Introduction to Peano Arithmetic

Gödel Incompleteness and Nonstandard Models

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Preface

The 1998-1999 Master Class Program in Mathematical Logic

These Lecture Notes contain the material of a series of lectures I gave in the Spring of 1999, in the Master Class Program in Mathematical Logic. This program was funded by the Mathematical Research Institute, a cooperation of the Mathematics Departments of the Universities of Utrecht, Nijmegen, and Groningen.

In the fall of 1998, students followed basic courses in core subjects of Logic: Model Theory, Recursion Theory, Proof Theory and Lambda Calculus. Moreover there was a series of introductory talks on varying topics in Logic, including Intuitionism, Term Rewriting, Naive Set Theory, the Language of Categories, the P=NP?-problem, and Provability Logic; the series was called Logic Panorama.

The second semester featured more advanced courses: Type Theory, Peano Arithmetic, and the Logic of Sheaves. The students also followed a seminar on Descriptive Set Theory.

Apart from doing exams for the lecture courses, the students had to write two essays: one, elementary, on a subject of their choice from the Panorama series; the other, the so-called "test problem", required them to demonstrate an ability to read research texts independently, furnish missing details, and solve a (simple) problem.

I believe that in all, this program provided a balanced and thorough introduction to the subject, and gave would-be research students and excellent basis on which to start a research career.

Responsability for the Program was taken by the Logic Groups of the Universities of Utrecht and Nijmegen. Teachers were Henk Barendregt, Wil Dekker, Herman Geuvers, Ieke Moerdijk, Jaap van Oosten, Harold Schellinx and Wim Veldman. The following teachers (apart from those already mentioned) took part in the Panorama Program and/or supervised the writing of essays: Tibor Beke, Hans Bodlaender, François Métayer, Erik Palmgren, Anne Troelstra, Albert Visser and Hans Zantema.

The course on Peano Arithmetic (PA)

Naturally divided into two parts, the course treats Gödel's Incompleteness Theorems and gives an introduction to the Model Theory of PA. In spite of the clear separation between working in an axiomatic theory and considering models of the theory, there are themes running through the entire course, giving unity to the treatment. These are: the formalization of elementary number theory in PA, the arithmetization of syntax, the natural stratification of sentences in the arithmetical hierarchy, and the issue of definability, coming up over and over again.

These themes are in fact central to Logic as a scientific discipline: the student will meet them everywhere, in different guises. I am therefore convinced that the study of Peano Arithmetic provides the student with basic skills he will be using continuously, in every area of Logic.

Many people are of opinion that syntax is boring and coding troublesome, and that these matters should therefore be glossed over in a hand-waving manner. It is quite ironic that this belief is shared by many logicians, whereas syntax is the raw material of Logic itself! Very often the result of a hand-waving treatment is, that students feel insecure about syntactical matters, and have no clear understanding of the problems involved in formalization. Of course, the problem of treating syntax needs reflection of a special kind. If presented in a well thought-out way, the theory of coding and syntax can be elegant and rewarding in itself (besides being indispensible). I hope that in these notes I have succeeded in bringing this to light.

Mathematically, Peano Arithmetic is attractive because of the many applications of Model Theory and Recursion Theory it offers; permitting to see these subjects 'at work'.

Now let me briefly outline the contents of the course. The first chapter gives the definition of PA as an axiomatic theory, and treats the formalization of elementary number theory in it, up to the representability theorems for recursive and primitive recursive functions. The second chapter gives an account of Gödel's Incompleteness Theorems. The third and fourth chapters are concerned with the model theory of PA. Chapter 3 focusses on structural aspects of extensions of models. After a discussion of the ordered structure of (nonstandard) models and the Overspill Principle, the two basic kinds of extension (cofinal and end-extension) are treated: existence of proper elementary extensions of each kind, and Gaifman's Splitting Theorem. Chapter four is called 'Recursive Aspects of models of PA' (I couldn't think of a better name) and deals with theorems connected to the existence of the partial truth (or 'satisfaction') predicates for Σ_n-formulas, and the theory of coded sets. We have the classical theorems of Ryll-Nardzewski (PA is not finitely axiomatized) and Tennenbaum (no countable nonstandard model of PA is recursive); and then the beautiful results of Scott and Friedman.

There is lots of scope for follow-up courses in many directions. Among the topics I specifically regret not having been able to say anything, are weak subtheories of PA (there is an interesting model theory, and ramifications to complexity theory), and the algebraic structure of nonstandard models (an easy fact, recorded in the Appendix, indicates that these have interesting properties).

Prerequisites: these notes have been written for students who have been through basic mathematical education (the first two years of the university curriculum in mathematics) as well as the basics of model theory and recursion theory. Specifically, what is required from model theory is: elementary embeddings, the method of diagrams, the Omitting Types theorem. From recursion theory: the recursion theorem, r.e. sets, the arithmetic hierarchy, relative computability.

Literature An outstanding reference for models is Kaye's *Models of Peano Arithmetic* which also has most of the material in chapter 1, and which I have plagiarized happily. Another very helpful source was Smorynski's *Lecture Notes on Nonstandard Models of Arithmetic*, in Logic Colloquium '82.

For a good overview of the (modal) logical structure of ther Incompleteness Theorems, see Smorynski's *Self-Reference and Modal Logic*, and for various number-theoretical aspects, his *Logical Number Theory I*.

There are many good and accessible treatments of Gödel's First Incompleteness Theorem, but, rather embarrassingly for such a central result, not so many for the Second. For example, Smullyan's *Gödel's Incompleteness Theorems* does not give a proof of the Second Incompleteness Theorem! There is a good exposition in Girard's *Proof Theory and Logical Complexity*.

1 The Formal System of Peano Arithmetic

The system of first-order Peano Arithmetic or PA, is a theory in the language $\mathcal{L}_{\mathrm{PA}} = \{0, 1; +, \cdot\}$ where $0, 1$ are constants, and $+, \cdot$ binary function symbols. It has the following axioms:

1) $\forall x \neg(x + 1 = 0)$

2) $\forall xy(x + 1 = y + 1 \rightarrow x = y)$

3) $\forall x(x + 0 = x)$

4) $\forall xy(x + (y + 1) = (x + y) + 1)$

5) $\forall x(x \cdot 0 = 0)$

6) $\forall xy(x \cdot (y + 1) = (x \cdot y) + x)$

7) $\forall \vec{x}[(\varphi(0, \vec{x}) \wedge \forall y(\varphi(y, \vec{x}) \rightarrow \varphi(y + 1, \vec{x}))) \rightarrow \forall y \varphi(y, \vec{x})]$

Item 7 is meant to be an axiom for every formula $\varphi(y, \vec{x})$. These axioms are called *induction axioms.* Such a set of axioms, given by one or more generic symbols "φ" which range over all formulas, is called an *axiom scheme*; in our case we talk about the *induction scheme*.

So, PA is given by infinitely many axioms and we shall see that this infinitude is essential.

Clearly, the set $\mathbb{N}$ together with the elements 0,1 and usual addition and multiplication, is a model of PA, which we call the *standard model* and denote by $\mathcal{N}$. It is easy to see that PA has also non-standard models. First define, for every $n \in \mathbb{N}$, a term $\overline{n}$ of $\mathcal{L}_{\mathrm{PA}}$ by recursion: $\overline{0} = 0$ and $\overline{n+1} = \overline{n} + 1$ (mind you, this is not the identity function! E.g., $\overline{3} = ((0 + 1) + 1) + 1)$. Terms of the form $\overline{n}$ are called *numerals* and we shall use them a lot later on. Now let c be a new constant, and consider in the language $\mathcal{L}_{\mathrm{PA}} \cup \{c\}$ the set of axioms:

$$\{\text{axioms of PA}\} \cup \{\neg(c = \overline{n}) \mid n \in \mathbb{N}\}$$

Since every finite subset has a straighforward interpretation in $\mathbb{N}$, this is a consistent set of axioms and has therefore a model $\mathcal{M}$, which has a *nonstandard element* $c^{\mathcal{M}}$.

The theory PA is surprisingly strong: it can represent (in a suitable sense, soon to be made precise) all recursive functions, and most elementary number theory can be carried out in this system. Ironically though, it is exactly this strength that lies at the basis of its being *incomplete* as Gödel was the first to show. Since we wish to arrive at these famous *Incompleteness Theorems*, our first aim is to develop some elementary number theory in PA. Our first proposition establishes basic properties of addition and multiplication.

Proposition 1.1

i) $\mathrm{PA} \vdash \forall x(x = 0 \vee \exists y(x = y + 1))$

ii) PA $\vdash \forall xyz(x + (y + z) = (x + y) + z)$

iii) PA $\vdash \forall xy(x + y = y + x)$

iv) PA $\vdash \forall xyz(x + z = y + z \rightarrow x = y)$

v) PA $\vdash \forall xyz(x{\cdot}(y{\cdot}z) = (x{\cdot}y){\cdot}z)$

vi) PA $\vdash \forall xy(x{\cdot}y = y{\cdot}x)$

vii) PA $\vdash \forall xyz(x{\cdot}(y + z) = (x{\cdot}y) + (x{\cdot}z))$

viii) PA $\vdash \forall xyz(\neg(z = 0) \wedge x{\cdot}z = y{\cdot}z \rightarrow x = y)$

Proof. All of these are proved using the induction axioms. For i), let $\varphi(x)$ be $x = 0 \vee \exists y(x = y + 1)$. Clearly, PA $\vdash \varphi(0) \wedge \forall y\varphi(y + 1)$, so PA $\vdash \forall x\varphi(x)$.

For ii), use "induction on z" that is, let $\varphi(z)$ be the formula $\forall xy(x+(y+z) = (x + y) + z)$. Then PA $\vdash \varphi(0)$ by axiom 3, and PA $\vdash \varphi(z) \rightarrow \varphi(z + 1)$ by axiom 4, since

$$\varphi(z) \vdash (x + (y + z)) + 1 = x + ((y + z) + 1) = x + (y + (z + 1))$$

The proof of the other statements is a useful exercise (sometimes, as in iii), you will need to perform a double induction). ∎

Exercise 1. Prove statements iii)-viii) of proposition 1.1.

Proposition 1.2 *Let $\varphi(x, y)$ be the formula $\exists z(x+(z+1) = y)$. Then in* PA*, φ defines a discrete linear order with least element which satisfies the least number principle, i.e.*

i) PA $\vdash \neg\varphi(x, x)$

ii) PA $\vdash \varphi(x, y) \wedge \varphi(y, z) \rightarrow \varphi(x, z)$

iii) PA $\vdash \varphi(x, y) \vee x = y \vee \varphi(y, x)$

iv) PA $\vdash x = 0 \vee \varphi(0, x)$

v) PA $\vdash \varphi(x, y) \rightarrow (y = x + 1 \vee \varphi(x + 1, y))$

vi) PA $\vdash \exists w\psi(w) \rightarrow \exists y(\psi(y) \wedge \forall x(\varphi(x, y) \rightarrow \neg\psi(x)))$

vii) PA $\vdash \varphi(x, x + 1)$

Exercise 2. Prove proposition 1.2

The scheme vi) of proposition 1.2 is called the *least number principle* LNP.

Exercise 3. Prove that LNP is equivalent to the scheme of induction, in the following sense: let PA$'$ be the theory with the first 6 axioms of PA, and the statements of proposition 1.2 as axioms. Then PA and PA$'$ are equivalent theories.

The order defined in proposition 1.2 is so important that we introduce a new symbol for it: henceforth we write $x < y$ for $\exists z(x + (z + 1) = y)$. We shall also use the abbreviations $\exists x < y$ and $\forall x < y$ for $\exists x(x < y \wedge \ldots)$ and $\forall x(x < y \rightarrow \ldots)$, respectively. We shall write $x \leq y$ for $x = y \vee x < y$, and $x \neq y$ for $\neg(x = 0)$. This process of introducing abbreviations will continue throughout; it is absolutely essential if we want to write meaningful formal statements (but, especially later when we shall also introduce function symbols, we shall have to make sure that the properties of the meant functions are provable in PA).

Exercise 4. Prove the principle of *well-founded induction*, that is:

$$\mathrm{PA} \vdash \forall w(\forall v < w \psi(v) \rightarrow \psi(w)) \rightarrow \forall w \psi(w)$$

Exercise 5. Prove:

$$\mathrm{PA} \vdash \forall xy(y \neq 0 \rightarrow x \leq x \cdot y)$$

1.1 Elementary Number Theory in PA

The starting point for our treatment of elementary number theory in PA is the theorem of *Euclidean division*.

Theorem 1.3 (Division with remainder)

$$\mathrm{PA} \vdash \forall xy(y \neq 0 \rightarrow \exists ab(x = a \cdot y + b \wedge 0 \leq b < y))$$

Moreover, PA *proves that such a, b are unique.*

Proof. By induction on x. Clearly, $0 = 0 \cdot y + 0$; if $x = a \cdot y + b \wedge 0 \leq b < y$ then by 1.2v), $b+1 < y \vee b+1 = y$. If $b+1 < y$, $x+1 = a \cdot y + (b+1)$ and if $b+1 = y$, $x + 1 = (a + 1) \cdot y + 0$.

For uniqueness, suppose $x = a \cdot y + b = a' \cdot y + b'$ with $0 \leq b, b' < y$. If $a < a'$ then $a + 1 \leq a'$ hence

$$a' \cdot y \geq a \cdot y + y > a \cdot y + b = x$$

with a contradiction. So $a' \leq a$ and by symmetry, $a = a'$. Then $b = b'$ follows by 1.1iv). ■

In the notation of theorem 1.3, we call b the *remainder of x on division by y*, and a the *integer part of x divided by y*.

Again, we introduce shorthand notation:

$$\begin{array}{rcl} x|y & \equiv & \exists z(x \cdot z = y) \\ \mathrm{irred}(x) & \equiv & \forall v \leq x(v|x \rightarrow v = 1 \vee v = x) \\ \mathrm{prime}(x) & \equiv & x > 1 \wedge \forall yz(x|(y \cdot z \rightarrow x|y \vee x|z) \end{array}$$

Furthermore, since $\mathrm{PA} \vdash \forall xy \exists! z((z = 0 \wedge x < y) \vee x = z + y)$, we may introduce a function symbol $-$ to the language, with axiom

$$\forall xy((x < y \wedge x - y = 0) \vee (x = y + (x - y)))$$

I hope the notations are familiar. The notions "irreducible" and "prime" element are from ring theory.

Proposition 1.4

$$\mathrm{PA} \vdash \forall x(x > 1 \to (\mathrm{irred}(x) \leftrightarrow \mathrm{prime}(x)))$$

Proof. If $\mathrm{prime}(x)$ and $v|x$ so $v \cdot z = x$ then either $x|v$ whence $v = x$, or $x|z$ whence $v = 1$. So $\mathrm{irred}(x)$. Conversely suppose $\mathrm{irred}(x)$ and $x > 1$. Let $P(v)$ be the formula

$$\forall yz \leq v(y \cdot z \leq v \wedge x|(y \cdot z) \to x|y \vee x|z)$$

We show $\forall w(\forall v < wP(v) \to P(w))$, so by well-founded induction we may conclude $\forall wP(w)$ which clearly implies $\mathrm{prime}(x)$.

So suppose $\forall v < wP(v)$ and $y, z \leq w$ such that $y \cdot z \leq w$, $x|(y \cdot z)$, $x \nmid y$, $x \nmid z$. Then $y, z > 1$ and using 1.3 we may assume $y < x$ since otherwise replace y by its remainder on division by x. Again using 1.3, let $x = a \cdot y + b$ with $0 \leq b < y$. If $b = 0$ then by irreducibility of x, $y = 1 \vee y = x$, a contradiction in both cases. If $b > 0$ we have

$$b \cdot z = (x - a \cdot y) \cdot z = x \cdot z - a \cdot y \cdot z$$

so $x|(b \cdot z)$, $x \nmid b$, $x \nmid z$ and $b \cdot z < y \cdot z \leq w$; contradiction with $\forall v < wP(v)$. Therefore $P(w)$, and we are done. ∎

Proposition 1.5 $\mathrm{PA} \vdash \forall x(x > 1 \to \exists v(\mathrm{prime}(v) \wedge v|x))$

Proof. If $x > 1$, since $x|x$ we have $\exists w(w > 1 \wedge w|x)$. By LNP, there is a least such w. The least such w is irreducible, hence prime by proposition 1.4. ∎

Exercise 6. Prove that "PA proves the existence of infinitely many primes", i.e. the statement

$$\forall x \exists y(x < y \wedge \mathrm{prime}(y))$$

[Hint: first prove, by induction in PA, $\forall x \exists y > 0 \forall i(1 \leq i \leq x \to i|y)$. Given such y, consider $y + 1$ and apply proposition 1.5]

We define two predicates, "x is a power of the prime v" and "x is a prime power" respectively:

$$\begin{aligned} \mathrm{pow}(x, v) &\equiv x \geq 1 \wedge \mathrm{prime}(v) \wedge \forall w \leq x(w > 1 \wedge w|x \to v|w) \\ \mathrm{pp}(x) &\equiv \exists v \leq x \, \mathrm{pow}(x, v) \end{aligned}$$

Exercise 7.

a) $\mathrm{PA} \vdash \forall xv(\mathrm{pow}(x, v) \to \mathrm{pow}(x \cdot v, v))$

b) $\mathrm{PA} \vdash \forall xyv(\mathrm{pow}(x, v) \wedge \mathrm{pow}(y, v) \to x|y \vee y|x)$

c) $\mathrm{PA} \vdash \forall xyv(\mathrm{pow}(x, v) \wedge \mathrm{pow}(y, v) \wedge x < y \to (x \cdot v)|y)$

For prime(v), we want to define for each number $y > 0$ its v-part, that is the highest power of v that divides y. We denote this by $y \upharpoonright v$, and we assume as axiom:

$$\mathrm{pow}(y \upharpoonright v, v) \wedge (y \upharpoonright v)|y \wedge (y \upharpoonright v)\cdot v \nmid y$$

Of course, to be able to do this we have to prove that

$$\mathrm{PA} \vdash \forall yv \exists ! z((z = 0 \wedge (y = 0 \vee \neg \mathrm{prime}(v))) \vee \mathrm{pow}(z, v) \wedge z|y \wedge z\cdot v \nmid y)$$

If $\mathrm{pow}(y, v)$ take $z = y$. Otherwise, $\exists w \le y(w|y \wedge v \nmid w)$ hence $\exists z \le y \exists w \le y(y = w\cdot z \wedge v \nmid w)$, so by LNP there is a least such z. Then $\mathrm{pow}(z, v)$ and $z|y$. If $z\cdot v|y$ so $y = w'\cdot z\cdot v = w\cdot z$, then $w'\cdot v = w$, contradiction with $v \nmid w$. So z exists; its uniqueness follows from the Exercise above.

The following lemma states that $x|y$ iff every prime power which divides x also divides y.

Lemma 1.6

$$\mathrm{PA} \vdash \forall xy(x|y \leftrightarrow \forall v \le x(\mathrm{pp}(v) \wedge v|x \rightarrow v|y))$$

Proof. The direction from left to right is trivial, as is the case $y = 0 \vee x = 1$ in the other direction. For a contradiction, let $x > 1$ be least such that

$$\exists y \ge 1(\forall v \le x(\mathrm{pp}(v) \wedge v|x \rightarrow v|y) \wedge x \nmid y)$$

and take the least such y. Its remainder on division by x satisfies the same property, so we may assume $y < x$. Let $x = a\cdot y + b$ with $0 \le b < y$. If $0 < b$ we have a contradiction with the minimality of y. So $b = 0$ and $x = a\cdot y$. Suppose $a > 1$. Then a has a prime divisor v by 1.5. Since $\mathrm{pp}(v)$ and $v|x$, $v|y$. But now we have

$$\mathrm{pp}((y \upharpoonright v)\cdot v) \wedge (y \upharpoonright v)\cdot v|x \wedge (y \upharpoonright v)\cdot v \nmid y$$

which is a contradiction. ∎

We can now define the least common multiple and greatest common divisor of two numbers, and prove their basic properties in PA.

Let $x, y \ge 1$. Since $x|x\cdot y$ and $y|x\cdot y$ there is a unique least $w > 0$ with $x|w \wedge y|w$; we denote this w by $\mathrm{lcm}(x, y)$. Clearly, $\mathrm{lcm}(x, y) \le x\cdot y$.

