb{A}^4$ is a morphism.

Exercise 11.4.2 Consider the vector space $\wedge^m k^n \cong k^{\binom{n}{m}}$. Define $\iota : \mathrm{Gr}(m, n) \to \mathbb{P}(\wedge^m k^n) = \mathbb{P}^N$, where $N := \binom{n}{m} - 1$ by sending $\langle v_1, \ldots, v_m \rangle_k$ to $\langle v_1 \wedge \cdots \wedge v_m \rangle_k$. The claim is that ι is a well-defined

[2] Topologically, a singular conic is the union of two spheres S^2 glued along a common point, so $\chi(C_{\mathrm{sing}}) = 2\chi(S^2) - 1 = 3$.

morphism, $\iota(\mathrm{Gr}(m,n)) \subset \mathbb{P}^N$ is closed, and ι is an isomorphism onto $\iota(\mathrm{Gr}(m,n))$ (i.e. ι is a closed embedding). In this exercise, we consider the case $m = 2$ and $n = 4$. The morphism ι is known as the *Plücker embedding* for $\mathrm{Gr}(m,n)$.

i. Let (e_1, e_2, e_3, e_4) be the standard basis for k^4, then $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\wedge^2 k^4$ and we obtain corresponding homogeneous coordinates $\{x_{ij}\}_{i<j}$ for $\mathbb{P}(\wedge^2 k^4) = \mathbb{P}^5$. Prove that ι is a well-defined morphism. *Hint: Use the charts U_I for $\mathrm{Gr}(2,4)$ and U_{ij} for $\mathbb{P}^5$.*

ii. Show that $\iota(\mathrm{Gr}(2,4)) = Z_{\mathrm{proj}}(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$. The latter equation is known as a *Plücker relation*.

iii. Prove that ι is an isomorphism onto its image. We deduce that $\mathrm{Gr}(2,4)$ is isomorphic to a quadric hypersurface in $\mathbb{P}^5$.

Exercise 11.4.3 Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial and let $X = Z_{\mathrm{proj}}(f)$.

i. Let $P \in X$. Suppose $\frac{\partial f}{\partial x_i}\big|_P = 0$ for all $i = 0, \ldots, n$. Show that $f(P) = 0$. *Hint: First prove Euler's formula* $\deg(f) \cdot f = \sum_{i=0}^n x_i \frac{\partial f}{\partial x_i}$.

ii. Suppose f is not divisible by g^2 for some homogeneous polynomial $g \in k[x_0, \ldots, x_n]$ with $\deg(g) > 0$ (i.e. f has no multiple factors). Show that X is smooth if and only if $Z_{\mathrm{proj}}\big(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\big) = \varnothing$.

Exercise 11.4.4 Consider the Fermat cubic $X = Z_{\mathrm{proj}}(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$.

i. Prove that X is smooth. You may use Exercise 11.4.3.

ii. Let $L \subset \mathbb{P}^3$ be a line. Then, up to permuting coordinates,

$$L = Z_{\mathrm{proj}}(x_0 - (a_2x_2 + a_3x_3), x_1 - (b_2x_2 + b_3x_3))$$

for some $a_2, a_3, b_2, b_3 \in k$. Write down four polynomial equations in a_i, b_i which are satisfied if and only if $L \subset X$.

iii. Suppose $L \subset X$. Prove that $a_2, a_3, b_2, b_3 \in k$ cannot be all zero.

iv. Suppose $L \subset X$ and $a_2 = 0$. Solve for a_3, b_2, b_3 and deduce that the line $L = Z_{\mathrm{proj}}(x_0 + \zeta^i x_3, x_1 + \zeta^j x_2)$ lies on X for any $0 \leq i, j \leq 2$, where ζ is a primitive third root of unity.

v. Prove that X contains 27 lines.

Exercise 11.4.5 Finish the proof of Proposition 11.2.2 in the case L, L_1, L_1' pass through one point.

Exercise 11.4.6 Let $L, M, N \subset \mathbb{P}^3$ be three mutually disjoint lines, i.e. no two of them lie in a plane.

i. Prove that there are infinitely many lines intersection L, M, N. *Hint: Take $P \in L$ and consider the plane spanned by P and M.*

ii. Show that there exist homogeneous coordinates on $\mathbb{P}^3$ such that $L = Z_{\mathrm{proj}}(x_0, x_1)$ and $M = Z_{\mathrm{proj}}(x_2, x_3)$.

iii. Show that there exist homogeneous coordinates on $\mathbb{P}^3$ such that L, M are as above and in addition $N = Z_{\mathrm{proj}}(x_0 - x_2, x_1 - x_3)$.

iv. Use the homogeneous coordinates obtained in (iii) in order to show that there exists a unique quadric surface containing L, M, N and this quadric surface is given by $Q := Z_{\mathrm{proj}}(x_0x_3 - x_1x_2)$.

