Defining Functors by Well-founded Recursion

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Suppose we have a well-founded poset $(X,<)$ and a category $\mathcal{C}$; how can we construct functors from X to $\mathcal{C}$ by well-founded recursion?

Recall that a poset $(X,<)$ is *well-founded* if for every subset $Y \subseteq X$ we have: if for every $x \in X$, the statement $\{y \in X \,|\, y < x\} \subseteq Y$ implies $x \in Y$, then $Y = X$.

Well-founded posets admit the following well-known principle of *definition by well-founded recursion*: if Y is a set and F is a function from the power-set of Y to Y, then there is a unique function $G : X \to Y$ such that for every $x \in X$ the equality

$$G(x) \;=\; F(\{G(y) \,|\, y < x\})$$

holds.

In order to generalize this to *functors* from $(X,<)$ (regarded as a category) to a category $\mathcal{C}$, we need, for the "induction step", to be able to extend a functor defined on $\{y \,|\, y < x\}$ to a functor which is defined on x (and on all inequalities $y < x$). That is, we need a *cocone* for the original functor (for a functor $G : \mathcal{C} \to \mathcal{D}$, a cocone for G consists of an objct D of $\mathcal{D}$ and a natural transformation from G to the constant functor with value D; D is called the *vertex* of the cocone).

Suppose G is a functor $Y \to \mathcal{C}$ where Y is an initial segment (downwards closed subset) of X. We write $G_{<y}$ for the restriction of G to $\{z \,|\, z < y\}$.

Proposition 0.1 *Let $(X,<)$ be a well-founded poset, $\mathcal{C}$ a category, and F a function which, to every functor from an initial segment of $(X,<)$ to $\mathcal{C}$, assigns a cocone for that functor.*

Then there exists a unique functor $G : (X,<) \to \mathcal{C}$ with the property that for every $x \in X$ the following hold:

i) *$G(x)$ is the vertex of $F(G_{<x})$*

ii) *For all $y < x$, the arrow $G(y < x)$ is the component of (the natural transformation of) $F(G_{<x})$ at y.*

Condition i) is, of course, only needed to cater for the case that $G_{<x}$ is the empty functor.

Proof. Let $Y \subseteq X$ consist of those elements $y \in X$ such that there is a unique functor G_y from $\{x \in X \mid x \leq y\}$ to $\mathcal{C}$ which satisfies i) and ii) for all $x \leq y$. Then clearly, Y is downwards closed and by the uniqueness condition in the definition of Y, if $z \leq y \in Y$ then G_z is the restriction of G_y to the down-segment of z.

Therefore, if $\{y \in X \mid y < x\} \subseteq Y$, then all functors G_y agree pairwise on their common domain, and therefore amalgamate to a functor $G_{<x}$ defined on $\{y \in X \mid y < x\}$, which satisfies i) and ii) for all $y < x$. Extend $G_{<x}$ to a functor G_x defined on $\{y \in X \mid y \leq x\}$ by letting $G_x(x)$ be the vertex of $F(G_{<x})$, and $G_x(y < x)$ the component of $F(G_{<x})$ at y. Then obviously, G_x satisfies i) and ii). Moreover, G_x is unique with this property: if also H would satisfy this property then by assumption on $y < x$, the restriction of H to the segment $\leq y$ would have to be G_y, and therefore the restriction of H to the segment $< x$ would be $G_{<x}$; but then, since H satisfies i) and ii), $H = G_x$. So $x \in Y$. By well-founded induction we conclude that $Y = X$.

Again, all functors G_x agree pairwise on their common domain, so amalgamate to a functor G defined on X satisfying i) and ii). If also H is such a functor, then by the uniqueness of G_x we have that the restriction of H to the segment $\leq x$ coincides with G_x for all x; that is, $H = G$. So G is unique. ∎