Writing $x\cdot y = a\cdot\mathrm{lcm}(x, y) + b$, $0 \le b < \mathrm{lcm}(x, y)$ we see that $x|b \wedge y|b$ so if $b > 0$ we get a contradiction with the minimality of $\mathrm{lcm}(x, y)$. So $x\cdot y = a\cdot\mathrm{lcm}(x, y)$ for a unique a, which we denote by $\mathrm{gcd}(x, y)$. Writing $\mathrm{lcm}(x, y) = y\cdot z$, we have $x\cdot y = \mathrm{gcd}(x, y)\cdot y\cdot z$ so $x = \mathrm{gcd}(x, y)\cdot z$ and $\mathrm{gcd}(x, y)|x$; similarly, $\mathrm{gcd}(x, y)|y$.

Exercise 8. Define yourself the function symbols $\max(x, y)$ and $\min(x, y)$ and prove their basic properties in PA. Prove furthermore:

a) $\mathrm{PA} \vdash \mathrm{prime}(v) \rightarrow \mathrm{lcm}(x, y) \upharpoonright v = \max(x \upharpoonright v, y \upharpoonright v)$

b) $\mathrm{PA} \vdash \mathrm{prime}(v) \rightarrow \mathrm{gcd}(x, y) \upharpoonright v = \min(x \upharpoonright v, y \upharpoonright v)$

Proposition 1.7

a) PA $\vdash \forall xyu(x, y \geq 1 \wedge x|u \wedge y|u \rightarrow \mathrm{lcm}(x,y)|u)$

b) PA $\vdash \forall xyu(x, y \geq 1 \wedge u|x \wedge u|y \rightarrow u|\mathrm{gcd}(x,y))$

Proof. For a), consider the remainder of u on division by $\mathrm{lcm}(x,y)$; if it is non-zero, it is $< \mathrm{lcm}(x,y)$ and still a common multiple of x and y.

For b), use proposition 1.6. Let $\mathrm{pow}(z,v) \wedge z|u$. Then $z|(x \upharpoonright v) \wedge z|(y \upharpoonright v)$ so $z|(\mathrm{gcd}(x,y) \upharpoonright v)$ (by the Exercise), so $z|\mathrm{gcd}(x,y)$. By 1.6, $u|\mathrm{gcd}(x,y)$. ∎

Exercise 9. Prove:

a) PA $\vdash \forall xy \geq 1 \forall x'y'(x = x' \cdot \mathrm{gcd}(x,y) \wedge y = y' \cdot \mathrm{gcd}(x,y) \rightarrow \mathrm{gcd}(x',y') = 1)$

b) PA $\vdash \forall xyab(y = a \cdot x + b \wedge 0 \leq b < x \rightarrow \mathrm{gcd}(x,y) = \mathrm{gcd}(x,b))$

Theorem 1.8 (Bézout's Theorem for PA)

$$\text{PA} \vdash \forall xy \geq 1 \exists a \leq y, b \leq x(a \cdot x = b \cdot y + \mathrm{gcd}(x,y)))$$

Proof. By induction on x. For $x = 1$ take $a = 1, b = 0$.

For $x > 1$ let $y = c \cdot x + d$, $0 \leq d < x$. Dividing this equation by $\mathrm{gcd}(x,y)$ we have $y' = c \cdot x' + d'$ with $d' < x' \leq x$ and $\mathrm{gcd}(x', d') = 1$; by induction hypothesis we have

$$u \cdot d' = v \cdot x' + 1$$

for suitable u, v; so $v \cdot x' = u \cdot d' - 1$. Squaring both sides gives

$$a' \cdot x' = b' \cdot d' + 1$$

for some a', b'; multiplying by $\mathrm{gcd}(x,y)$ gives

$$(a' + b' \cdot c) \cdot x = b' \cdot y + \mathrm{gcd}(x,y)$$

Finally, let $(a' + b' \cdot c) = c' \cdot y + a''$, $0 \leq a'' < y$. Then

$$a'' \cdot x = (b' - c' \cdot x) \cdot y + \mathrm{gcd}(x,y)$$

with $a'' < y$ and since $(b' - c' \cdot x) \cdot y \leq a'' \cdot x < x \cdot y$, we have $(b' - c' \cdot x) < x$. ∎

Theorem 1.8 plays a central role in the development of a rudimentary *coding of sequences* in PA, which was in fact Gödel's first crucial idea for the proof of his Incompleteness Theorems.

For a good understanding of what follows, it is useful first to see the algebraic trick underlying it. Suppose we are given a sequence of numbers $x_0, \ldots, x_{n-1}$.

Let $m = \max(x_0, \ldots, x_{n-1}, n)!$. Then for all i, j with $0 \leq i < j < n$ we have that the numbers $m(i+1)+1$ and $m(j+1)+1$ are relatively prime, for if p is a prime number which divides both of them, it divides their difference which is $m(j-i)$. Since p is prime, it follows that $p|m$, but also $p|(i+1)m+1$, a contradiction. Since $x_i < (i+1)m+1$ for all i, we have by the Chinese remainder theorem a number a such that

$$a \equiv x_i \bmod m(i+1)+1$$

for all i. The number a, or rather the pair (a,m), codes the sequence $x_0, \ldots, x_{n-1}$ in a sense.

The following theorem establishes three essential properties of this coding in PA: for every x, there is a sequence starting with x; every sequence can be extended; and a technical condition necessary later on.

We use the following abbreviations: $\mathrm{rm}(x,y)$ denotes the remainder of x on division by y, and $(a,m)_i$ denotes $\mathrm{rm}(a, m\cdot(i+1)+1)$.

Theorem 1.9

i) $\mathrm{PA} \vdash \forall x \exists a, m((a,m)_0 = x)$

ii) $\mathrm{PA} \vdash \forall yxam \exists bn(\forall i < y((a,m)_i = (b,n)_i) \wedge (b,n)_y = x)$

iii) $\mathrm{PA} \vdash \forall ami((a,m)_i \leq a)$

Proof. For i), take $m = x$ and $a = 2x+1$; then

$$\mathrm{rm}(a, m\cdot(0+1)+1) = \mathrm{rm}(2x+1, x+1) = x$$

iii) is trivial, so we are left to prove ii). Let us observe:

$$\begin{array}{ll}
\mathrm{PA} \vdash \forall yxam \exists u(\forall i < y((a,m)_i < u) \wedge x < u \wedge y < u) & (1)\\
\mathrm{PA} \vdash \forall u \exists v \geq 1 \forall i \leq u\, (i \geq 1 \rightarrow i|v) & (2)\\
\mathrm{PA} \vdash \forall uv(\forall i \leq u(i \geq 1 \rightarrow i|v) \rightarrow & \\
\quad \forall ij(0 \leq i < j \leq u \rightarrow \gcd((i+1)\cdot v+1, (j+1)\cdot v+1) = 1)) & (3)
\end{array}$$

((1) is proved by induction on y, (2) by induction on u, and (3) by formalizing the informal argument given above, using the properties about gcd that we know)

So, given y, x, a, m, take successively u satisfying (1) and v satisfying (2) for u; put $n = v$. We have:

$$\begin{gathered}
\forall i < y((a,m)_i < (i+1)\cdot n+1)\\
x < (y+1)\cdot n+1\\
\forall ij(0 \leq i < j \leq y \rightarrow \gcd((i+1)\cdot n+1, (j+1)\cdot n+1) = 1)
\end{gathered}$$

and we want to find b such that

$$(\forall i < y((a,m)_i = (b,n)_i)) \wedge x = (b,n)_y$$

To do this we employ induction. Suppose for $k < y$ there is b' satisfying

$$(\forall i < k((a,m)_i = (b',n)_i)) \wedge x = (b',n)_y$$

We want to find b satisfying

$$(\forall i \leq k((a,m)_i = (b,n)_i)) \wedge x = (b,n)_y$$

Now it is easy to show that for all $k < y$,

$$\exists w((y+1)\cdot n + 1|w \wedge \forall i < k((i+1)\cdot n + 1|w) \wedge \mathrm{gcd}(w,(k+1)\cdot n+1) = 1)$$

(use induction on k and the properties of n). Take such w. Then by 1.8, there is $u \leq (k+1)\cdot n + 1$ such that

$$\mathrm{rm}(u\cdot w,(k+1)\cdot n+1) = 1$$

Put $b = b' + u\cdot w\cdot(b'\cdot n\cdot(k+1) + (a,m)_k)$. Then $(b,n)_y = (b',n)_y = x$ since $(y+1)\cdot n+1|w$, and $i < k \rightarrow (b,n)_i = (b',n)_i = (a,m)_i$ since $(i+1)\cdot n+1|w$. Finally,

$$\begin{array}{rcl}(b,n)_k & = & \mathrm{rm}(b,(k+1)\cdot n+1) \\ & = & \mathrm{rm}(b' + b'\cdot n\cdot(k+1) + (a,m)_k,(k+1)\cdot n+1) \\ & = & \mathrm{rm}(b'\cdot((k+1)\cdot n+1) + (a,m)_k,(k+1)\cdot n+1) \\ & = & (a,m)_k\end{array}$$

which completes the induction step and the proof. ∎

We shall shortly see (in Theorem 1.13 below) how to use theorem 1.9 to define every primitive recursive function in PA, after the necessary definitions to make precise what this means. But to give the idea already now, let's "define" the exponential function $x,y \mapsto x^y$. Let $\theta(x,y,z)$ be the formula

$$\exists am((a,m)_0 = 1 \wedge \forall i < y((a,m)_{i+1} = x\cdot(a,m)_i) \wedge (a,m)_y = z)$$

Exercise 10. Prove that $\mathrm{PA} \vdash \forall xy\exists! z\theta(x,y,z)$. Introduce a function symbol exp to $\mathcal{L}_{\mathrm{PA}}$, with axiom $\forall xy\theta(x,y,\exp(x,y))$. Prove:

$$\begin{array}{l}\mathrm{PA} \vdash \forall xyy'(\exp(x,y+y') = \exp(x,y)\cdot\exp(x,y')) \\ \mathrm{PA} \vdash \forall xyy'(\exp(x,y\cdot y') = \exp(\exp(x,y),y')) \\ \mathrm{PA} \vdash \forall xv(\mathrm{pow}(x,v) \rightarrow \exists y < x(x = \exp(v,y)))\end{array}$$

And try your hand at:

Exercise 11. Formulate and prove in PA the theorem of unique prime factorization.

1.2 Representing Recursive Functions in PA

Definition 1.10 An $\mathcal{L}_{\mathrm{PA}}$-formula φ is called a Δ_0-formula if all quantifiers are bounded in φ, that is of the form $\forall x < t$ or $\exists x < t$, for a term t not containing the variable x. A formula φ is a Σ_1-formula if it is of the form $\exists y_1 \ldots y_t\psi$ with ψ a Δ_0-formula. We also write $\varphi \in \Delta_0$, $\varphi \in \Sigma_1$.

Exercise 12. Prove the *Collection Principle* in PA:

$$\text{PA} \vdash \forall i < t \exists v \psi \rightarrow \exists v \forall i < t \exists u < v \psi$$

and deduce that if φ is equivalent to a Σ_1-formula, so is $\forall i < t\varphi$.

We now discuss the so-called "Σ_1-completeness" of PA: the statement that PA proves all Σ_1-sentences which are true in the standard model $\mathcal{N}$. Recall the definition of the numerals $\overline{n}$ from page 1.

Exercise 13. Prove:

$$\begin{array}{rl} \text{PA} \vdash \overline{n} + \overline{m} = \overline{k} \Leftrightarrow n + m = k & \text{for all } n, m, k \in \mathbb{N} \\ \text{PA} \vdash \overline{n} \cdot \overline{m} = \overline{k} \Leftrightarrow n \cdot m = k & \text{for all } n, m, k \in \mathbb{N} \\ \text{PA} \vdash \overline{n} < \overline{m} \Leftrightarrow n < m & \text{for all } n, m \in \mathbb{N} \\ \text{PA} \vdash \forall x(x < \overline{n} \leftrightarrow x = \overline{0} \vee \ldots \vee x = \overline{n-1}) & \text{for all } n > 0 \end{array}$$

From this exercise we can see by induction on the $\mathcal{L}_{\text{PA}}$-term $t(x_1, \ldots, x_k)$ with variables $x_1, \ldots, x_k$: if $t^{\mathcal{N}}$ is its interpretation in the model $\mathcal{N}$, as function $\mathbb{N}^k \rightarrow \mathbb{N}$, then for all $n_1, \ldots, n_k \in \mathbb{N}$:

$$\text{PA} \vdash t(\overline{n_1}, \ldots, \overline{n_k}) = \overline{t^{\mathcal{N}}(n_1, \ldots, n_k)}$$

Exercise 14. (Σ_1-completeness of PA) Prove that for every Δ_0-formula φ with free variables $x_1, \ldots, x_k$ and all $n_1, \ldots, n_k \in \mathbb{N}$:

$$\text{PA} \vdash \varphi(\overline{n_1}, \ldots, \overline{n_k}) \Leftrightarrow \mathcal{N} \models \varphi[n_1, \ldots, n_k]$$

and deduce that the same equivalence holds for Σ_1-formulas. Conclude that a Σ_1-sentence is provable in PA if and only if it is true in $\mathcal{N}$.

Warning. The equivalence does *not* hold for negations of Σ_1-formulas, as we shall soon see!

Definition 1.11 Let $A \subseteq \mathbb{N}^k$ a k-ary relation. An $\mathcal{L}_{\text{PA}}$-formula $\varphi(x_1, \ldots, x_k)$ of k free variables is said to *represent A (numeralwise)* if for all $n_1, \ldots, n_k \in \mathbb{N}$ we have:

$$\begin{array}{lcll} (n_1, \ldots, n_k) \in A & \Rightarrow & \text{PA} \vdash \varphi(\overline{n_1}, \ldots, \overline{n_k}) & \text{and} \\ (n_1, \ldots, n_k) \notin A & \Rightarrow & \text{PA} \vdash \neg\varphi(\overline{n_1}, \ldots, \overline{n_k}) & \end{array}$$

Let $F : \mathbb{N}^k \rightarrow \mathbb{N}$ a k-ary function. An $\mathcal{L}_{\text{PA}}$-formula $\varphi(x_1, \ldots, x_k, z)$ of $k+1$ free variables represents F numeralwise if for all $n_1, \ldots, n_k \in \mathbb{N}$:

$$\begin{array}{c} \text{PA} \vdash \varphi(\overline{n_1}, \ldots, \overline{n_k}, \overline{F(n_1, \ldots, n_k)}) \quad \text{and} \\ \text{PA} \vdash \exists! z \varphi(\overline{n_1}, \ldots, \overline{n_k}, z) \end{array}$$

Exercise 15. If $F : \mathbb{N}^k \rightarrow \mathbb{N}$ is numeralwise represented then so is its graph, considered as $k+1$-ary relation.

We say that a relation or function is Σ_1-represented if there is a Σ_1-formula representing it. Later, we shall see that if a function is represented at all, it must be Σ_1-represented, and recursive (and vice versa).

Definition 1.12 A function $F : \mathbb{N}^k \to \mathbb{N}$ is called *provably recursive* in PA if it is represented by a Σ_1-formula $\varphi(x_1, \ldots, x_k, z)$ for which

$$\mathrm{PA} \vdash \forall x_1 \ldots x_k \exists! z \varphi(x_1, \ldots, x_k, z)$$

Theorem 1.13 *Every primitive recursive function is provably recursive in* PA.

Proof. We prove this by induction on the generation of the primitive recursive function. The basic functions $\lambda x_1 \cdots x_k.x_i$, $\lambda x.x+1$ and $\lambda x.0$ are clearly provably recursive.

If $F(\vec{x})$ is defined by composition from $G, H_1, \ldots, H_m$, so

$$F(\vec{x}) = G(H_1(\vec{x}), \ldots, H_m(\vec{x}))$$

suppose by induction hypothesis that $G, H_1, \ldots, H_m$ are represented by the Σ_1-formulas $\psi, \chi_1, \ldots, \chi_m$ respectively. Then F is represented by the formula

$$\varphi(\vec{x}, z) \equiv \exists z_1 \cdots z_m (\chi_1(\vec{x}, z_1) \wedge \cdots \wedge \chi_m(\vec{x}, z_m) \wedge \psi(z_1, \ldots, z_m, z))$$

which is equivalent to a Σ_1-formula; that $\mathrm{PA} \vdash \forall \vec{x} \exists! z \varphi(\vec{x}, z)$ follows from the corresponding property for $\psi, \chi_1, \ldots, \chi_m$.

The crucial induction step is primitive recursion; it is here that we use theorem 1.9. Suppose that $F(\vec{x}, y)$ is defined by primitive recursion from G and H, so

$$F(\vec{x}, 0) = G(\vec{x}) \text{ and } F(\vec{x}, y+1) = H(\vec{x}, F(\vec{x}, y), y)$$

By induction hypothesis, G and H are Σ_1-represented by $\psi(\vec{x}, z)$ and $\chi(\vec{x}, u, v, w)$ respectively. Then F is represented by the formula $\varphi(\vec{x}, y, u)$ defined as

$$\exists am(\psi(\vec{x}, (a,m)_0) \wedge \forall i < y\, \chi(\vec{x}, (a,m)_i, i, (a,m)_{i+1}) \wedge (a,m)_y = u)$$

To be sure, this should really be seen as an *abbreviation*, since there is no term $(a,m)_i$ in $\mathcal{L}_{\mathrm{PA}}$, so e.g. $\psi(\vec{x}, (a,m)_0)$ is shorthand for

$$\exists c, d < a(a = c{\cdot}(m+1) + d \wedge 0 \leq d < m+1 \wedge \psi(\vec{x}, d))$$

but still one sees that the formula φ is equivalent to a Σ_1-formula. The proof that $\mathrm{PA} \vdash \forall \vec{x}, y \exists! u \varphi(\vec{x}, y, u)$ is done by induction (in PA!) on u, where one uses the properties listed in theorem 1.9. The details of this proof, as well as the proof that φ represents F, are left to the reader. ∎

Exercise 16. Carry out the filling in of missing details in the proof of theorem 1.13.

The study of the class of all functions which are provably recursive in PA, is important for the *proof theory* of PA. It is an old result that the provably recursive functions in PA are the ε_0*-recursive* functions. This refers to an ordinal hierarchy of total recursive functions, and ε_0 is the least ordinal α such that there exists a recursive binary relation $\prec$ on $\mathbb{N}$ with the properties:

- $(\mathbb{N}, \prec)$ is a well-order of order-type α;
- PA does not prove the scheme

$$\forall x(\forall y \prec x\psi(y) \to \psi(x)) \to \forall x\psi(x)$$

(where, of course, we use a Σ_1-formula representing $\prec$ in PA)

There are several equivalent definitions of ε_0; another one is: the least ordinal which is closed under the operation $\beta \mapsto \omega^\beta$.

We do not enter this study in this course, but just point out that there are lots of provably total functions which are not primitive recursive. To give the simplest possible case:

Exercise 17. Prove that the Ackermann function:

$$\begin{array}{rcl} A(0,x) & = & x+1 \\ A(n+1,0) & = & A(n,1) \\ A(n+1,x+1) & = & A(n,A(n+1,x)) \end{array}$$

is provably recursive in PA.

Theorem 1.14 *Every total recursive function is* Σ_1*-represented in* PA.

Proof. By basic recursion theory, there is a primitive recursive predicate T, a primitive recursive function U such that for every k-ary recursive function F we have a number e such that:

$$F(n_1,\ldots,n_k) = m \Leftrightarrow \exists y(T(e,n_1,\ldots,n_k,y) \wedge U(y) = m)$$

The set $\{(n_1,\ldots,n_k,y,m) \mid T(e,n_1,\ldots,n_k,y) \wedge U(y) = m\}$ is primitive recursive and so, by 1.13, represented by a Σ_1-formula $\varphi(x_1,\ldots,x_k,y,w)$, which we can write as

$$\exists z_1 \ldots z_l P(x_1,\ldots,x_k,y,w,z_1,\ldots,z_l)$$

for a Δ_0-formula P.