v. Use the homogeneous coordinates obtained in (iii) in order to show that the union of all lines intersecting L, M, N is precisely Q. *Hint: For one inclusion, take any* $P = (0 : 0 : \lambda : \mu) \in L$*, write down an equation for the plane spanned by* P *and* M*, etc.*

Exercise 11.4.7 Let $X := Z_{\mathrm{proj}}(f) \subset \mathbb{P}^3$, where $f \in k[x_0, x_1, x_2, x_3]$ is homogeneous of degree 3. Suppose X is irreducible and singular at $P \in X$. We refer to X as a *singular cubic surface*. Prove that there exists a line $P \in L \subset X$. *Hint: Use homogeneous coordinates such that* $P = (0 : 0 : 0 : 1)$ *and use Exercise 11.4.3.* Extra: show that the maximal number of lines in X through P is 6 (and this is the generic number of lines in X passing through P). *Hint: Bézout's theorem.*

Exercise 11.4.8 Let $X := Z_{\mathrm{proj}}(f) \subset \mathbb{P}^4$ where $f \in k[x_0, x_1, x_2, x_3, x_4]$ is homogeneous of degree 3. Suppose X is smooth, then we refer to X as a *cubic threefold*. Prove that for any point $P \in X$, there exists a line $L \subset X \subset \mathbb{P}^4$ passing through P. *Hint: Choose homogeneous coordinates such that* $P = (0 : 0 : 0 : 0 : 1)$ *and reduce to the case of cubic surfaces.* Therefore X can be covered by lines $\mathbb{P}^1$.

Exercise 11.4.9 Suppose $X \subset \mathbb{P}^3$ is a cubic surface and $L, M \subset X \subset \mathbb{P}^3$ are disjoint lines. Define a map $\psi : X \setminus (L \cup M) \to L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ as follows. For any point $P \in X \setminus (L \cup M)$, consider the plane $H \subset \mathbb{P}^3$ containing P, L. Then there exists a unique line $N \subset H$ intersecting L and M (why?) and we set $\psi : P \mapsto (N \cap L, N \cap M)$.

i. Prove that ψ extends to a morphism $\psi : X \to L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$. *Hint: Choose homogeneous coordinates such that* $L = Z_{\mathrm{proj}}(x_0, x_1)$*,* $M = Z_{\mathrm{proj}}(x_2, x_3)$*, write* f *in terms of coordinates (using that* $L, M \subset X$*), and write* ψ *down explicitly.*

ii. Prove that there exist 5 lines in X which are contracted to points by ψ.

Bibliography

[AM] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[Beau] A. Beauville. *Complex algebraic surfaces.* Second edition. London Mathematical Society Student Texts, 34. Cambridge University Press, Cambridge, 1996.

[Hart] R. Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics 52. Springer, New York, 1977.

[KK] L. Kaup and B. Kaup. *Holomorphic Functions of Several Variables*. De Gruyter Studies in Mathematics.

[DS] F. Diamond and J.M. Shurman. *A first course in modular forms.* Springer, 2005.

[Lang] S. Lang. *Algebra.* Addison-Wesley. 2nd and 3rd editions.

[Looij] E. Looijenga. *A first course on algebraic geometry.* Available at: `http://www.staff.science.uu.nl/~looij101/AG2014.pdf`

[Moo] B. Moonen. *Introduction to Algebraic Geometry*. Available at: `http://www.math.ru.nl/~bmoonen/AlgGeom/alggeom.pdf`

[Mum] D. Mumford.*The red book of varieties and schemes.* Second, expanded edition. Includes the Michigan lectures (1974) on curves and their Jacobians. With contributions by Enrico Arbarello. Lecture Notes in Mathematics, 1358. Springer-Verlag, Berlin, 1999.

[Reid] M. Reid. *Undergraduate algebraic geometry*. London Mathematical Society Student Texts, 12. Cambridge University Press, Cambridge, 1988.

[Reid2] M. Reid. *Chapters on algebraic surfaces.* Complex algebraic geometry (Park City, UT, 1993), 3–159, IAS/Park City Math. Ser., 3, Amer. Math. Soc., Providence, RI, 1997.

[Spri] T.A. Springer. *Linear algebraic groups.* Reprint of the 1998 second edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009.

[Stev] P. Stevenhagen. *Algebra 1, 2, en 3.* Available at: `http://websites.math.leidenuniv.nl/algebra/`

[Tae] L. Taelman. *Modules and Categories.* Lecture notes, available at: `https://staff.fnwi.uva.nl/l.d.j.taelman/ca.pdf`.

[Var] V.S. Varadarajan. *Lie groups, Lie algebras, and their representations.* Reprint of the 1974 edition. Graduate Texts in Mathematics, 102. Springer-Verlag, New York, 1984.