If $R(z,\vec{x},w)$ is the Δ_0-formula $\exists y < z\exists z_1 < z \cdots \exists z_l < zP$, then clearly

$$\mathrm{PA} \vdash \exists yw\varphi(\vec{x},y,w) \leftrightarrow \exists zwR(z,\vec{x},w)$$

Finally, let $S(z,\vec{x},w)$ be the Δ_0-formula

$$w < z \wedge R(z,\vec{x},w) \wedge \forall u < z\neg\exists v < uR(u,\vec{x},v)$$

Then $\mathrm{PA} \vdash \exists zwR(z,\vec{x},w) \leftrightarrow \exists! z\exists wS(z,\vec{x},w)$ by LNP.

I claim that the Σ_1-formula $\exists zS(z,\vec{x},w)$ represents the function F. First, for $n_1,\ldots,n_k \in \mathbb{N}$ is

$$\exists zS(z,\overline{n_1},\ldots,\overline{n_k},\overline{F(n_1,\ldots,n_k)})$$

a true Σ_1-formula, hence provable in PA by Σ_1-completeness. To show that

$$\mathrm{PA} \vdash \exists! w\exists zS(z,\overline{n_1},\ldots,\overline{n_k},w)$$

let $a \in \mathbb{N}$ such that $S(\overline{a}, \overline{n_1}, \ldots, \overline{n_k}, \overline{F(n_1, \ldots, n_k)})$ is true. By unicity of z in S we have

$$\mathrm{PA} \vdash \forall zw(S(z, \overline{n_1}, \ldots, \overline{n_k}, w) \to z = \overline{a} \wedge w < \overline{a})$$

and since $\mathrm{PA} \vdash \forall w < \overline{a}\, (w = \overline{0} \vee \cdots \vee w = \overline{a-1})$, we have

$$\begin{array}{ll} \mathrm{PA} \vdash \overline{F(n_1, \ldots, n_k)} < \overline{a} & \text{and} \\ \mathrm{PA} \vdash \neg S(\overline{a}, \overline{n_1}, \ldots, \overline{n_k}, \overline{b}) & \text{for all } b < a,\, b \neq F(n_1, \ldots, n_k) \end{array}$$

since $S \in \Delta_0$. So, $\mathrm{PA} \vdash \exists! w \exists z S(z, \overline{n_1}, \ldots, \overline{n_k}, w)$. ■

Exercise 18. In the next chapter we shall see that there are Σ_1-sentences which are false in $\mathcal{N}$ but consistent with PA. Use this to show that the following implication does *not* hold: for a Σ_1-formula $\varphi(w)$ with only free variable w, if $\exists! w \varphi(w)$ is true in $\mathcal{N}$, then $\mathrm{PA} \vdash \exists! w \varphi(w)$.

Exercise 19. Prove that every recursive set is Σ_1-represented in PA.

Exercise 20. Let $D_1, D_2, D_3, \ldots$ be a sequence of definitions of primitive recursive functions with the properties that for every k, the function f_k defined by D_k is either a basic function or defined from functions f_l with $l < k$, and every primitive recursive function is f_k for some k.

Introduce, for every k, a new function symbol F_k and an axiom φ_k, corresponding to the definition D_k of f_k.

Let PA' be the theory in the language $\mathcal{L}_{\mathrm{PA}} \cup \{F_1, F_2, \ldots\}$, axiomatized by the axioms of PA, together with the axioms φ_k, and the scheme of induction extended to the full new language.

Prove that there is a mapping $(\cdot)^*$ from $\mathcal{L}_{\mathrm{PA}'}$-formulas to $\mathcal{L}_{\mathrm{PA}}$-formulas, which is the identity on $\mathcal{L}_{\mathrm{PA}}$-formulas, such that

$$\begin{array}{c} \mathrm{PA}' \vdash \varphi \leftrightarrow (\varphi)^* \\ \mathrm{PA}' \vdash \varphi \Rightarrow \mathrm{PA} \vdash (\varphi)^* \end{array}$$

for all $\mathcal{L}_{\mathrm{PA}'}$-formulas φ. Conclude that PA' is conservative over PA.

Exercise 21. Devise a coding of the definitions D_k in the previous exercise, and show that a *recursive* sequence $D_1, D_2, \ldots$ exists with the required properties. Can it be primitive recursive?

1.3 A Primitive Incompleteness Theorem

The representability of recursive functions allows us to prove already that PA is not a complete theory (this, however, is not quite Gödel's theorem; the latter gives more information). We have to leave one detail to the reader's imagination (it will be fully treated in the next chapter, but it is easy): for every $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(w)$ with exactly one free variable w, the set

$$\{n \in \mathbb{N} \mid \mathrm{PA} \vdash \varphi(\overline{n})\}$$

is *recursively enumerable*.

Now we do know, that for every recursively enumerable set $X \subseteq \mathbb{N}$, there is a Σ_1-formula $\varphi(w)$, such that for all $n \in \mathbb{N}$:

$$n \in X \Leftrightarrow \mathrm{PA} \vdash \varphi(\overline{n})$$

(Use the characterization of r.e. sets as projections of recursive sets, representability of recursive sets in PA, and Σ_1-completeness of PA)

Now, let X be a nonrecursive, r.e. set and suppose the Σ_1-sentence φ defines X in this sense. Let $Y = \{n \in \mathbb{N} \mid \mathrm{PA} \vdash \neg\varphi(\overline{n})\}$. Then since PA is consistent, X and Y are disjoint r.e. sets and since X is not recursive, Y is not the complement of X. Take $m \notin X \cup Y$. Since $\mathrm{PA} \vdash \varphi(\overline{m})$ implies $m \in X$ and $\mathrm{PA} \vdash \neg\varphi(\overline{m})$ implies $m \in Y$, we see that none of these can hold; therefore, $\varphi(\overline{m})$ is a sentence which is independent of PA.

The following exercise is a result which will be needed in the next chapter. We call a formula $\varphi(x_1, \ldots, x_k)$ Δ_1, or *a Δ_1-formula*, if both φ and $\neg\varphi$ are equivalent (in PA) to a Σ_1-formula.

Exercise 22. Show that the proof of theorem 1.13 can be adapted to give the following stronger result: for every primitive recursive function $F : \mathbb{N}^k \to \mathbb{N}$ there is a Δ_1-formula $\varphi_F(x_1, \ldots, x_{k+1})$ which represents F and is such that

$$\mathrm{PA} \vdash \forall x_1 \cdots x_k \exists! x_{k+1} \varphi_F(x_1, \ldots, x_{k+1})$$

2 Gödel Incompleteness

2.1 Coding of Formulas and Diagonalization

Let us recall a primitive recursive coding of pairs and sequences from basic recursion theory.

Let $j(n,m) = \frac{(n+m)^2+3n+m}{2}$.

Exercise 23. Prove that j defines a bijection: $\mathbb{N}^2 \to \mathbb{N}$, and that there are primitive recursive functions $j_0, j_1 : \mathbb{N} \to \mathbb{N}$ such that $x = j(j_0(x), j_1(x))$, $j_0(j(x,y)) = x$ and $j_1(j(x,y)) = y$.

We have primitive recursive bijections $j^m : \mathbb{N}^m \to \mathbb{N}$ for $m \geq 1$, defined recursively by

$$j^1(x) = x \quad j^{m+1}(x_1, \dots, x_{m+1}) = j(j^m(x_1, \dots, x_m), x_{m+1})$$

and primitive recursive j_i^m $(1 \leq i \leq m)$ such that

$$j^m(j_1^m(x), \dots, j_m^m(x)) = x \quad \text{and} \quad j_i^m(j^m(x_1, \dots, x_m)) = x_i$$

Moreover, the function

$$F(x,y,z) = \begin{cases} 0 & \text{if } y = 0 \text{ or } y > x \\ j_y^x(z) & \text{else} \end{cases}$$

is primitive recursive.

Let $\mathbb{N}^{<\omega}$ be the set of finite sequences of natural numbers. We have a bijection $\langle\cdot\rangle : \mathbb{N}^{<\omega} \to \mathbb{N}$ given by

$$\begin{aligned} \langle\rangle &= 0 && \text{(empty sequence)} \\ \langle x_0, \dots, x_{m-1}\rangle = j(m-1, j^m(x_0, \dots, x_{m-1})) &&& \text{for } m > 0 \end{aligned}$$

We call $\langle x_0, \dots, x_{m-1}\rangle$ the *code* of the sequence $x_0, \dots, x_{m-1}$. There are primitive recursive functions lh and $(\cdot)_i$, such that for every x, $\mathrm{lh}(x)$ gives the length of the sequence coded by x, and $(x)_i$ is the i-th element of the sequence coded by x:

$$\begin{aligned} \mathrm{lh}(x) &= \begin{cases} 0 & x = 0 \\ j_1(x-1)+1 & x > 0 \end{cases} \\ (x)_i &= \begin{cases} j_{i+1}^{\mathrm{lh}(x)}(j_2(x-1)) & x > 0 \text{ and } 0 \leq i \leq \mathrm{lh}(x) \\ 0 & \text{else} \end{cases} \end{aligned}$$

We use sequence encoding to assign to any formula φ of $\mathcal{L}_{\mathrm{PA}}$ a code $\ulcorner\varphi\urcorner \in \mathbb{N}$ and this in such a way that all relevant operations on formulas translate into primitive recursive functions on codes.

We assume that in our language, variables are numbered $v_0, v_1, \dots$. Consider the following "code book" (from now on we take $<$ as a primitive symbol of $\mathcal{L}_{\mathrm{PA}}$):

0	1	v	$+$	$\cdot$	$=$	$<$	$\wedge$	$\vee$	$\to$	$\neg$	$\forall$	$\exists$
0	1	2	3	4	5	6	7	8	9	10	11	12

For each term t define its code $\ulcorner t \urcorner$ by recursion on t: $\ulcorner 0 \urcorner = \langle 0 \rangle$, $\ulcorner 1 \urcorner = \langle 1 \rangle$, $\ulcorner v_i \urcorner = \langle 2, i \rangle$; $\ulcorner t + s \urcorner = \langle 3, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$, $\ulcorner t \cdot s \urcorner = \langle 4, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$.

It is now immediate that the properties "x is the code of a term", "x codes a constant", "the variable v_i occurs in the term coded by x", etcetera, are all primitive recursive in their arguments.

Likewise, we define codes for formulas: $\ulcorner t = s \urcorner = \langle 5, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$, $\ulcorner t < s \urcorner = \langle 6, \ulcorner t \urcorner, \ulcorner s \urcorner \rangle$, $\ulcorner \varphi \wedge \psi \urcorner = \langle 7, \ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \rangle$, $\ulcorner \varphi \vee \psi \urcorner = \langle 8, \ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \rangle$ and so on; $\ulcorner \forall v_i \varphi \urcorner = \langle 11, i, \ulcorner \varphi \urcorner \rangle$ and $\ulcorner \exists v_i \varphi \urcorner = \langle 12, i, \ulcorner \varphi \urcorner \rangle$.

And we have that the properties "x codes a formula", "the main connective of the formula coded by x is $\wedge$", "the variable v_i occurs freely in the formula coded by x" and so forth, are primitive recursive in their arguments.

Exercise 24. Verify this for some of the mentioned properties.

Exercise 25. Verify that the property "x codes a formula φ and y codes a term t and t is free for v_i in φ" is primitive recursive in x, y, i; and show that there is a primitive recursive function Sub, such that

$$\mathrm{Sub}(x, y, i) = \begin{cases} \ulcorner \varphi[s/v_i] \urcorner & \text{if } y = \ulcorner \varphi \urcorner \text{ and } x = \ulcorner s \urcorner \\ 0 & \text{else} \end{cases}$$

Exercise 26. Convince yourself that the properties "x is the code of a Δ_0-formula" and "x codes a Σ_1-formula" are primitive recursive.

Having done this work, we now arrive at the *second* main idea of Gödel, the Diagonalization Lemma.
We say that φ is a Π_1-formula if it is of the form $\forall y_1 \cdots \forall y_n \psi$ with $\psi \in \Delta_0$.

Lemma 2.1 (Diagonalization Lemma) *For any $\mathcal{L}_{\mathrm{PA}}$-formula φ with free variable v_0 there is an $\mathcal{L}_{\mathrm{PA}}$-formula ψ with the same free variables as φ except v_0, such that*

$$\mathrm{PA} \vdash \psi \leftrightarrow \varphi[\overline{\ulcorner \psi \urcorner}/v_0]$$

Moreover, if $\varphi \in \Pi_1$ then ψ can be chosen to be Π_1 too.

Proof. Recall the function $\mathrm{Sub}(x, y, i)$ from Exercise 25. It is primitive recursive hence so is $\lambda xy.\mathrm{Sub}(x, y, 0)$; let S be a Σ_1-formula representing this function in PA. Let T be a Σ_1-formula representing the primitive recursive function $n \mapsto \ulcorner \overline{n} \urcorner$. Then we have

$$\begin{array}{rr} \forall nm \in \mathbb{N}.\mathrm{PA} \vdash S(\overline{n}, \overline{m}, \overline{\mathrm{Sub}(n, m, 0)}) & (1) \\ \forall n \in \mathbb{N}.\mathrm{PA} \vdash T(\overline{n}, \overline{\ulcorner \overline{n} \urcorner}) & (2) \\ \mathrm{PA} \vdash \forall xy \exists! z S(x, y, z) & (3) \\ \mathrm{PA} \vdash \forall x \exists! y T(x, y) & (4) \end{array}$$

Now let φ have v_0 free. Define the formula C by

$$C \equiv \forall xy(T(v_0, x) \wedge S(x, v_0, y) \rightarrow \varphi[y/v_0])$$

and let ψ be defined by

$$\psi \equiv C[\overline{\ulcorner C \urcorner}/v_0] \qquad (5)$$

Clearly, if $\varphi \in \Pi_1$ then so are C and ψ. Now we have by (2) and (4),

$$\mathrm{PA} \vdash \forall y(\exists x(T(\overline{\ulcorner C \urcorner}, x) \wedge S(x, \overline{\ulcorner C \urcorner}, y)) \leftrightarrow S(\overline{\ulcorner \overline{\ulcorner C \urcorner} \urcorner}, \overline{\ulcorner C \urcorner}, y))$$

and (1) and (3) give us

$$\mathrm{PA} \vdash \forall y(S(\overline{\ulcorner \overline{\ulcorner C \urcorner} \urcorner}, \overline{\ulcorner C \urcorner}, y) \leftrightarrow y = \overline{\ulcorner C[\overline{\ulcorner C \urcorner}/v_0] \urcorner})$$

By (5) then,

$$\mathrm{PA} \vdash \forall y(\exists x(T(\overline{\ulcorner C \urcorner}, x) \wedge S(x, \overline{\ulcorner C \urcorner}, y)) \leftrightarrow y = \overline{\ulcorner \psi \urcorner})$$

so

$$\begin{aligned} \mathrm{PA} \vdash \psi &\leftrightarrow \forall y(\exists x(T(\overline{\ulcorner C \urcorner}, x) \wedge S(x, \overline{\ulcorner C \urcorner}, y)) \rightarrow \varphi[y/v_0]) \\ &\leftrightarrow \forall y(y = \overline{\ulcorner \psi \urcorner} \rightarrow \varphi[y/v_0]) \\ &\leftrightarrow \varphi[\overline{\ulcorner \psi \urcorner}/v_0] \end{aligned}$$

∎

Remark. One should compare the proof of Lemma 2.1 with the proofs of very similar theorems, such as the recursion theorem, or the fixpoint theorem in λ-calculus.

I include the following corollary, which is analogous to Smullyan's "simultaneous recursion theorem", or Bekić' Lemma in Domain Theory, for its own interest. We shall not apply it.

Corollary 2.2 (Simultaneous Diagonalization) *Let φ and ψ be formulas both having the variables v_0, v_1 free. Then there are formulas θ and χ, such that θ has the same free variables as φ minus v_0, v_1, and ditto for χ and ψ, such that*

$$\mathrm{PA} \vdash \theta \leftrightarrow \varphi[\overline{\ulcorner \theta \urcorner}/v_0, \overline{\ulcorner \chi \urcorner}/v_1]$$
$$\mathrm{PA} \vdash \chi \leftrightarrow \psi[\overline{\ulcorner \theta \urcorner}/v_0, \overline{\ulcorner \chi \urcorner}/v_1]$$

And, if $\varphi, \psi \in \Pi_1$, so are θ, χ.

Proof. Let T be the same formula as in the proof of Lemma 2.1, and S_1 similar, that is: S_1 now represents substitution for the variable v_1. So $\mathrm{PA} \vdash S_1(\overline{\ulcorner s \urcorner}, \overline{\ulcorner \varphi \urcorner}, \overline{\ulcorner \varphi[s/v_1] \urcorner})$, etc. Let φ and ψ be given. First, apply Lemma 2.1 to find θ_1 such that

$$\mathrm{PA} \vdash \theta_1 \leftrightarrow \forall zy(T(v_1, z) \wedge S_1(z, \overline{\ulcorner \theta_1 \urcorner}, y) \rightarrow \varphi[y/v_0, v_1])$$

and then χ such that

$$\mathrm{PA} \vdash \chi \leftrightarrow \forall xy(T(\overline{\ulcorner \chi \urcorner}, z) \wedge S_1(z, \overline{\ulcorner \theta_1 \urcorner}, y) \rightarrow \psi[y/v_0, \overline{\ulcorner \chi \urcorner}/v_1])$$

Put $\theta \equiv \theta_1[\overline{\ulcorner \chi \urcorner}/v_1]$. Then as in the proof of Lemma 2.1, we have:

$$\mathrm{PA} \vdash T(\overline{\ulcorner \chi \urcorner}, \overline{\ulcorner \overline{\ulcorner \chi \urcorner} \urcorner}) \wedge S_1(\overline{\ulcorner \overline{\ulcorner \chi \urcorner} \urcorner}, \overline{\ulcorner \theta_1 \urcorner}, \overline{\ulcorner \theta_1[\overline{\ulcorner \chi \urcorner}/v_1] \urcorner})$$
$$\mathrm{PA} \vdash \forall y(\exists z(T(\overline{\ulcorner \chi \urcorner}, z) \wedge S_1(z, \overline{\ulcorner \theta_1 \urcorner}, y)) \leftrightarrow y = \overline{\ulcorner \theta \urcorner})$$
$$\mathrm{PA} \vdash \theta \leftrightarrow \theta_1[\overline{\ulcorner \chi \urcorner}/v_1] \leftrightarrow \varphi[\overline{\ulcorner \theta \urcorner}, \overline{\ulcorner \chi \urcorner}]$$

and so, also $\mathrm{PA} \vdash \chi \leftrightarrow \psi[\overline{\ulcorner \theta \urcorner}, \overline{\ulcorner \chi \urcorner}]$. ∎

2.2 Coding of Proofs and Gödel's First Incompleteness Theorem

Just as we have coded formulas, we can code proofs in PA by natural numbers. Since the idea is essentially the same, we give only a sketch. First, we have to decide which proof system we use; let's use natural deduction. Again we make a code book, now of construction steps for natural deduction trees (I have *not* tried to make the system as economical as possible!):

Ass	0	$\vee$I − r	5	$\forall$E	10	$\bot$	15
$\wedge$I	1	$\vee$I − l	6	$\exists$I	11	$\neg\neg$	16
$\wedge$E − r	2	$\to$E	7	$\exists$E	12		
$\wedge$E − l	3	$\to$I	8	$\neg$I	13		
$\vee$E	4	$\forall$I	9	$\neg$E	14		

We view natural deduction proofs as labelled trees; every node is labelled by a formula, and by a rule. Most connectives have an *introduction* and an *elimination* rule, sometimes more than one, for example the rule $\wedge$E − r (conjunction elimination to the right) infers ψ from $\phi \wedge \psi$. The rule $\neg$E infers $\bot$ from $\phi, \neg\phi$; the rule $\bot$ infers ψ from $\bot$, the rule $\neg\neg$ infers ψ from $\neg\neg\psi$. The rule Ass (assumption) is the only starting rule: it allows one to construct a one-node tree, labelled with a formula φ. I hope that the meaning of every rule is now clear.

Now every tree has a set of so-called *open* (or undischarged) assumptions. An assumption is a formula which labels a leaf of the tree. Assumptions are discharged with the steps $\to$ I, $\neg$I, $\vee$E and $\exists$E. We follow the so-called *crude discharge convention*: that is, whenever we introduce $\varphi \to \psi$ by $\to$ I, we discharge *all* assumptions φ above this application.

Let us outline the coding of trees. The tree with one node, labelled φ, gets code $\langle 0, \ulcorner\varphi\urcorner\rangle$; suppose D_1, D_2 are trees with roots labelled by φ, ψ respectively; the tree resulting from D_1 and D_2 by applying $\wedge$I gets code $\langle 1, \ulcorner D_1\urcorner, \ulcorner D_2\urcorner, \ulcorner\varphi \wedge \psi\urcorner\rangle$, where $\ulcorner D_1\urcorner$ denotes the code of D_1. If D_2 results from D_1 by applying $\wedge E - r$, so the root of D_1 is labelled $\varphi \wedge \psi$ and the root of D_2 is labelled ψ, we have $\ulcorner D_2\urcorner = \langle 2, \ulcorner D_1\urcorner, \ulcorner\psi\urcorner\rangle$. If D_4 results from D_1, D_2, D_3 by $\vee$-elimination, that is: the root of D_1 is labelled $\varphi \vee \psi$, D_2 and D_3 have χ at the root, and D_4 also has χ at the root, whereby in D_2, all open assumptions φ are discharged and in D_3 all open assumptions ψ are discharged, we have $\ulcorner D_4\urcorner = \langle 4, \ulcorner D_1\urcorner, \ulcorner D_2\urcorner, \ulcorner D_3\urcorner, \ulcorner\chi\urcorner\rangle$.

I hope the process is now clear: the length of $\ulcorner D\urcorner$ is $n + 2$ where n is the number of branches from the root (in fact, always $n \leq 3$), the first element of $\ulcorner D\urcorner$ is the code of the last rule applied, and the last element of $\ulcorner D\urcorner$ is the formula which labels the root of D. In this way, we can easily recover the whole tree D from its code $\ulcorner D\urcorner$. We can also define a primitive recursive function OA, which, given $\ulcorner D\urcorner$, gives a code for the set of undischarged assumptions of D. Therefore, we can, primitive recursively, check whether D is in fact a correct proof tree (for example, when introducing $\forall u\varphi(u)$ by $\forall$I from $\varphi(v)$, we need to know that the variable v does not occur in any undischarged assumption, and so on). The conclusion is that we have a primitive recursive predicate $\mathrm{NDT}(x, y)$:

$\mathrm{NDT}(x, y)$ says that y is the code of a formula and x is the code of a correct natural deduction tree with root labelled by the formula coded by y.

In order that x codes a proof in PA, we need to know that all open assumptions of the tree coded by x are axioms of PA, or axioms of the predicate calculus governing the equality sign $=$: the axioms $u = u$, $u = v \wedge v = w \to u = w$ and $t = s \wedge \varphi[t/u] \to \varphi[s/u]$ (subject to the well-known conditions).

Exercise 27. Show that the predicate $\mathrm{Ax}(x)$: x is the code of an axiom of PA or the predicate calculus, is primitive recursive.

Let $\mathrm{Prf}(x, y)$ be the predicate: y is the code of a formula, and x is the code of a correct proof in PA of the formula coded by y:

$$\mathrm{Prf}(x, y) \leftrightarrow \mathrm{NDT}(x, y) \wedge \forall z \in \mathrm{OA}(x)\mathrm{Ax}(z)$$

Let $\overline{\mathrm{Prf}}$, $\overline{\mathrm{NDT}}$ and $\overline{\mathrm{Ax}}$ be Δ_1-formulas representing the predicates Prf, NDT, Ax in PA.

The predicate Prf is defined by a course-of-values recursion, and we can assume that PA proves this course of values recursion for the representing formula $\overline{\mathrm{Prf}}$. That is,

$$\mathrm{PA} \vdash \overline{\mathrm{Prf}}(x, y) \leftrightarrow C_0(x, y) \vee \cdots \vee C_{16}(x, y)$$

(referring to our code book of natural deduction rules), where $C_0(x, y)$ is the formula

$$x = \langle 0, y\rangle \wedge \overline{\mathrm{Ax}}(y)$$

$C_1(x, y)$ will be the formula

$$\exists abvw < x(y = \langle \bar{7}, v, w\rangle \wedge \overline{\mathrm{Prf}}(a, v) \wedge \overline{\mathrm{Prf}}(b, w) \wedge x = \langle \bar{1}, a, b, y\rangle)$$

and so on. In some cases, where open assumptions are discharged, we have to write conditions; e.g., C_8 (corresponding to $\to$I) will read:

$$\exists avw < x(x = \langle \bar{8}, a, y\rangle \wedge y = \langle \bar{9}, v, w\rangle \wedge \overline{\mathrm{NDT}}(a, w) \wedge \forall z \in \mathrm{OA}(a)(\overline{\mathrm{Ax}}(z) \vee z = v))$$

(slightly abusing notation: "$z \in \mathrm{OA}(a)$" means of course the intended formalization)

It is now straightforward to see that we have the following proposition:

Proposition 2.3

i) $\mathrm{PA} \vdash \varphi \Rightarrow \mathrm{PA} \vdash \exists x\overline{\mathrm{Prf}}(x, \overline{\ulcorner\varphi\urcorner})$

ii) $\mathrm{PA} \vdash \forall xy(\overline{\mathrm{Prf}}(x, \overline{\ulcorner\varphi \to \psi\urcorner}) \wedge \overline{\mathrm{Prf}}(y, \overline{\ulcorner\psi\urcorner}) \to \overline{\mathrm{Prf}}(\langle\bar{7}, x, y, \overline{\ulcorner\psi\urcorner}\rangle, \overline{\ulcorner\psi\urcorner}))$

We introduce an abbreviation: $\Box\varphi$ for $\exists x\overline{\mathrm{Prf}}(x, \overline{\ulcorner\varphi\urcorner})$. Proposition 2.3 now says:

D1	$\mathrm{PA} \vdash \varphi \Rightarrow \mathrm{PA} \vdash \Box\varphi$
D2	$\mathrm{PA} \vdash \Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$

Theorem 2.4 (Gödel's First Incompleteness Theorem) *Apply Lemma 2.1 to the formula* $\neg\exists x\overline{\mathrm{Prf}}(x, v_0)$*, to obtain a* Π_1*-sentence* G *such that*

$$\mathrm{PA} \vdash G \leftrightarrow \neg\Box G$$

Then G *is independent of* PA.

Proof. Since $\overline{\mathrm{Prf}}(x, y)$ is Δ_1, clearly G can be chosen to be Π_1. If $\mathrm{PA} \vdash G$ then by D1, $\mathrm{PA} \vdash \Box G$, so $\mathrm{PA} \vdash \neg G$ by the choice of G. So PA is inconsistent, quod non.

On the other hand, if $\mathrm{PA} \vdash \neg G$ then $\mathrm{PA} \vdash \Box G$ by the choice of G. Then $\Box G$ is true in $\mathcal{N}$, which means that there is a proof of G, i.e. $\mathrm{PA} \vdash G$, and again PA is inconsistent. ∎

Remarks.

i) The sentence G is the famous "Gödel sentence". Roughly speaking it says "I am not provable", and it has therefore been compared with several liar paradoxes (see the work by Smullyan and Smorynski).

ii) The sentence G is true in $\mathcal{N}$, because if it were false, then $\neg G$ would be a true Σ_1-sentence, hence provable in PA by Σ_1-completeness.

iii) In the proof of Theorem 2.4, we have used the reasoning: "if $\mathrm{PA} \vdash \varphi$ then $\mathcal{N} \models \varphi$" (in fact, we only used this for the Σ_1-sentence $\neg G$). This is not satisfactory, because we would like to extend Gödel's method to consistent extensions of PA, which need not have this property, even for Σ_1-sentences (for example, $\mathrm{PA}\cup\{\neg G\}$ is such a theory). A way of avoiding this reasoning was found by Rosser, a few years after Gödel. Let $\varphi(v_0)$ be the formula
$$\forall x(\overline{\mathrm{Prf}}(x, v_0) \to \exists y < x\overline{\mathrm{Prf}}(y, \langle 10, v_0\rangle))$$
Check that $\varphi(v_0)$ is equivalent to a Π_1-formula! Apply Lemma 2.1 to $\varphi(v_0)$, to obtain a Π_1-sentence R such that

$$\mathrm{PA} \vdash R \leftrightarrow \forall x(\overline{\mathrm{Prf}}(x, \overline{\ulcorner R\urcorner}) \to \exists y < x\overline{\mathrm{Prf}}(y, \overline{\ulcorner\neg R\urcorner}))$$

We can show that R is independent of PA, just using that PA is consistent and Σ_1-complete. Suppose $\mathrm{PA} \vdash R$. By consistency of PA, $\mathrm{PA} \nvdash \neg R$, whence the sentence

$$\exists x(\overline{\mathrm{Prf}}(x, \overline{\ulcorner R\urcorner}) \wedge \forall y < x\neg\overline{\mathrm{Prf}}(y, \overline{\ulcorner\neg R\urcorner}))$$

is a true Σ_1-sentence, hence by Σ_1-completeness provable in PA. But this sentence is equivalent to $\neg R$, contradiction. Conversely, if $\mathrm{PA} \vdash \neg R$ we have for some $n \in \mathbb{N}$ that $\mathrm{PA} \vdash \overline{\mathrm{Prf}}(\bar{n}, \overline{\ulcorner\neg R\urcorner})$ and $\mathrm{PA} \vdash \forall y < \bar{n}\neg\overline{\mathrm{Prf}}(y, \overline{\ulcorner R\urcorner})$, since these are true Σ_1-sentences. It follows that $\mathrm{PA} \vdash \forall x(\overline{\mathrm{Prf}}(x, \overline{\ulcorner R\urcorner}) \to \exists y < x\overline{\mathrm{Prf}}(y, \overline{\ulcorner\neg R\urcorner}))$, that is $\mathrm{PA} \vdash R$. Again, a contradiction with the consistency of PA.

iv) The sentence $\neg\Box-$ is called the sentence expressing the consistency of PA, and often written as $\mathrm{Con}_{\mathrm{PA}}$. It is an easy consequence of D2 that $\mathrm{PA} \vdash \Box- \to \Box\psi$ for any ψ, so we have $\mathrm{PA} \vdash G \to \mathrm{Con}_{\mathrm{PA}}$. In the next section, we shall see that in fact, $\mathrm{PA} \vdash G \leftrightarrow \mathrm{Con}_{\mathrm{PA}}$, from which it follows that $\mathrm{PA} \nvdash \mathrm{Con}_{\mathrm{PA}}$. This is Gödel's Second Incompleteness Theorem: PA does not prove its own consistency".

A number of exercises to finish this section:

Exercise 28. Show that for any formula $\varphi(v)$ with one free variable v, the set

$$\{n \in \mathbb{N} \mid \mathrm{PA} \vdash \varphi[\bar{n}/v]\}$$

is recursively enumerable. Conclude that if a function is numeralwise representable in PA, it is recursive, hence Σ_1-representable.

Exercise 29. Define a function $F : \mathbb{N} \to \mathbb{N}$ by:

$$F(n) = \max\{\mu m.\mathcal{N} \models \theta[n, j_0(m), j_1(m)] \mid \theta \in \Theta(n)\} + 1$$

where $\Theta(n)$ is the set of all Δ_0-formulas $\theta(u, v, w)$ such that

$$\ulcorner\theta(u, v, w)\urcorner < n \text{ and } \exists y < n\mathrm{Prf}(y, \ulcorner\forall u\exists v\exists w\theta(u, v, w)\urcorner)$$

(and the maximum of the empty set is 0).

i) Show that F is total recursive;

ii) show that F cannot be provably recursive.

Exercise 30. (Tarski's theorem on the non-definability of truth). Apply Lemma 2.1 to show that there is no formula of $\mathcal{L}_{\mathrm{PA}}$ which defines the set of true $\mathcal{L}_{\mathrm{PA}}$-sentences, i.e. if

$$A = \{n \in \mathbb{N} \mid n \text{ is the code of a sentence } \varphi \text{ such that } \mathcal{N} \models \varphi\}$$

then there is no formula $\psi(v)$ such that for all $n \in \mathbb{N}$:

$$n \in A \Leftrightarrow \mathcal{N} \models \psi[n]$$

2.3 Formalized Σ_1-completeness and Gödel's Second Incompleteness Theorem

As we said in the preceding section, Gödel's Second Incompleteness Theorem asserts that "PA does not prove its own consistency". More formally: $\mathrm{PA} \nvdash \mathrm{Con}_{\mathrm{PA}}$ (recall that $\mathrm{Con}_{\mathrm{PA}}$ is the sentence $\neg\Box-$).

Recall that we had derived (proposition 2.3) the following rules governing the operation $\Box$:

D1	$\mathrm{PA} \vdash \varphi \;\Rightarrow\; \mathrm{PA} \vdash \Box\varphi$
D2	$\mathrm{PA} \vdash \Box(\varphi \to \psi) \wedge \Box\varphi \to \Box\psi$

Exercise 31. Prove that for *any* operation $\Box$, satisfying D1 anfd D2, one has:

$$\mathrm{PA} \vdash \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$$

Our aim in this section is to prove that we have a third rule:

$$\text{D3} \qquad\qquad \mathrm{PA} \vdash \Box\varphi \rightarrow \Box\Box\varphi$$

Let us see that this implies what we want:

Theorem 2.5 *For any operation* $\Box$ *satisfying* D1–D3 *and any* G *such that* $\mathrm{PA} \vdash G \leftrightarrow \neg\Box G$*, we have*

$$\mathrm{PA} \vdash G \leftrightarrow \neg\Box -$$

Proof. Since $\mathrm{PA} \vdash - \rightarrow G$, by D1 and D2 we have $\mathrm{PA} \vdash \Box - \rightarrow \Box G$, so $\mathrm{PA} \vdash G \rightarrow \neg\Box G \rightarrow \neg\Box -$.

For the converse implication, we have from D2 and the assumption on G, $\mathrm{PA} \vdash \Box G \rightarrow \Box(\neg\Box G)$; by D3 we have $\mathrm{PA} \vdash \Box G \rightarrow \Box\Box G$. Combining the two, we have $\mathrm{PA} \vdash \Box G \rightarrow \Box -$, so $\mathrm{PA} \vdash \neg G \rightarrow \Box G \rightarrow \Box -$, whence $\mathrm{PA} \vdash \neg\Box - \rightarrow G$.
■

Corollary 2.6 (Gödel's Second Incompleteness Theorem)

$$\mathrm{PA} \nvdash \mathrm{Con}_{\mathrm{PA}}$$

Proof. Immediate. ■

The rule D3, which we want to prove, is in fact a consequence of a more general theorem, which is known as "Formalized Σ_1-completeness". This is because $\Box\varphi$ is a Σ_1-sentence.

Theorem 2.7 (Formalized Σ_1-completeness of PA) *For every* Σ_1*-sentence of* PA*,*

$$\mathrm{PA} \vdash \varphi \rightarrow \Box\varphi$$

The rest of this section is devoted to the proof of theorem 2.7. Let us recall how we proved ordinary Σ_1-completeness. We proved that for any Δ_0-formula $\varphi(v_0, \ldots, v_{k-1})$ and for every k-tuple of natural numbers $n_0, \ldots, n_{k-1}$:

$$(\dagger) \qquad \mathcal{N} \models \varphi[n_0, \ldots, n_{k-1}] \;\Rightarrow\; \mathrm{PA} \vdash \varphi[\overline{n_0}/v_0, \ldots, \overline{n_{k-1}}/v_{k-1}]$$

We follow a similar line in the formalized case. We now assume that $\mathcal{L}_{\mathrm{PA}}$ is augmented with function symbols $\langle \cdot, \ldots, \cdot \rangle$, lh, $(\cdot)_i$ for the manipulation of sequences. We also take a function symbol T, representing the primitive recursive function $n \mapsto \ulcorner \overline{n} \urcorner$; and we want function symbols S_f and S_t representing the primitive recursive substitution operations on formulas and terms, respectively:

$$S_f(y,x) = \begin{cases} \ulcorner\varphi[s_0/v_0,\ldots,s_{k-1}/v_{k-1}]\urcorner & \text{if } y \text{ is a code for } \varphi, \\ & \mathrm{lh}(x)=k, \text{ and for each } i<k \\ & (x)_i \text{ is a code for } s_i \\ 0 & \text{else} \end{cases}$$

$$S_t(y,x) = \begin{cases} \ulcorner t[s_0/v_0,\ldots,s_{k-1}/v_{k-1}]\urcorner & \text{if } y \text{ is a code for } t, \\ & \mathrm{lh}(x)=k, \text{ and for each } i<k \\ & (x)_i \text{ is a code for } s_i \\ 0 & \text{else} \end{cases}$$

As before, we may assume that PA proves the recursions for these functions. In particular, we may assume that the sentences

$$\begin{gathered} T(0) = \overline{\langle 0\rangle} \\ T(x+1) = \langle \bar{3}, T(x), \overline{\langle 1\rangle}\rangle \\ S_t(\langle \bar{3}, \overline{\ulcorner t\urcorner}, \overline{\ulcorner s\urcorner}\rangle, x) = \langle \bar{3}, S_t(\overline{\ulcorner t\urcorner}, x), S_t(\overline{\ulcorner s\urcorner}, x)\rangle \\ S_t(\langle \bar{4}, \overline{\ulcorner t\urcorner}, \overline{\ulcorner s\urcorner}\rangle, x) = \langle \bar{4}, S_t(\overline{\ulcorner t\urcorner}, x), S_t(\overline{\ulcorner s\urcorner}, x)\rangle \\ S_f(\langle \bar{5}, \overline{\ulcorner t\urcorner}, \overline{\ulcorner s\urcorner}\rangle, x) = \langle \bar{5}, S_t(\overline{\ulcorner t\urcorner}, x), S_t(\overline{\ulcorner s\urcorner}, x)\rangle \\ \vdots \end{gathered}$$

are provable in PA. The formalization of statement (†) above is:

Lemma 2.8 *For every Δ_0-formula $\varphi(v_0,\ldots,v_{k-1})$ we have:*

$$\mathrm{PA} \vdash \forall x_0\cdots x_{k-1}(\varphi(\vec{x}) \to \exists y \overline{\mathrm{Prf}}(y, S_f(\overline{\ulcorner\varphi\urcorner}, \langle T(x_0),\ldots,T(x_{k-1})\rangle)))$$

The proof of Lemma 2.8 goes via the auxiliary lemmas 2.9, 2.10 and 2.11 below.

Lemma 2.9

$$\begin{aligned} &\mathrm{PA} \vdash \forall xy\exists z\overline{\mathrm{Prf}}(z, \langle \bar{5}, T(x+y), S_t(\overline{\ulcorner v_0+v_1\urcorner}, \langle T(x),T(y)\rangle)\rangle) \\ &\mathrm{PA} \vdash \forall xy\exists z\overline{\mathrm{Prf}}(z, \langle \bar{5}, T(x\cdot y), S_t(\overline{\ulcorner v_0\cdot v_1\urcorner}, \langle T(x),T(y)\rangle)\rangle) \end{aligned}$$

Proof. Check, that these statements are formalizations of the statements that $\mathrm{PA} \vdash \overline{n+m} = \overline{n}+\overline{m}$ and $\mathrm{PA} \vdash \overline{n\cdot m} = \overline{n}\cdot\overline{m}$.

By the recursion equations for S_t we have that

$$S_t(\overline{\ulcorner v_0+v_1\urcorner}, \langle T(x),T(y)\rangle) = \langle \bar{3}, T(x), T(y)\rangle$$

so we must prove

$$\exists z\overline{\mathrm{Prf}}(z, \langle \bar{5}, T(x+y), \langle \bar{3}, T(x), T(y)\rangle\rangle)$$

which we do by induction on y. For $y=0$, $T(y) = \overline{\langle 0\rangle}$ and we observe that

$$\langle \bar{5}, T(x), \langle \bar{3}, T(x), \overline{\langle 0\rangle}\rangle\rangle = S_f(\overline{\ulcorner v_0 = v_0+0\urcorner}, \langle T(x)\rangle)$$

Since $\forall v_0(v_0 = v_0+0)$ is the universal closure of a PA-axiom, we have by one step (∀E),

$$\exists z\overline{\mathrm{Prf}}(z, S_f(\overline{\ulcorner v_0 = v_0+0\urcorner}, \langle T(x)\rangle))$$

For the induction step, assume

$$\exists z \overline{\mathrm{Prf}}(z, \langle \bar{5}, T(x+y), \langle \bar{3}, T(x), T(y)\rangle\rangle)$$

Then by applying a substitution axiom for equality, also

$$\exists z \overline{\mathrm{Prf}}(z, \langle \bar{5}, \langle \bar{3}, T(x+y), \overline{\langle 1\rangle}\rangle, \langle \bar{3}, \langle \bar{3}, T(x), T(y)\rangle, \overline{\langle 1\rangle}\rangle\rangle)$$

By an application of the axiom $\forall uv((u+v)+1 = u+(v+1))$ we have

$$\exists z \overline{\mathrm{Prf}}(z, \langle \bar{5}, \langle \bar{3}, \langle \bar{3}, T(x), T(y)\rangle, \overline{\langle 1\rangle}\rangle, \langle \bar{3}, T(x), \langle \bar{3}, T(y), \overline{\langle 1\rangle}\rangle\rangle\rangle)$$

But $\langle \bar{3}, T(y), \overline{\langle 1\rangle}\rangle = T(y+1)$ by the recursion equations for T, which also give $\langle \bar{3}, T(x+y), \overline{\langle 1\rangle}\rangle = T(x+(y+1)) = T((x+y)+1)$, so by applying transitivity of equality we get

$$\exists z \overline{\mathrm{Prf}}(z, \langle \bar{5}, T(x+(y+1)), \langle \bar{3}, T(x), T(y+1)\rangle\rangle)$$

as desired.

The proof of the second statement is similar (and uses the first!). ∎

The proof of lemma 2.9 was, of course, quite unreadable, but the point is that one has a precise idea of what one is doing. One cannot write, for example, that $\langle \bar{3}, T(x), T(y)\rangle = \ulcorner T(x) + T(y)\urcorner$; but, $T(x)$ and $T(y)$ are, "in PA", codes for *terms* $\tilde{x}$ and $\tilde{y}$, so that "$\langle \bar{3}, T(x), T(y)\rangle = \ulcorner \tilde{x} + \tilde{y}\urcorner$" but again this is imprecise, because our coding acts on real terms only. The following notational convention gives a precise way of getting some clarification: for any formula $\varphi(v_0, \ldots, v_{k-1})$, we let

$$\ulcorner \varphi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}})\urcorner$$

be an abbreviation for $S_f(\overline{\ulcorner \varphi \urcorner}, \langle T(x_0), \ldots, T(x_{k-1})\rangle)$. We write

$$\boxdot \varphi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}})$$

for $\exists z \overline{\mathrm{Prf}}(z, \ulcorner \varphi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}})\urcorner)$. With these conventions, Lemma 2.9 becomes:

$$\begin{array}{l} \mathrm{PA} \vdash \forall xy \boxdot (\widetilde{x+y} = \widetilde{x} + \widetilde{y}) \\ \mathrm{PA} \vdash \forall xy \boxdot (\widetilde{x \cdot y} = \widetilde{x} \cdot \widetilde{y}) \end{array}$$

It is now straightforward (by induction on the term) to show that for any term $t(v_0, \ldots, v_{k-1})$ we have:

$$\mathrm{PA} \vdash \forall x_0 \cdots x_{k-1} \boxdot \widetilde{t(x_0, \ldots, x_{k-1})} = t(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}})$$

Exercise 32. Carry out this proof.

The following lemma is an immediate consequence.

Lemma 2.10 *For terms* $t(v_0, \ldots, v_{k-1})$ *and* $s(v_0, \ldots, v_{k-1})$ *we have*

$$\begin{array}{l} \mathrm{PA} \vdash \forall x_0 \cdots x_{k-1}(t(\vec{x}) = s(\vec{x}) \to \boxdot(t(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}) = s(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}))) \\ \mathrm{PA} \vdash \forall x_0 \cdots x_{k-1}(t(\vec{x}) < s(\vec{x}) \to \boxdot(t(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}) < s(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}))) \end{array}$$

We are now ready for the final induction.

Lemma 2.11 *Let Φ be the set of formulas $\varphi(v_0, \ldots, v_{k-1})$ for which*

$$\mathrm{PA} \vdash \forall x_0 \cdots x_{k-1}(\varphi(x_0, \ldots, x_{k-1}) \to \Box\varphi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}))$$

Then Φ contains all formulas of form $t = s$ and $t < s$, and Φ is closed under conjunction, disjunction and bounded quantification.

Proof. That Φ contains all formulas $t = s$ and $t < s$, is lemma 2.10. The induction steps for $\wedge$ and $\vee$ are easy.

Now suppose $\varphi(v_0, \ldots, v_{k-1})$ has the form $\exists v_k < v_0 \psi(v_0, \ldots, v_k)$, for $\psi \in \Phi$. Then $\forall x_0 \cdots x_{k-1}(\varphi(\vec{x}) \to \Box\varphi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}))$ is equivalent (in PA) to

$$\forall x_0 \cdots x_k(x_k < x_0 \wedge \psi(x_0, \ldots, x_k) \to \Box(\exists v_k < \widetilde{x_0}\, \psi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}, v_k)))$$

Since $\psi \in \Phi$, $v_k < v_0 \in \Phi$ and by the induction step for $\wedge$, we have

$$\mathrm{PA} \vdash \forall x_0 \cdots x_k(x_k < x_0 \wedge \psi(x_0, \ldots, x_k) \to \Box(\widetilde{x_k} < \widetilde{x_0} \wedge \psi(\widetilde{x_0}, \ldots, \widetilde{x_k})))$$

so the desired conclusion follows by an application of $\exists$I.
Now suppose φ is $\forall v_k < v_0 \psi(v_0, \ldots, v_k)$ with $\psi \in \Phi$. We prove the implication:

$$\forall v_k < x_0 \psi(x_0, \ldots, x_{k-1}, v_k) \to \Box(\forall v_k < \widetilde{x_0} \psi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}, v_k))$$

by induction on x_0. For x_0 it holds trivially; for the induction step we observe that

$$\forall v_k < x_0 + 1 \psi \leftrightarrow \forall v_k < x_0 \psi \wedge \psi(x_0, \ldots, x_{k-1}, x_0)$$

so that

$$\forall v_k < v_0 \psi \to \Box(\forall v_k < \widetilde{x_0} \psi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}, v_k) \wedge \psi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}, \widetilde{x_0}))$$

We also have $\forall x_0 \Box\, (\widetilde{x_0 + 1} = \widetilde{x_0} + 1)$ and

$$\forall x_0 \Box\, (\forall v_k (v_k < \widetilde{x_0} + 1 \leftrightarrow v_k < \widetilde{x_0} \vee v_k = \widetilde{x_0}))$$

so we obtain the desired implication

$$\forall v_k < x_0 + 1 \psi \to \Box \forall v_k < \widetilde{x_0} \psi(\widetilde{x_0}, \ldots, \widetilde{x_{k-1}}, v_k)$$

∎

Exercise 33.

i) Show that lemma 2.11 is sufficient to prove Lemma 2.8. That is, show that the set Φ contains all Δ_0-formulas;

ii) show that, in turn, Lemma 2.8 implies Theorem 2.7.

Remark The proof of Gödel's Incompleteness Theorems can be carried out for any recursively enumerable extension of PA. By this we mean: a theory, formulated in a language which is coded in a recursive way, and with axioms whose codes form an r.e. set. In fact, we don't need the full force of PA here. Any recursively enumerable theory T which has enough arithmetic to represent (and prove the recursion equations of) the necessary primitive recursive functions, can formulate its own consistency Con_T, and if T is consistent, then $T \nvdash \mathrm{Con}_T$.

An important example is ZFC: set theory with the axiom of Choice. Here is an example of an application of Gödel's Second Incompleteness Theorem to ZFC. A cardinal number κ is called *strongly inaccessible* if $\kappa > \aleph_0$, κ is regular, and $\forall\lambda < \kappa(2^\lambda < \kappa)$. One can prove, in ZFC, that if κ is strongly inaccessible, then V_κ is a model of ZFC. Therefore, in ZFC, if κ is strongly inaccessible, ZFC is consistent. By Gödel's Second Incompleteness Theorem, ZFC $\nvdash$ I where I is the statement: there is a strongly inaccessible cardinal. But one may wish to know whether ZFC+I is consistent. The question becomes: assuming $\mathrm{Con}_{\mathrm{ZFC}}$, can we prove $\mathrm{Con}_{\mathrm{ZFC+I}}$? Again *no*, for we have seen that $\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}_{\mathrm{ZFC}}$, so if $\mathrm{ZFC} + \mathrm{Con}_{\mathrm{ZFC}} \vdash \mathrm{Con}_{\mathrm{ZFC+I}}$, then $\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}_{\mathrm{ZFC+I}}$ which contradicts the Second Incompleteness Theorem, applied to the theory ZFC+I.

Another application of Theorem 2.6 to an extension of PA is *Löb's Theorem.* Löb's theorem says that although the formula $\Box\varphi \to \varphi$ is true in $\mathcal{N}$, it is provable in PA only if φ is provable in PA:

Theorem 2.12 (Löb's Theorem) *If* $\mathrm{PA} \vdash \Box\varphi \to \varphi$, *then* $\mathrm{PA} \vdash \varphi$.

Proof. If $\mathrm{PA} \nvdash \varphi$ then $\mathrm{PA} + \neg\varphi$ is consistent, so by the Second Incompleteness Theorem, applied to $\mathrm{PA}+\neg\varphi$, $\mathrm{PA}+\neg\varphi \nvdash \mathrm{Con}_{\mathrm{PA}+\neg\varphi}$. But now, in PA, $\mathrm{Con}_{\mathrm{PA}+\neg\varphi}$ is equivalent to $\neg\Box\varphi$. So we have $\mathrm{PA} + \neg\varphi \nvdash \neg\Box\varphi$, whence $\mathrm{PA} \nvdash \Box\varphi \to \varphi$. ∎

Exercise 34. Prove Löb's Theorem directly from Lemma 2.1, by taking a sentence ψ such that

$$\mathrm{PA} \vdash \psi \leftrightarrow \Box(\psi \to \varphi)$$

Use the properties D1–D3.

Exercise 35. As before, but now take ψ satisfying

$$\mathrm{PA} \vdash \psi \leftrightarrow (\Box\psi \to \varphi)$$

3 Models of PA: Introduction

3.1 The theory PA^- and end-extensions

From now on, we take the symbol $<$ as part of the language $\mathcal{L}_{\mathrm{PA}}$, so every $\mathcal{L}_{\mathrm{PA}}$-structure $\mathcal{M}$ carries a binary relation $<^{\mathcal{M}}$.

I repeat that the symbol $\mathcal{N}$ will *always* denote the standard model.

We shall find it useful to consider some $\mathcal{L}_{\mathrm{PA}}$-structures that are not models of PA, but of a weaker theory PA^-, which we therefore now introduce.

Definition 3.1 PA^- is the $\{+, \cdot; <; 0, 1\}$-theory with axioms stating that:

1) $+$ and $\cdot$ are commutative and associative and $\cdot$ distributes over $+$;

2) $\forall x(x{\cdot}0 = 0 \wedge x{\cdot}1 = x \wedge x + 0 = x)$

3) $<$ is a linear order satisfying $\forall x(0 \leq x)$ and $\forall x(0 < x \leftrightarrow 1 \leq x)$

4) $\forall xyz(x < y \rightarrow x + z < y + z)$

5) $\forall xyz(0 < z \wedge x < y \rightarrow x{\cdot}z < y{\cdot}z)$

6) $\forall xy(x < y \rightarrow \exists z(x + z = y))$

So, every model of PA^- is a linear order. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{L}_{\mathrm{PA}}$-structures and $\mathcal{M}_1$ is a substructure of $\mathcal{M}_2$, we say that $\mathcal{M}_1$ is an *initial segment* of $\mathcal{M}_2$, or $\mathcal{M}_2$ is an *end-extension* of $\mathcal{M}_1$, if for all $m \in \mathcal{M}_2$ and $n \in \mathcal{M}_1$, if $\mathcal{M}_2 \models m < n$ then $m \in \mathcal{M}_1$. Notation: $\mathcal{M}_1 \subseteq_e \mathcal{M}_2$.

If $\mathcal{M}$ is any model of PA^-, the function $n \mapsto \overline{n}^{\mathcal{M}} : \mathbb{N} \rightarrow \mathcal{M}$ is an embedding of $\mathcal{L}_{\mathrm{PA}}$-structures.

Exercise 36. Prove this, and prove also that this mapping embeds $\mathcal{N}$ as initial segment in $\mathcal{M}$.

If ? is a class of formulas, and $\mathcal{M}_1$ a $\mathcal{L}_{\mathrm{PA}}$-substructure of $\mathcal{M}_2$, we say that $\mathcal{M}_1$ is a ?*-elementary* substructure of $\mathcal{M}_2$, notation: $\mathcal{M}_1 \prec_{\Gamma} \mathcal{M}_2$, if for every $\varphi(v_1, \ldots, v_k) \in$? and all k-tuples $m_1, \ldots, m_k \in \mathcal{M}_1$,

$$\mathcal{M}_1 \models \varphi[m_1, \ldots, m_k] \Leftrightarrow \mathcal{M}_2 \models \varphi[m_1, \ldots, m_k]$$

Exercise 37. Let $\mathcal{M}_1 \subseteq_e \mathcal{M}_2$ and $\mathcal{M}_1$, $\mathcal{M}_2$ models of PA^-. Show that $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$.

Exercise 38. Show that for any inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$ of models of PA, that $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$ implies $\mathcal{M}_1 \prec_{\Delta_1} \mathcal{M}_2$.

Exercise 39. Show that PA^- proves all true Σ_1-sentences.

Exercise 40. Show that for $\mathcal{L}_{\mathrm{PA}}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$: if $\mathcal{M}_1 \subseteq_e \mathcal{M}_2$ and $\mathcal{M}_2$ is a model of PA^-, then $\mathcal{M}_1$ is a model of PA^-.

3.2 Cuts, Overspill and Underspill

Let $\mathcal{M}$ be a model of PA. A *cut* of $\mathcal{M}$ is a nonempty subset $I \subseteq \mathcal{M}$ such that $x < y$ and $y \in I$ implies $x \in I$, and $x \in I$ implies $x + 1 \in I$. A cut I is *proper* if $I \neq \mathcal{M}$. The following easy lemma is of fundamental importance in the study of nonstandard models of PA.

Lemma 3.2 *Let $\mathcal{M}$ be a model of* PA*, and $I \subset \mathcal{M}$ a proper cut. Then I is not definable in parameters from $\mathcal{M}$, that is: there is no $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(v_1, \ldots, v_{k+1})$ such that for some $m_1, \ldots, m_k \in \mathcal{M}$:*

$$I = \{m \in \mathcal{M} \mid \mathcal{M} \models \varphi[m_1, \ldots, m_k, m]\}$$

Proof. Since I is nonempty, $0 \in I$. Moreover, $m \in I$ implies $m + 1 \in I$. Were I definable by φ in parameters $m_1, \ldots, m_k$ as in the Lemma, then since $\mathcal{M}$ satisfies induction, we would have $I = \mathcal{M}$. ∎

Corollary 3.3 (Overspill Lemma) *Let $\mathcal{M}$ be a model of* PA *and $I \subset \mathcal{M}$ a proper cut. If $m_1, \ldots, m_k \in \mathcal{M}$ and $\mathcal{M} \models \varphi[m_1, \ldots, m_k, b]$ for every $b \in I$, then there is $c \in \mathcal{M} \setminus I$ such that*

$$\mathcal{M} \models \forall y \leq c\varphi[m_1, \ldots, m_k, y]$$

Proof. Certainly, for all $c \in I$ we have $\mathcal{M} \models \forall y \leq c\varphi[m_1, \ldots, m_k, y]$; so if such $c \in \mathcal{M} \setminus I$ would not exist, we would have

$$I = \{c \mid \mathcal{M} \models \forall y \leq c\varphi[m_1, \ldots, m_k, y]\}$$

contradicting the non-definability of I of Lemma 3.2. ∎

Corollary 3.4 *Again let $\mathcal{M}$ be a model of* PA *and $I \subset \mathcal{M}$ a proper cut. Suppose that for φ, $m_1, \ldots, m_k \in \mathcal{M}$ we have: for all $x \in I$ there is $y \in I$ with*

$$\mathcal{M} \models y \geq x \wedge \varphi[m_1, \ldots, m_k, y]$$

Then for each $c \in \mathcal{M} \setminus I$ there is $b \in \mathcal{M} \setminus I$ with

$$\mathcal{M} \models b < c \wedge \varphi[m_1, \ldots, m_k, b]$$

Proof. Apply Corollary 3.3 to the formula

$$\exists y(x \leq y < c \wedge \varphi[m_1, \ldots, m_k, y])$$

∎

Corollary 3.5 (Underspill Lemma) *Let $\mathcal{M}$ a model of* PA *and $I \subset \mathcal{M}$ a proper cut.*

i) If for all $c \in \mathcal{M} \setminus I$, $\mathcal{M} \models \varphi[m_1, \ldots, m_k, c]$, then there is $b \in I$ such that $\mathcal{M} \models \forall x \geq b\varphi[m_1, \ldots, m_k, x]$;

ii) if for all $c \in \mathcal{M} \setminus I$ there is $x \in \mathcal{M} \setminus I$ with $\mathcal{M} \models x < c \wedge \varphi[m_1, \ldots, m_k, x]$, then for all $b \in I$ there is $y \in I$ with $\mathcal{M} \models b < y \wedge \varphi[m_1, \ldots, m_k, y]$.

Exercise 41. Prove Corollary 3.5.

3.3 The ordered Structure of Models of PA

We study now the order-type of models of PA; that is, their $\{<\}$-reduct.

If A and B are two linear orders, we order the set $A \times B$ lexicographically, that is: $(a, b) < (a', b')$ iff $a < a'$ or $a = a' \wedge b < b'$. $A \times B$ is then also a linear order, and the picture is: replace every $a \in A$ by a copy of B. By $A + B$ we mean the ordered set which is the disjoint union of A and B, and in which every element of A is below every element of B.

Theorem 3.6 *Let $\mathcal{M}$ be a nonstandard model of* PA. *Then as ordered set, $\mathcal{M} \cong \mathbb{N} + A \times \mathbb{Z}$ where A is a dense, linear order without end-points. Therefore, if $\mathcal{M}$ is countable, $\mathcal{M} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$*

Proof. $\mathcal{M}$ has $\mathcal{N}$ as initial segment, so $\mathcal{M} \cong \mathbb{N} + X$ for some linear order X. For nonstandard $a \in \mathcal{M}$, let $Z(a)$ the set of elements of $\mathcal{M}$ which differ from a by a standard element: $a' \in Z(a)$ iff for some $n \in \mathbb{N}$, $\mathcal{M} \models a' + \overline{n} = a \vee a + \overline{n} = a'$. If $a, b \in \mathcal{M}$ are nonstandard elements and $a \notin Z(b)$, then $Z(a) \cap Z(b) = \emptyset$, and if moreover $a < b$, we have $x < y$ for every $x \in Z(a)$ and $y \in Z(b)$. Since clearly, every $Z(a)$ is order-isomorphic to $\mathbb{Z}$, we have $\mathcal{M} \cong \mathbb{N} + A \times \mathbb{Z}$, where A is the collection of all sets $Z(a)$, ordered by: $Z(a) < Z(b)$ iff $a < b$.

Now A is dense, for given a, b nonstandard, if $Z(a) < Z(b)$ then $Z(a) < Z([\frac{a+b}{2}]) < Z(b)$ (check!).

A has no endpoints: for every nonstandard a we have $Z([\frac{a}{2}]) < Z(a) < Z(a + a)$ (check this too!).

The final statement of the theorem follows from the well-known fact that every countable dense linear order without end-points is order-isomorphic to $\mathbb{Q}$. ∎

We shall now see some examples of proper cuts of a nonstandard model $\mathcal{M}$. For us, the interesting cuts are initial segments, that is: cuts which are closed under the operations $+, \cdot$ in $\mathcal{M}$ (such cuts are then $\mathcal{L}_{\mathrm{PA}}$-substructures of $\mathcal{M}$, and hence models of PA^-, if $\mathcal{M}$ is).

Examples.

1) Let $\mathcal{M}$ be a nonstandard model of PA, and $a \in \mathcal{M}$ nonstandard. By $a^{\mathbb{N}}$ we mean the set

$$\{m \in \mathcal{M} \mid \text{for some } n \in \mathbb{N},\, \mathcal{M} \models m < a^n\}$$

Convince yourself that $a^{\mathbb{N}}$ is closed under the operations $+, \cdot$ of $\mathcal{M}$. Moreover, $a \in a^{\mathbb{N}}$. It is easy to see, that $a^{\mathbb{N}}$ is the smallest initial segment of $\mathcal{M}$ that contains a. It is also easy to see, that $a^{\mathbb{N}} \neq \mathcal{M}$, for $a^a \notin a^{\mathbb{N}}$. By the same token, $a^{\mathbb{N}}$ is not a model of PA.

2) Let $a \in \mathcal{M}$ be nonstandard as before. By $a^{1/\mathbb{N}}$ we mean the set

$$\{m \in \mathcal{M} \mid \text{for all } n \in \mathbb{N},\, \mathcal{M} \models m^n < a\}$$

Again, $a^{1/\mathbb{N}}$ is closed under $+, \cdot$ and is a proper initial segment since $a \notin a^{1/\mathbb{N}}$. Since $\mathbb{N} \subseteq a^{1/\mathbb{N}}$, for every $n \in \mathbb{N}$ we have $\mathcal{M} \models n^n < a$; by the Overspill Lemma, there is a nonstandard element $c \in \mathcal{M}$ such that $\mathcal{M} \models c^c < a$. Clearly then, $c \in a^{1/\mathbb{N}} \setminus \mathbb{N}$.

The following exercises both require use of the Overspill Lemma.

Exercise 42. Show that for $a \in \mathcal{M}$ nonstandard, $m \in \mathcal{M} \setminus a^{\mathbb{N}}$ if and only if $a^c < m$ for some nonstandard $c \in \mathcal{M}$.

Exercise 43. Let $a \in \mathcal{M}$ be nonstandard.

a) Show that for each $n \in \mathbb{N}$ there is $b \in \mathcal{M}$ such that $\mathcal{M} \models b^n \leq a < (b+1)^{n+1}$. Show that for each such b, $\mathcal{M} \models b^b > a$;

b) show that $a^{1/\mathbb{N}}$ is not a model of PA, by showing that there is $c \in a^{1/\mathbb{N}}$ with $\mathcal{M} \models c^c > a$.

The following exercise explains the name "cut".

Exercise 44. Let $\mathcal{M}$ be a countable nonstandard model of PA and $I \subseteq \mathcal{M}$ a proper cut which is not the standard cut $\mathbb{N}$. Suppose that I is closed under $+$. Then under the identification $\mathcal{M} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$ of 3.6, I corresponds to $\mathbb{N} + A \times \mathbb{Z}$, where $A \subset \mathbb{Q}$ is a *Dedekind cut*: a set of form $\{q \in \mathbb{Q} \mid q < r\}$ for some real number r.

Exercise 45. Let $\mathcal{M}$ be a nonstandard model of PA; by theorem 3.6, write $\mathcal{M} \cong \mathbb{N} + A \times \mathbb{Z}$ as ordered structures, with A a dense linear order without end-points. Show that A cannot be order-isomorphic to the real line $\mathbb{R}$ [Hint: let $m \in \mathcal{M}$ be nonstandard and consider the set $\{Z(m \cdot \bar{n}) \mid n \in \mathbb{N}\}$ as subset of A].

Theorem 3.7 *Let $\mathcal{M}$ be a countable, nonstandard model of* PA. *Then $\mathcal{M}$ has $2^{\aleph_0}$ proper cuts which are closed under $+$ and $\cdot$.*

Proof. Define an equivalence relation on the set of nonstandard elements of $\mathcal{M}$ by: $a \sim b$ iff for some $n \in \mathbb{N}$,

$$a \leq b < a^n \text{ or } b \leq a < b^n$$

Clearly, this is an equivalence relation, and the set A of $\sim$-equivalence classes of $\mathcal{M} \setminus \mathbb{N}$ is linearly ordered by $[a] <_A [b]$ iff $a < b$ in $\mathcal{M}$. Suppose $[a] <_A [b]$. Then $a^n < b$ for each $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, there is x with $a^n < x < x^{n+2} < b$; that is, the formula

$$\exists x(a^y < x < x^{y+2} < b)$$

is satisfied (in $\mathcal{M}$) by all standard elements y. By the Overspill Lemma, there is a nonstandard c such that for some $x \in \mathcal{M}$,

$$a^c < x < x^c < b$$

It follows that $[a] <_A [x] <_A [b]$. So the ordering $(A, <_A)$ is dense, and by a similar overspill argument one deduces that it has no end points.

Therefore, since $\mathcal{M}$ was countable, there is an isomorphism $(A, <_A) \cong (\mathbb{Q}, <)$ and hence a surjective, $\leq$-preserving map

$$\mathcal{M} \setminus \mathbb{N} \to (\mathbb{Q}, <)$$

The inverse image of each Dedekind cut in $\mathbb{Q}$ defines a proper cut in $\mathcal{M}$, which is closed under $+$ and $\cdot$. Since there are $2^{\aleph_0}$ Dedekind cuts in $\mathbb{Q}$, the theorem is proved. ∎

3.4 Cofinal extensions; MRDP Theorem and Gaifman's Splitting Theorem

Initial segments are one extreme of inclusions of models; the other extreme is the notion of a *cofinal* submodel. If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are models of PA^-, we say that $\mathcal{M}_1$ is cofinal in $\mathcal{M}_2$, or $\mathcal{M}_2$ is a *cofinal extension* of $\mathcal{M}_1$, if for every $m \in \mathcal{M}_2$ there is $m' \in \mathcal{M}_1$ such that $m < m'$ in $\mathcal{M}_2$. Notation: $\mathcal{M}_1 \subseteq_{\mathrm{cf}} \mathcal{M}_2$.

We extend the notions of Σ_1 and Π_1-formulas to arbitrary n, by putting inductively: a formula is Σ_{n+1} iff it is of form $\exists \vec{y}\psi$ with $\psi \in \Pi_n$; a formula is Π_{n+1} iff it is of form $\forall \vec{y}\psi$ with $\psi \in \Sigma_n$. Clearly, every formula is (up to equivalence in predicate logic) Σ_n for some n. In the definition of Σ_n and Π_n we allow the string $\vec{y}$ to be empty, so that every Σ_n-formula is automatically Σ_{n+1} and Π_{n+1}. First an easy lemma which gives a simplified condition for when an extension is Σ_n-elementary.

Lemma 3.8 *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be an inclusion of $\mathcal{L}_{\mathrm{PA}}$-structures. If $n > 0$ and for each Σ_n-formula $\theta(\vec{x})$ and every tuple $\vec{a}$ of elements of $\mathcal{M}_1$ we have*

$$\mathcal{M}_2 \models \theta[\vec{a}] \Rightarrow \mathcal{M}_1 \models \theta[\vec{a}]$$

then $\mathcal{M}_1 \prec_{\Sigma_n} \mathcal{M}_2$.

Proof. For the converse direction, let $\theta(\vec{x}) \equiv \exists \vec{y}\varphi(\vec{x}, \vec{y})$ (with $\varphi \in \Pi_{n-1}$) and suppose $\mathcal{M}_1 \models \theta[\vec{a}]$ so $\mathcal{M}_1 \models \varphi[\vec{a}, \vec{b}]$ for some tuple $\vec{b}$ of elements of $\mathcal{M}_1$. Since $\neg\varphi$ is trivially Σ_n, we cannot have $\mathcal{M}_2 \models \neg\varphi[\vec{a}, \vec{b}]$; so $\mathcal{M}_2 \models \varphi[\vec{a}, \vec{b}]$ hence $\mathcal{M}_2 \models \theta[\vec{a}]$. ∎

Theorem 3.9 *Let $\mathcal{M}_1 \subseteq_{\mathrm{cf}} \mathcal{M}_2$ be a cofinal extension of models of PA^- such that $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$. If $\mathcal{M}_1$ is a model of PA then $\mathcal{M}_1 \prec \mathcal{M}_2$.*

Proof. First we prove, using the criterion of lemma 3.8, that $\mathcal{M}_1 \prec_{\Sigma_2} \mathcal{M}_2$; and then that for $n \geq 2$, if $\mathcal{M}_1 \prec_{\Sigma_n} \mathcal{M}_2$ then $\mathcal{M}_1 \prec_{\Sigma_{n+1}} \mathcal{M}_2$.

Let $\theta(\vec{x})$ be a Σ_2-formula, $\theta(\vec{x}) \equiv \exists \vec{y} \forall \vec{z} \psi(\vec{x}, \vec{y}, \vec{z})$ with $\psi \in \Delta_0$, and suppose for $\vec{a} \in \mathcal{M}_1$ that $\mathcal{M}_2 \models \theta[\vec{a}]$, so there is $\vec{b} = b_1, \ldots, b_k$ in $\mathcal{M}_2$ such that $\mathcal{M}_2 \models \forall \vec{z}\psi[\vec{a}, \vec{b}, \vec{z}]$. Now $\mathcal{M}_1 \subseteq_{\mathrm{cf}} \mathcal{M}_2$, so there is $b \in \mathcal{M}_1$ with $b_1, \ldots, b_k < b$; then $\mathcal{M}_2 \models \exists \vec{y} < b \forall \vec{z} \psi[\vec{a}, \vec{y}, \vec{z}]$. Then certainly for all $c \in \mathcal{M}_1$ we have

$$\mathcal{M}_2 \models \exists \vec{y} < b \forall \vec{z} < c \psi[\vec{a}, \vec{y}, \vec{z}]$$

This is a Δ_0-formula, so because $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$ we have

$$\mathcal{M}_1 \models \forall w \exists \vec{y} < b \forall \vec{z} < w \psi[\vec{a}, \vec{y}, \vec{z}]$$

Now we use the assumption that $\mathcal{M}_1$ is a model of PA and satisfies therefore the Collection Principle: it follows, that

$$\mathcal{M}_1 \models \exists \vec{y} < b \forall \vec{z} \psi[\vec{a}, \vec{y}, \vec{z}]$$

(since its negation $\forall \vec{y} < b \exists \vec{z} \neg \psi$ implies, by Collection, $\exists w \forall \vec{y} < b \exists \vec{z} < w \neg \psi$)
In particular, $\mathcal{M}_1 \models \exists \vec{y} \forall \vec{z} \psi[\vec{a}, \vec{y}, \vec{z}]$. By lemma 3.8 we may conclude that $\mathcal{M}_1 \prec_{\Sigma_2} \mathcal{M}_2$.

For the inductive step, now assume $\mathcal{M}_1 \prec_{\Sigma_n} \mathcal{M}_2$ for $n \geq 2$. Then since $\mathcal{M}_1$ is a model of PA and $\mathcal{M}_1 \prec_{\Sigma_2} \mathcal{M}_2$, the pairing function is a bijection from $\mathcal{M}_2^2$ to $\mathcal{M}_2$ (because this is expressed by a Π_2-formula which is true in $\mathcal{M}_1$). This has for effect that we can contract strings of quantifiers into single quantifiers, so for a Π_{n+1}-formula $\psi(\vec{x})$ we may assume it has the form $\psi \equiv \forall y \exists z \varphi(\vec{x}, y, z)$ with $\varphi \in \Pi_{n-1}$.

Suppose for $\vec{a} \in \mathcal{M}_1$ that $\mathcal{M}_1 \models \psi[\vec{a}]$. In order to show $\mathcal{M}_2 \models \psi[\vec{a}]$, we show that for each $b \in \mathcal{M}_1$, $\mathcal{M}_2 \models \forall y < b \exists z \varphi[\vec{a}, y, z]$, which suffices since $\mathcal{M}_1 \subseteq_{\mathrm{cf}} \mathcal{M}_2$.

Recall Theorem 1.9; since $\mathcal{M}_1 \models \forall y \exists z \varphi$ and $\mathcal{M}_1$ is a model of PA, by the induction axioms of PA we have

$$\mathcal{M}_1 \models \exists a, m \forall y < b \forall z (z = (a, m)_y \rightarrow \varphi[\vec{a}, y, z])$$

But this is Σ_n (check!), so

$$\mathcal{M}_2 \models \exists a, m \forall y < b \forall z (z = (a, m)_y \rightarrow \varphi[\vec{a}, y, z])$$

Since certainly $\mathcal{M}_2 \models \forall a, m \forall y \exists z (z = (a, m)_y)$ (because this is a Π_2-formula), we have that $\mathcal{M}_2 \models \forall y < b \exists z \varphi[\vec{a}, y, z]$, as desired.

We have proved: $\mathcal{M}_1 \models \psi[\vec{a}] \Rightarrow \mathcal{M}_2 \models \psi[\vec{a}]$ for every Π_{n+1}-formula $\psi(\vec{x})$ and every tuple $\vec{a}$ from $\mathcal{M}_1$; so $\mathcal{M}_2 \models \psi[\vec{a}] \Rightarrow \mathcal{M}_1 \models \psi[\vec{a}]$ for every Σ_{n+1}-formula $\psi(\vec{x})$ and every tuple $\vec{a}$ from $\mathcal{M}_1$; by lemma 3.8, we are done. ∎

The following result we need, although very easy to state, is quite deep, and we won't prove it. It is the famous Matiyasevich-Robinson-Davis-Putnam Theorem, which was used to show that Hilbert's 10th Problem cannot be solved (there is no algorithm which decides for an arbitrary polynomial $P(\vec{x})$ with integer coefficients and an arbitrary number of unknowns, whether the equation $P(\vec{x}) = 0$ has a solution in the integers).

Theorem 3.10 (MRDP Theorem) *For every Σ_1-formula $\varphi(\vec{x})$ there is a formula $\psi(\vec{x})$ of form $\exists \vec{y} \chi(\vec{x}, \vec{y})$ with χ quantifier-free, such that*

$$\mathrm{PA} \vdash \forall \vec{x} (\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}))$$

The MRDP Theorem means we can eliminate bounded quantifiers from Σ_1-formulas. The following exercise gives its relevance to Hilbert's 10th Problem.

Exercise 46. Show that for every quantifier-free $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(y,\vec{x})$ there are polynomials $P(y,\vec{x})$ and $Q(y,\vec{x})$ such that for all tuples $\vec{n}$ of natural numbers: $\mathcal{N} \models \exists y \varphi[y,\vec{n}]$ if and only if the equation $P(y,\vec{n}) = Q(y,\vec{n})$ has a solution in $\mathbb{N}$.

Corollary 3.11 *Any inclusion between models of* PA *is* Δ_0*-elementary.*

Proof. Let $\theta(\vec{x})$ be Δ_0. Since both θ and $\neg\theta$ are Σ_1, by the MRDP Theorem there are quantifier-free formulas φ and ψ such that

$$\mathrm{PA} \vdash \forall\vec{x}(\theta(\vec{x}) \leftrightarrow \exists\vec{y}\varphi(\vec{x},\vec{y}))$$
$$\mathrm{PA} \vdash \forall\vec{x}(\neg\theta(\vec{x}) \leftrightarrow \exists\vec{z}\psi(\vec{x},\vec{z}))$$

Now let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be an inclusion of models of PA. If, for $\vec{a} \in \mathcal{M}_1$, $\mathcal{M}_1 \models \theta[\vec{a}]$ then for certain $\vec{b} \in \mathcal{M}_1$, $\mathcal{M}_1 \models \varphi[\vec{a},\vec{b}]$. Since φ is quantifier-free, $\mathcal{M}_2 \models \varphi[\vec{a},\vec{b}]$ and so $\mathcal{M}_2 \models \theta[\vec{a}]$, since $\mathcal{M}_2$ is a model of PA. The argument in the other direction uses the equivalence for $\neg\theta$, and is the same. ∎

Theorem 3.12 (Gaifman's Splitting Theorem) *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be an inclusion of models of* PA. *Then there is a unique model K with $\mathcal{M}_1 \subseteq_{\mathrm{cf}} K \subseteq_e \mathcal{M}_2$. Moreover, $\mathcal{M}_1 \prec K$, so K is a model of* PA *too.*

Proof. Clearly, there is at most one K with $\mathcal{M}_1 \subseteq_{\mathrm{cf}} K \subseteq_e \mathcal{M}_2$; we have to take

$$K = \{m \in \mathcal{M}_2 \,|\, \text{for some } n \in \mathcal{M}_1, m < n\}$$

Then K is a $\mathcal{L}_{\mathrm{PA}}$-substructure of $\mathcal{M}_2$, as well as an initial segment of it, so K is a model of PA^- and $K \prec_{\Delta_0} \mathcal{M}_2$. Since $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$ by Corollary 3.11, also $\mathcal{M}_1 \prec_{\Delta_0} K$ (check this!). Theorem 3.9 now gives $\mathcal{M}_1 \prec K$. ∎

Corollary 3.13 *Every nonstandard model of* PA *has proper elementary cofinal extensions.*

Proof. Let $\mathcal{M}$ be a nonstandard model of PA. Let $\mathcal{L}'$ be $\mathcal{L}_{\mathrm{PA}}$ augmented with constants $\underline{m}$ for every $m \in \mathcal{M}$, as well as a new constant c. Let $b \in \mathcal{M}$ be nonstandard and consider the theory

$$\mathrm{Th}(\mathcal{M}) \cup \{c \neq \underline{m} \,|\, m \in \mathcal{M}\} \cup \{c < \underline{b}\}$$

By compactness, this theory has a model $\mathcal{M}'$ which is an elementary extension of $\mathcal{M}$; applying theorem 3.12 to the inclusion $\mathcal{M} \subseteq \mathcal{M}'$ gives $\mathcal{M} \subseteq_{\mathrm{cf}} K \subseteq_e \mathcal{M}'$ with $\mathcal{M} \prec K$. Moreover, $c \in K \setminus \mathcal{M}$, so the extension is proper. ∎

3.5 Prime Models and Existence of Elementary End-extensions

In this section we shall ultimately see that every model $\mathcal{M}$ of PA has a proper elementary end-extension. For *countable* $\mathcal{M}$, this is a relatively easy Omitting Types argument, given below; but the general case needs a more sophisticated approach. We shall review the theory of prime models of complete theories extending PA, and then, by a rather tricky argument, find a proper elementary end-extension of any given model $\mathcal{M}$ as a particular prime model. First, let us do the countable case. From now on, $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$ always denotes the language $\mathcal{L}_{\mathrm{PA}}$ augmented with constants from the model $\mathcal{M}$. Let c be a new constant, and consider, in the language $\mathcal{L}_{\mathrm{PA}}(\mathcal{M}) \cup \{c\}$, the theory $T_{\mathcal{M}}(c)$:

$$T_{\mathcal{M}}(c) = \{\theta \in \mathcal{L}_{\mathrm{PA}}(\mathcal{M}) \mid \mathcal{M} \models \theta\} \cup \{c > m \mid m \in \mathcal{M}\}$$

For every $a \in \mathcal{M}$, let $\Sigma_a(x)$ be the type

$$\Sigma_a(x) = \{x < a\} \cup \{x \neq b \mid b \in \mathcal{M}\}$$

Every model of $T_{\mathcal{M}}(c)$ is a proper elementary extension of $\mathcal{M}$, and it is an end-extension if and only if it omits each $\Sigma_a(x)$. Since $\mathcal{M}$ is countable, we may, by the Extended Omitting Types Theorem, conclude that there is such a model, provided we can show that $T_{\mathcal{M}}(c)$ locally omits each $\Sigma_a(x)$.

Suppose that there is an $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-formula $\varphi(u, v)$ such that:

$$\begin{array}{ll} (1) & T_{\mathcal{M}}(c) \vdash \varphi(u,c) \to u < a \\ (2) & \text{For all } b \in \mathcal{M} : T_{\mathcal{M}}(c) \vdash \varphi(u,c) \to u \neq b \end{array}$$

By definition of $T_{\mathcal{M}}(c)$, (1) implies that there is $n_1 \in \mathcal{M}$ such that

$$(3) \quad \mathcal{M} \models \forall x > n_1 \forall u(\varphi(u,x) \to u < a))$$

And similarly (2) implies that for every $b \in \mathcal{M}$ there is $n_b \in \mathcal{M}$ such that $\mathcal{M} \models \forall x > n_b \forall u(\varphi(u,x) \to u \neq b))$. By induction in $\mathcal{M}$, it follows that

$$(4) \quad \mathcal{M} \models \forall z \exists y \forall x > y \forall u(\varphi(u,x) \to u > z))$$

If n_2 is such that $\mathcal{M} \models \forall x > n_2 \forall u(\varphi(u,x) \to u > a))$, then for $n = \max(n_1, n_2)$ we have

$$\mathcal{M} \models \forall x > n \forall u \neg\varphi(u,x)$$

and therefore, $T_{\mathcal{M}}(c) \vdash \forall u \neg\varphi(u,c)$. So we see that our assumption leads to the conclusion that $\varphi(u,c)$ is inconsistent with $T_{\mathcal{M}}(c)$, which therefore locally omits $\Sigma_a(x)$.

Since the Omitting Types theorem is false for uncountable languages and for uncountably many types (see, e.g., Chang & Keisler), the general case turns out to be more complicated.

3.5.1 Prime Models

Let $\mathcal{M}$ be a model of PA and $A \subseteq \mathcal{M}$. By $K(\mathcal{M};A)$ we denote the set of elements of $\mathcal{M}$ which are definable over A. That is, those elements a for which there is a formula $\theta_a(x,u_1,\ldots,u_n)$ of $\mathcal{L}_{\mathrm{PA}}$ and elements $a_1,\ldots,a_n \in A$ such that

$$\mathcal{M} \models \forall x(\theta_a(x,a_1,\ldots,a_n) \leftrightarrow x = a)$$

Let $\mathcal{L}_{\mathrm{PA}}(A)$ the language with constants from A added, and $\mathrm{Th}(\mathcal{M})_A$ the $\mathcal{L}_{\mathrm{PA}}(A)$-theory which is true in $\mathcal{M}$.

Theorem 3.14

i) *$K(\mathcal{M};A)$ is an $\mathcal{L}_{\mathrm{PA}}(A)$-substructure of $\mathcal{M}$, and $A \subseteq \mathcal{L}_{\mathrm{PA}}(A) \prec \mathcal{M}$ as $\mathcal{L}_{\mathrm{PA}}(A)$-structures;*

ii) *For every model $\mathcal{M}'$ of $\mathrm{Th}(\mathcal{M})_A$ there is a unique $\mathcal{L}_{\mathrm{PA}}(A)$-elementary embedding from $K(\mathcal{M};A)$ into $\mathcal{M}'$;*

iii) *$K(\mathcal{M};A)$ has no proper $\mathcal{L}_{\mathrm{PA}}(A)$-elementary substructures and no nontrivial $\mathcal{L}_{\mathrm{PA}}(A)$-automorphisms.*

Proof. i) Certainly $A \subseteq K(\mathcal{M};A)$ since every $a \in A$ is defined over A by the formula $x = a$. If a and b are defined by $\mathcal{L}_{\mathrm{PA}}(A)$-formulas $\theta_a(x)$ and $\theta_b(x)$ respectively, then $a+b$ is defined by $\exists zw(\theta_a(z) \wedge \theta_b(w) \wedge x = z + w)$; similarly $a{\cdot}b$ is defined over A. So $K(\mathcal{M};A)$ is an $\mathcal{L}_{\mathrm{PA}}(A)$-substructure of $\mathcal{M}$. To see that $K(\mathcal{M};A) \prec \mathcal{M}$ we employ the Tarski-Vaught test. Let $\exists x\varphi$ be an $\mathcal{L}_{\mathrm{PA}}(A)$-sentence which is true in $\mathcal{M}$. Since $\mathcal{M}$ satisfies the least number principle, we have

$$\mathcal{M} \models \exists x(\varphi(x) \wedge \forall y < x \neg\varphi(y))$$

The formula $\varphi(x) \wedge \forall y < x\neg\varphi(y)$ now defines an element of $K(\mathcal{M};A)$ which satisfies φ, so $K(\mathcal{M};A) \models \exists x\varphi$

ii) For every $a \in K(\mathcal{M};A)$ let $\theta_a(x)$ be an $\mathcal{L}_{\mathrm{PA}}(A)$-formula defining a. For a model $\mathcal{M}'$ of $\mathrm{Th}(\mathcal{M})_A$, send a to the unique element a' of $\mathcal{M}'$ such that $\mathcal{M}' \models \theta_a(a')$. This defines a mapping $h : K(\mathcal{M};A) \to \mathcal{M}'$. This does not depend on the choices for θ_a, because if a is also defined by ζ_a, then $\mathcal{M}$ and $\mathcal{M}'$ satisfy the formula $\forall x(\theta_a(x) \leftrightarrow \zeta_a(x))$. One sees that h is an embedding of $\mathcal{L}_{\mathrm{PA}}(A)$-structures, and the proof that it is elementary, is by a similar application of the Tarski-Vaught test as in i). Finally, h must be unique with these properties, since $h(a)$ must satisfy $\theta_a(x)$.

iii) Since every $\mathcal{L}_{\mathrm{PA}}(A)$-automorphism of $K(\mathcal{M};A)$ is an $\mathcal{L}_{\mathrm{PA}}(A)$-elementary embedding, there can be at most one such by ii); so the identity function is the only one.

If $\mathcal{M}' \prec K(\mathcal{M};A)$ is a proper $\mathcal{L}_{\mathrm{PA}}(A)$-elementary substructure, by ii) there is a unique $\mathcal{L}_{\mathrm{PA}}(A)$-elementary embedding $h : K(\mathcal{M};A) \to \mathcal{M}'$. Composing with the identity gives an elementary embedding of $K(\mathcal{M};A)$ into itself. By ii), there is only one such, which is the identity. But this cannot factor through a proper subset, of course. ∎

From the proof of theorem 3.14 we see that if $\mathcal{M}'$ is a model of $\mathrm{Th}(\mathcal{M})_A$ and $A' \subseteq \mathcal{M}'$ is the set of interpretations of the constants from A, then the unique $h : K(\mathcal{M}; A) \to \mathcal{M}'$ takes values in $K(\mathcal{M}'; A')$. By symmetry, we must have that the models $K(\mathcal{M}; A)$ and $K(\mathcal{M}'; A')$ are isomorphic. Therefore, the model $K(\mathcal{M}; A)$ is determined by the theory $\mathrm{Th}(\mathcal{M})_A$, and does not depend on $\mathcal{M}$ or A.

If $A = \emptyset$, we write $K(\mathcal{M})$ for $K(\mathcal{M}; A)$. In view of the remark above, for every consistent, complete $\mathcal{L}_{\mathrm{PA}}$-theory T extending PA we have a prime model K_T which we can take to be $K(\mathcal{M})$ for any model $\mathcal{M}$ of T.

Exercise 47. This exercise recalls some notions from Model Theory. Given a complete theory T in a countable language $\mathcal{L}$, we say that an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ is *complete* in T if it is consistent with T and for any other $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$, either $T \vdash \forall x_1 \cdots x_n(\varphi(\vec{x}) \to \psi(\vec{x}))$ or
$T \vdash \forall x_1 \cdots x_n(\varphi(\vec{x}) \to \neg\psi(\vec{x}))$ (Equivalently, $T \cup \{\varphi(c_1, \ldots, c_n)\}$ is a complete $\mathcal{L} \cup \{c_1, \ldots, c_n\}$-theory, where $c_1, \ldots, c_n$ are new constants). The theory T is called *atomic* if for every $\mathcal{L}$-formula $\varphi(\vec{x})$ which is consistent with T, there is a complete formula $\psi(\vec{x})$ such that $T \vdash \forall\vec{x}(\psi(\vec{x}) \to \varphi(\vec{x}))$.

Show that every complete extension of PA is atomic.

Exercise 48. Let T be a complete, consistent extension of PA in a language $\mathcal{L}_{\mathrm{PA}} \cup C$, where C is a new set of constants. Let $\mathcal{M}$ be a model of T and $A \subseteq \mathcal{M}$ the set of interpretations of the constants from C. Assume that for $c \neq c' \in C$, $T \vdash c \neq c'$. Show that for every $\mathcal{L}_{\mathrm{PA}} \cup C$-type $\Sigma(x)$ which is consistent with T, $K(\mathcal{M}; A)$ realizes $\Sigma(x)$ if and only if T locally realizes $\Sigma(x)$.

3.5.2 Conservative Extensions and MacDowell-Specker Theorem

The MacDowell-Specker Theorem asserts what we announced as our main result for this section: every model of PA has a proper elementary end-extension. The way we shall prove it, it comes out as a corollary of another theorem.

If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ is an inclusion of models of PA, we say that $\mathcal{M}_2$ is a *conservative extension* of $\mathcal{M}_1$, if for every subset X of $\mathcal{M}_2$, if X is definable in $\mathcal{M}_2$ in parameters from $\mathcal{M}_2$ (that is: there is $\theta(x, u_1, \ldots, u_n)$ and $a_1, \ldots, a_2 \in \mathcal{M}_2$ such that $X = \{m \in \mathcal{M}_2 \,|\, \mathcal{M}_2 \models \theta(x, a_1, \ldots, a_n)\}$) then $X \cap \mathcal{M}_1$ is definable in $\mathcal{M}_1$ in parameters from $\mathcal{M}_1$.

The theorem we shall prove, is:

Theorem 3.15 *Every model of* PA *has a proper elementary conservative extension.*

Let us see that this implies what we want:

Lemma 3.16 *Every conservative extension is an end-extension.*

Proof. Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ a conservative extension; let $a \in \mathcal{M}_1, b \in \mathcal{M}_2$ and suppose $b < a$. The set $\{m \in \mathcal{M}_2 \,|\, m \leq b\}$ is clearly definable in $\mathcal{M}_2$ with parameter b, so $\{m \in \mathcal{M}_1 \,|\, m \leq b\}$ is definable in parameters from $\mathcal{M}_1$, say

$$\{m \in \mathcal{M}_1 \,|\, m \leq b\} = \{m \in \mathcal{M}_1 \,|\, \mathcal{M}_1 \models \theta(m, a_1, \ldots, a_n)\}$$

Since $a \in \mathcal{M}_1$ and $b < a$ we have $\mathcal{M}_1 \models \forall x(\theta(x, a_1, \ldots, a_n) \to x \leq a)$. By the least number principle in $\mathcal{M}_1$, there is a least $a' \in \mathcal{M}_1$ such that

$$\mathcal{M}_1 \models \forall x(\theta(x, a_1, \ldots, a_n) \to x \leq a')$$

It follows that $\mathcal{M}_1 \models \theta(a', a_1, \ldots, a_n)$, so $a' \leq b$. But if $a' < b$ then $a' + 1 \leq b$ whence $\mathcal{M}_1 \models \theta(a' + 1, a_1, \ldots, a_n)$, but of course $\mathcal{M}_1 \not\models a' + 1 \leq a'$. Therefore we must have $a' = b$, so $b \in \mathcal{M}_1$, as desired. ∎

Hence, for the record:

Corollary 3.17 (MacDowell-Specker) *Every model of* PA *has a proper elementary end-extension.*

We now embark on the proof of theorem 3.15. We introduce the abbreviation $Qx\, \varphi(x)$ for $\forall y \exists x(x > y \wedge \varphi(x))$ ("there exist unboundedly many x satisfying $\varphi(x)$").

Lemma 3.18 *Let $\mathcal{M}$ be a model of* PA*, $\varphi(x)$ an $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-formula such that $\mathcal{M} \models Qx\varphi(x)$, and $\theta(x, y)$ an arbitrary $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-formula. Then there is an $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-formula $\psi(x)$ with the properties:*

i) $\mathcal{M} \models Qx\psi(x)$

ii) $\mathcal{M} \models \forall x(\psi(x) \to \varphi(x))$

iii) $\mathcal{M} \models \forall y \neg(Qx(\psi(x) \wedge \theta(x, y)) \wedge Qx(\psi(x) \wedge \neg\theta(x, y)))$

Proof. An equivalent for item iii) is:

$$\mathcal{M} \models \forall y \exists z(\forall x > z(\psi(x) \to \theta(x, y)) \vee \forall x > z(\psi(x) \to \neg\theta(x, y)))$$

The idea of the proof is as follows. We shall construct an $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-formula $\chi(y, x)$ such that

(1) $\mathcal{M} \models \forall x(\chi(0, x) \leftrightarrow (\varphi(x) \wedge (\theta(x, 0) \leftrightarrow Qv(\varphi(v) \wedge \theta(v, 0)))))$
(2) $\mathcal{M} \models \forall yx(\chi(y+1, x) \leftrightarrow \chi(y, x) \wedge (\theta(x, y+1) \leftrightarrow Qv(\chi(y, v) \wedge \theta(v, y+1))))$

For the moment, assume that $\chi(y, x)$ has been defined. It follows, by induction in $\mathcal{M}$, that $\mathcal{M} \models \forall y Qx\chi(y, x)$; for $Qx\chi(y, x)$ implies $Qx(\chi(y, x) \wedge \theta(x, y + 1)) \vee Qx(\chi(y, x) \wedge \neg\theta(x, y + 1))$, so $Qx\chi(y + 1, x)$. We note also, that $\mathcal{M} \models \forall yx(\chi(y, x) \to \varphi(x) \wedge \forall v \leq y\chi(v, x))$.

In order to define $\psi(x)$ from $\chi(y, x)$ we use theorem 1.9. We write $(s)_i$ instead of $(a, m)_i$ as in that theorem, putting $s = j(a, m)$:

$$(s)_i = \mathrm{rm}(j_1(s), (i + 1)j_2(s) + 1)$$

Let us also write $x = \mu z\varphi(z)$ for $\varphi(x) \wedge \forall y < x \neg\varphi(y)$.

Since $\forall y Q x \chi(y,x)$ holds in $\mathcal{M}$, we have by induction on z and theorem 1.9 that the sentence

$$\forall z \exists s((s)_0 = \mu x \chi(0,x) \wedge \forall i < z((s)_{i+1} = \mu x(x > (s)_i \wedge \chi(i+1,x))))$$

is true in $\mathcal{M}$; write this as $\forall z \exists s \Phi(z,s)$. Define

$$\psi(x) \equiv \exists s(\Phi(x,s) \wedge \exists i \leq x (s)_i = x) \tag{3}$$

Then $\mathcal{M} \models Q x \psi(x)$, so statement i) of the Lemma is satisfied. Statement ii), that $\forall x(\psi(x) \to \varphi(x))$, follows from $\forall yx(\chi(y,x) \to \varphi(x))$. As to statement iii), first note that if $w \leq z \wedge \Phi(z,s) \wedge \Phi(w,t)$, then $\forall v \leq w((s)_v = (t)_v)$. So for all $z \geq y$, if $\Phi(z,s)$ then $\forall w(y \leq w \leq z \to \chi(y,(s)_w)$. So if $\Phi(y,s) \wedge \psi(x) \wedge x \geq (s)_y$ then $\theta(y,x) \leftrightarrow \theta(y,(s)_y)$, which ensures that statement iii) holds.

It remains to define the formula $\chi(y,x)$ and prove the equivalences (1) and (2). Again, we use the sequence coding $(s)_i$. Let $P(s,y)$ be the formula

$$\forall u \leq y((s)_u = 0 \leftrightarrow Qz(\varphi(z) \wedge \theta(z,u) \wedge \forall v < u(\theta(z,v) \leftrightarrow (s)_v = 0)))$$

and define $\chi(y,x)$ as

$$\exists s(P(s,y) \wedge \forall u \leq y(\theta(x,u) \leftrightarrow (s)_u = 0) \wedge \varphi(x))$$

Since $P(s,0) \leftrightarrow ((s)_0 = 0 \leftrightarrow Qz(\varphi(z) \wedge \theta(z,0)))$, we have

$$\psi(0,x) \leftrightarrow \varphi(x) \wedge (\theta(x,0) \leftrightarrow Qz(\varphi(z) \wedge \theta(z,0)))$$

so (1) holds.

For (2), first note that $P(s,y) \wedge P(t,y)$ implies $\forall u \leq y((s)_u = 0 \leftrightarrow (t)_u = 0)$; from this and the definition of $\chi(y,x)$ it follows directly that

$$\begin{aligned} P(s,y) \to \forall u \leq y \forall x(\psi(u,x) \leftrightarrow \\ \varphi(x) \wedge \forall v \leq u(\theta(x,v) \leftrightarrow (s)_v = 0)) \end{aligned} \tag{4}$$

holds. We prove the equivalence of (2):

$\to$: Suppose $\chi(y+1,x)$, so

$$P(s,y+1) \wedge \forall u \leq y+1(\theta(x,u) \leftrightarrow (s)_u = 0) \wedge \varphi(x)$$

for some s. Applying (4) with $y+1$ for y we have

$$\forall z(\chi(y+1,z) \leftrightarrow \varphi(z) \wedge \forall v \leq y+1(\theta(z,v) \leftrightarrow (s)_v = 0))$$

so $\varphi(x) \wedge (\theta(x,y+1) \leftrightarrow (s)_{y+1} = 0)$. Combining this with the definition of $P(s,y+1)$, the fact that $\chi(y,x)$ implies $\varphi(x) \wedge \forall v \leq y\chi(v,x)$, and applying (4) again (inside the part $Qz(\ldots)$), we get

$$\chi(y,x) \wedge (\theta(x,y+1) \leftrightarrow Qz(\theta(z,y+1) \wedge \chi(y,z)) \tag{5}$$

$\leftarrow$: Conversely, assume (5) and $P(s,y)$. By theorem 1.9 there is t such that $\forall u \leq y((s)_u = (t)_u$, and

$$(t)_{y+1} = 0 \leftrightarrow Qz(\varphi(z) \wedge \theta(z, y+1) \wedge \forall v \leq y(\theta(z,v) \leftrightarrow (s)_v = 0))$$

Then $P(t, y+1)$ holds. We have to show:

$$\forall u \leq y+1(\theta(x,u) \leftrightarrow (t)_u = 0) \wedge \varphi(x)$$

Since $\chi(y,x)$ we have $\varphi(x)$, and for $u \leq y$ this is clear, since $P(s,y)$. For $u = y+1$ we have:

$$\begin{aligned}
\theta(x, y+1) &\leftrightarrow Qz(\theta(z, y+1) \wedge \chi(y,z)) \\
&\leftrightarrow Qz(\varphi(z) \wedge \theta(z, y+1) \wedge \forall v \leq y \\
&\qquad (\theta(z,v) \leftrightarrow (t)_v = 0)) \\
&\leftrightarrow (t)_{y+1} = 0
\end{aligned}$$

(the first equivalence by (5); the second by (4); the third by definition of t) We have proved the equivalence (2), and hence the lemma. ∎

We finish the proof of Theorem 3.15. Fix an enumeration $\theta_0(c, \vec{y}^{(0)}), \theta_1(c, \vec{y}^{(1)}), \ldots$ of all formulas of $\mathcal{L}_{\mathrm{PA}} \cup \{c\}$ (so $\theta_i(x, \vec{y}^{(i)})$ is an $\mathcal{L}_{\mathrm{PA}}$-formula and $\vec{y}^{(i)}$ is the list of free variables of $\theta_i(c, \vec{y}^{(i)})$). We construct a sequence of $\mathcal{L}_{\mathrm{PA}}$-formulas $\varphi_0(x), \varphi_1(x), \ldots$ in one free variable x, such that $\mathcal{M} \models Qx\varphi_i(x)$ for all i, as follows. Put $\varphi_0(x) \equiv x = x$. Given $\varphi_i(x)$ such that $\mathcal{M} \models Qx\varphi_i(x)$, we apply lemma 3.18 to find $\varphi_{i+1}(x)$ such that:

$$\begin{aligned}
&\mathcal{M} \models Qx\, \varphi_{i+1}(x) \\
&\mathcal{M} \models \forall x(\varphi_{i+1}(x) \rightarrow \varphi_i(x)) \\
&\mathcal{M} \models \forall \vec{y}^{(i)} \exists z(\forall x > z(\varphi_{i+1}(x) \rightarrow \theta_i(x, \vec{y}^{(i)})) \vee \\
&\qquad \forall x > z(\varphi_{i+1}(x) \rightarrow \neg\theta_i(x, \vec{y}^{(i)})))
\end{aligned}$$

Consider the $\mathcal{L}_{\mathrm{PA}}(\mathcal{M}) \cup \{c\}$-theory T given by the axioms

$$\begin{aligned}
&\{\theta(\vec{a}) \in \mathcal{L}_{\mathrm{PA}}(\mathcal{M}) \mid \mathcal{M} \models \theta(\vec{a})\} \cup \\
&\quad \{c > a \mid a \in \mathcal{M}\} \cup \{\varphi_i(c) \mid i \in \mathbb{N}\}
\end{aligned}$$

Since every finite subset of this has an interpretation in $\mathcal{M}$, T is consistent. Let $\mathcal{M}'$ be a model of T and let $K = K(\mathcal{M}'; \mathcal{M} \cup \{c\})$. We have $\mathcal{M} \prec \mathcal{M}'$ as $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-structures, $\mathcal{M} \subseteq K$ and $K \prec \mathcal{M}'$ as $\mathcal{L}_{\mathrm{PA}}(\mathcal{M}) \cup \{c\}$-structures; it follows that $\mathcal{M} \prec K$ as $\mathcal{L}_{\mathrm{PA}}(\mathcal{M})$-structures. Also, $c \in K \setminus \mathcal{M}$, so K is a proper elementary extension of $\mathcal{M}$. We want to show that the extension $\mathcal{M} \subseteq K$ is conservative.

Suppose s subset $S \subseteq K$ is defined by $S = \{k \mid K \models \theta(k, b_1, \ldots b_n)\}$ with $b_1, \ldots, b_n \in K$. By definition of K, every b_i is defined in $\mathcal{M}'$ by a formula $\eta_i(v, a_1, \ldots, a_k, c)$ with $a_1, \ldots, a_k \in \mathcal{M}$. Now the formula

$$\exists v_1 \cdots v_n (\bigwedge_{i=1}^{n} \eta_i(v_i, y_1, \ldots, y_k, x) \wedge \theta(y_0, v_1, \ldots, v_n))$$

is an $\mathcal{L}_{\mathrm{PA}}$-formula, so occurs in our enumeration as $\theta_j(x, \vec{y}^{(j)})$, with $\vec{y}^{(j)} = y_0, \ldots, y_k$. We claim:

$$
\begin{aligned}
& d \in \mathcal{M} \cap S \Leftrightarrow \\
& \mathcal{M} \models \exists w \forall x > w(\varphi_{j+1}(x) \rightarrow \theta_j(x, d, a_1, \ldots, a_k)))
\end{aligned}
$$

so that $\mathcal{M} \cap S$ is definable in $\mathcal{M}$ over $\mathcal{M}$. Observe, that for $d \in \mathcal{M}$, $d \in S$ if and only if $K \models \theta(d, b_1, \ldots, b_1)$, if and only if $K \models \theta_j(c, d, a_1, \ldots, a_k)$.

By construction of φ_{j+1}, we have either

$$
\text{i)} \quad \mathcal{M} \models \exists w \forall x > w(\varphi_{j+1}(x) \rightarrow \theta_j(x, d, a_1, \ldots, a_k))
$$

or

$$
\text{ii)} \quad \mathcal{M} \models \exists w \forall x > w(\varphi_{j+1}(x) \rightarrow \neg\theta_j(x, d, a_1, \ldots, a_k))
$$

These are formulas with parameters in $\mathcal{M}$, so since $\mathcal{M} \prec K$, each one is satisfied in $\mathcal{M}$ if and only if it holds in K. So, i) is the case if and only if $K \models \theta_j(c, d, a_1, \ldots, a_k)$, if and only if $d \in S$, as desired.

4 Recursive Aspects of Models of PA

4.1 Partial Truth Predicates

A *truth predicate* for PA is a formula $\mathrm{Tr}(y, x)$ such that for all formulas $\varphi(v_0, \ldots, v_{k-1})$:

$$\text{(Tr)} \qquad \mathrm{PA} \vdash \forall s(\mathrm{Tr}(\ulcorner \varphi \urcorner, s) \leftrightarrow \varphi((s)_0, \ldots, (s)_{k-1}))$$

where $(s)_i$ refers, again, to sequence coding as used in the proof of lemma 3.18.

Exercise 49. Arguing in a similar way as in the proof of Tarski's theorem on the undefinability of truth (exercise on page 20), show that a truth predicate for PA cannot exist.

However, we do have *partial* truth predicates: for each $n \geq 1$ we have a Σ_n-formula $\mathrm{Tr}_n(y, x)$, such that the statement (Tr) holds for Tr_n and Σ_n-formulas φ. These partial truth predicates are very useful, and the rest of this section is devoted to their construction. In order to have a concise presentation, we shall freely employ recursion inside PA, using the fact that primitive recursive predicates and functions are Δ_1-representable in PA by formulas for which PA proves the recursive definition. We shall have to be explicit about the way we define our primitive recursive functions, though, and this takes some time.

We start by defining (in PA) a function $\mathrm{Eval}(t, s)$, such that for all terms $t(v_0, \ldots, v_{k-1})$,

$$\text{(Eval)} \qquad \mathrm{PA} \vdash \forall s(\mathrm{Eval}(\ulcorner t \urcorner, s) = t((s)_0, \ldots, (s)_{k-1})$$

For this, we need the recursion for the predicate "x is the code of a term".

Proposition 4.1 *There is a Δ_1-predicate* $\mathrm{Term}(x)$ *such that*

$$\begin{aligned} \mathrm{PA} \vdash \forall x(\mathrm{Term}(x) \leftrightarrow x = \langle 0 \rangle \vee x = \langle 1 \rangle \\ \vee \exists i < x(x = \langle \bar{2}, i \rangle \\ \vee \exists uv < x(\mathrm{Term}(u) \wedge \mathrm{Term}(v) \wedge x = \langle \bar{3}, u, v \rangle) \\ \vee \exists uv < x(\mathrm{Term}(u) \wedge \mathrm{Term}(v) \wedge x = \langle \bar{4}, u, v \rangle) \end{aligned}$$

Exercise 50. Prove Proposition 4.1.

Proposition 4.2 *There is a Δ_1-predicate* $\mathrm{Val}(y, x, z)$ *such that*

$$\begin{aligned} \mathrm{PA} \vdash \forall xyz(\mathrm{Val}(y, x, z) \leftrightarrow [(z = 0 \wedge \neg\mathrm{Term}(y)) \\ \vee (y = \langle 0 \rangle \wedge z = 0) \\ \vee (y = \langle 1 \rangle \wedge z = 1) \\ \vee \exists i < y(y = \langle \bar{2}, i \rangle \wedge z = (x)_i) \\ \vee \exists uv < y \exists ab(y = \langle \bar{3}, u, v \rangle \wedge \mathrm{Val}(u, x, a) \wedge \\ \mathrm{Val}(v, x, b) \wedge z = a + b) \\ \vee \exists uv < y \exists ab(y = \langle \bar{4}, u, v \rangle \wedge \mathrm{Val}(u, x, a) \wedge \\ \mathrm{Val}(v, x, b) \wedge z = a \cdot b) \end{aligned}$$

Exercise 51. Prove that the quantifiers $\exists ab$ in the recursion for Val can in fact be bounded. Prove proposition 4.2. Prove also:

$$\mathrm{PA} \vdash \forall yx \exists! z \mathrm{Val}(y, x, z)$$

In view of this we introduce a function symbol Eval, so that

$$\forall yx \mathrm{Val}(y, x, \mathrm{Eval}(y, x))$$

It is now easy to prove the equation (Eval), by a straightforward induction on the term t.

Exercise 52. Carry this out.

Our next step is the recursion for $\Delta_0\mathrm{Form}(x)$: "x is the code of a Δ_0-formula". We define an abbreviation: $[\exists v_k < s.u]$ stands for the term

$$\langle \overline{12}, k, \langle \bar{9}, \langle \bar{6}, \langle \bar{2}, k \rangle, s \rangle, u \rangle \rangle$$

so that for a term t and formula φ,

$$[\exists v_k < \overline{\ulcorner t \urcorner}.\overline{\ulcorner \varphi \urcorner}] = \overline{\ulcorner \exists v_k (v_k < t \wedge \varphi) \urcorner}$$

We have a similarly defined abbreviation $[\forall v_k < s.u]$. The following proposition should be obvious.

Proposition 4.3 *There is a Δ_1-predicate $\Delta_0\mathrm{Form}(x)$ such that*

$$\begin{array}{l} \mathrm{PA} \vdash \forall x(\Delta_0\mathrm{Form}(x) \leftrightarrow \exists uv < x(\mathrm{Term}(u) \wedge \mathrm{Term}(v) \wedge \\ \quad (x = \langle \bar{5}, u, v \rangle \vee x = \langle \bar{6}, u, v \rangle)) \\ \quad \vee \exists uv < x(\Delta_0\mathrm{Form}(u) \wedge \Delta_0\mathrm{Form}(v) \wedge \\ \quad (x = \langle \bar{7}, u, v \rangle \vee x = \langle \bar{8}, u, v \rangle \vee x = \langle \overline{10}, u \rangle)) \\ \quad \vee \exists uks < x(\Delta_0\mathrm{Form}(u) \wedge \mathrm{Term}(s) \wedge x = [\exists v_k < s.u]) \\ \quad \vee \exists uks < x(\Delta_0\mathrm{Form}(u) \wedge \mathrm{Term}(s) \wedge x = [\forall v_k < s.u])) \end{array}$$

Proposition 4.4 *There is a Δ_1-predicate $\mathrm{Tr}_0(y, x)$ such that for all Δ_0-formulas $\varphi(v_0, \ldots, v_{k-1})$,*

$$(\mathrm{Tr}_0) \qquad \mathrm{PA} \vdash \forall s(\mathrm{Tr}_0(\overline{\ulcorner \varphi \urcorner}, s) \leftrightarrow \varphi((s)_0, \ldots, (s)_{k-1}))$$

Proof. The function V, which for codes of formulas gives the largest index of a variable which occurs in the formula, is of course primitive recursive and provably recursive in PA. Sloppily, we define:

$$V(y) = \begin{cases} 0 & \text{if } \neg\mathrm{Form}(y) \\ k & \text{if } \mathrm{Form}(y) \wedge k = \max\{l \mid v_l \text{ occurs in } y\} \end{cases}$$

By a recursion analogous to the ones we have already seen, there is a Δ_1-predicate $\mathrm{Tr}_0(y,x)$ such that

$$\begin{aligned}
\mathrm{PA} \vdash \forall yx[&\mathrm{Tr}_0(y,x) \leftrightarrow \\
&\Delta_0\mathrm{Form}(y) \wedge \\
&[\exists uv < y(y = \langle \bar{5}, u, v\rangle \wedge \mathrm{Eval}(u,x) = \mathrm{Eval}(v,x)) \\
&\vee \exists uv < y(y = \langle \bar{6}, u, v\rangle \wedge \mathrm{Eval}(u,x) < \mathrm{Eval}(v,x)) \\
&\vee \exists uv < y(y = \langle \bar{7}, u, v\rangle \wedge \mathrm{Tr}_0(u,x) \wedge \mathrm{Tr}_0(v,x)) \\
&\vee \exists uv < y(y = \langle \bar{8}, u, v\rangle \wedge (\mathrm{Tr}_0(u,x) \vee \mathrm{Tr}_0(v,x))) \\
&\vee \exists u < y(y = \langle \overline{10}, u\rangle \wedge \neg\mathrm{Tr}_0(u,x)) \\
&\vee \exists uks < y(y = [\exists v_k < s.u] \wedge \exists i < \mathrm{Eval}(s,x)\exists w \\
&(\forall j \le V(y)(j \ne k \to (w)_j = (x)_j) \wedge (w)_k = i \wedge \mathrm{Tr}_0(u,w))) \\
&\vee \exists uks < y(y = [\forall v_k < s.u] \wedge \forall i < \mathrm{Eval}(s,x)\exists w \\
&(\forall j \le V(y)(j \ne k \to (w)_j = (x)_j) \wedge (w)_k = i \wedge \mathrm{Tr}_0(u,w)))\ ]]
\end{aligned}$$

Exercise 53. Prove that

$$\mathrm{PA} \vdash \forall xiku\exists w((w)_i = u \wedge \forall j < k(j \ne i \to (w)_j = (x)_j))$$

Prove also, that

$$\mathrm{PA} \vdash \forall yxv(\forall i \le V(y)((x)_i = (v)_i) \to (\mathrm{Tr}_0(y,x) \leftrightarrow \mathrm{Tr}_0(y,v)))$$

Using this, we see that in the recursion for Tr_0, the quantifier $\exists w$ might as well have been $\forall w$. The rest of the quantifiers are bounded, so Tr_0 is Δ_1. The statement (Tr_0) follows by induction on φ. ∎

In the final inductive definition of Tr_n, we define simultaneously formulas Tr_n and Tr_n^c that work for Σ_n and Π_n-formulas, respectively.

First, the recursions for the predicates saying that x codes a Σ_n or Π_n-formula. For clarity, we write $[\exists v_k.u]$ for $\langle \overline{12}, k, u\rangle$ and $[\forall v_k.u]$ for $\langle \overline{11}, k, u\rangle$.

We have, for each n, Δ_1-predicates $\Sigma_n\mathrm{Form}(x)$ and $\Pi_n\mathrm{Form}(x)$: let

$$\Sigma_0\mathrm{Form}(x) \equiv \Pi_0\mathrm{Form}(x) \equiv \Delta_0\mathrm{Form}(x)$$

If $\Sigma_n\mathrm{Form}(x)$ and $\Pi_n\mathrm{Form}(x)$ are defined, define $\Sigma_{n+1}\mathrm{Form}(x)$ and $\Pi_{n+1}\mathrm{Form}(x)$ recursively, so that

$$\begin{aligned}
\mathrm{PA} \vdash \Sigma_{n+1}\mathrm{Form}(x) &\leftrightarrow \Pi_n\mathrm{Form}(x) \vee \\
&\exists ku < x(x = [\exists v_k.u] \wedge \Sigma_{n+1}\mathrm{Form}(u)) \\
\mathrm{PA} \vdash \Pi_{n+1}\mathrm{Form}(x) &\leftrightarrow \Sigma_n\mathrm{Form}(x) \vee \\
&\exists ku < x(x = [\forall v_k.u] \wedge \Pi_{n+1}\mathrm{Form}(u))
\end{aligned}$$

We now come to the final definition of the predicates Tr_n and Tr_n^c. For $n = 0$, we let $\mathrm{Tr}_0^c \equiv \mathrm{Tr}_0$, which we have already defined. In the definition of Tr_{n+1} and Tr_{n+1}^c we use the function $V(y)$ defined in the proof of proposition 4.4.

Let $F_{n+1}(\sigma, j, y)$ be the formula

$$\Pi_n\mathrm{Form}((\sigma)_0) \wedge \forall i < j\exists k < (\sigma)_{i+1}((\sigma)_{i+1} = [\exists v_k.(\sigma)_i]) \wedge (\sigma)_j = y$$

From the recursion for $\Sigma_{n+1}\mathrm{Form}(y)$ one proves by well-founded induction that

$$\mathrm{PA} \vdash \forall y(\Sigma_{n+1}\mathrm{Form}(y) \leftrightarrow \exists\sigma\exists j F_{n+1}(\sigma, j, y))$$

Let $\mathrm{Tr}_{n+1}(y, x)$ be the formula

$$\begin{aligned}&\exists\sigma j(F_{n+1}(\sigma, j, y) \wedge \exists w(\mathrm{Tr}^c_n((\sigma)_0, w)\wedge\\ &\qquad \forall i \leq V(y)(\forall l < j((\sigma)_{l+1} \neq [\exists v_i.(\sigma)_l]) \to (w)_i = (x)_i)))\end{aligned}$$

Similarly, let $G_{n+1}(\sigma, j, y)$ be the formula

$$\Sigma_n\mathrm{Form}((\sigma)_0) \wedge \forall i < j\exists k < (\sigma)_{i+1}((\sigma)_{i+1} = [\forall v_k.(\sigma)_i]) \wedge (\sigma)_j = y$$

and define $\mathrm{Tr}^c_{n+1}(y, x)$ as

$$\begin{aligned}&\Pi_{n+1}\mathrm{Form}(y) \wedge \forall\sigma\forall j(G_{n+1}(\sigma, j, y) \to\\ &\qquad \forall w((\forall i \leq V(y)(\forall l < j((\sigma)_{l+1} \neq [\forall v_i.(\sigma)_l]) \to (w)_i = (x)_i) \to\\ &\qquad\qquad \mathrm{Tr}_n((\sigma)_0, w)))\end{aligned}$$

Exercise 54. Check that the predicates $\Sigma_{n+1}\mathrm{Form}$, $\Pi_{n+1}\mathrm{Form}$, F_{n+1} and G_{n+1} are Δ_1; hence by induction on n, that Tr_n is Σ_n and Tr^c_n is Π_n. Convince yourself that these formulas have the claimed property w.r.t. Σ_n-formulas and Π_n-formulas, respectively.

Our first application of the partial truth predicates Tr_n is, that "the arithmetical hierarchy does not collapse". That is, for each n there is a Σ_n-formula which is not equivalent to a Π_n-formula.

Proposition 4.5 (Kleene) *The formula* Tr_n *is, in* PA, *not equivalent to a* Π_n*-formula.*

Proof. This is similar to the Hierarchy Theorem in Recursion Theory. It is easy to define, in PA, a provably recursive function $[\cdot]$ such that $([x])_0 = x$.

Now if Tr_n were equivalent to a Π_n-formula, there would be a Σ_n-formula $\theta(v_0)$ such that

$$\mathrm{PA} \vdash \forall x(\theta(x) \leftrightarrow \neg\mathrm{Tr}(x, [x]))$$

It follows, that

$$\mathrm{PA} \vdash \theta[\overline{\ulcorner\theta\urcorner}/v_0] \leftrightarrow \mathrm{Tr}_n(\overline{\ulcorner\theta\urcorner}, [\overline{\ulcorner\theta\urcorner}]) \leftrightarrow \neg\theta[\overline{\ulcorner\theta\urcorner}/v_0]$$

which contradicts the consistency of PA. ∎

Exercise 55. Show that in fact, for no model $\mathcal{M}$ of PA, Tr_n is, in $\mathcal{M}$, equivalent to a Π_n-formula.

4.2 PA is not finitely axiomatized

In this section we apply the partial truth predicates Tr_n to show that PA, or in fact every consistent extension of PA, is not finitely axiomatized.

Let $\mathcal{M}$ be a model of PA and $A \subseteq \mathcal{M}$. By $K^n(\mathcal{M}; A)$ we mean the subset of $\mathcal{M}$ consisting of elements which are Σ_n-definable in $\mathcal{M}$ in parameters from A: those $a \in \mathcal{M}$ such that for some Σ_n-formula $\theta(x, y_1, \ldots, y_k)$ and $a_1, \ldots, a_k \in A$,

$$\mathcal{M} \models \forall x(\theta(x, a_1, \ldots, a_k) \leftrightarrow x = a)$$

Exercise 56. Show that for $n > 0$, $K^n(\mathcal{M}; A)$ is a substructure of $\mathcal{M}$ which contains A.

We have the following analogue of Theorem 3.14.

Proposition 4.6 *Let $\mathcal{M}$ be a model of* PA *and $A \subseteq \mathcal{M}$. Then for all $n \geq 1$, $K^n(\mathcal{M}; A) \prec_{\Sigma_n} \mathcal{M}$ as $\mathcal{L}_{\mathrm{PA}}(A)$-structures.*

Proof. We write K for $K^n(\mathcal{M}; A)$. Let us first show that $K \prec_{\Delta_0} \mathcal{M}$. Since K is a substructure of $\mathcal{M}$, equations between terms in parameters from K will hold in K if and only if they hold in $\mathcal{M}$. Furthermore, if $c_1, c_2 \in K$ and $c_1 < c_2$ in $\mathcal{M}$, and $\theta_1(x, \vec{a})$ and $\theta_2(y, \vec{b})$ are Σ_n-formulas defining c_1 and c_2 in parameters from A, the formula

$$\exists x \exists y(\theta_1(x, \vec{a}) \wedge \theta_2(y, \vec{b}) \wedge x + (z + 1) = y)$$

is Σ_n and defines a unique element c_3 of K for which $c_1 + (c_3 + 1) = c_2$; so $c_1 < c_2$ in K. The converse is easy, so the equivalence $K \models \varphi \Leftrightarrow \mathcal{M} \models \varphi$ holds for all quantifier-free sentences φ with parameters from K. Now suppose the equivalence holds for $\varphi \in \Delta_0$, and consider $\exists x < t\varphi$. If $\mathcal{M} \models \exists x < t(\vec{a})\varphi(x, \vec{a})$ then by the least number principle in $\mathcal{M}$,

$$\mathcal{M} \models \exists x(x < t(\vec{a}) \wedge \varphi(x, \vec{a}) \wedge \forall y < x \neg\varphi(y, \vec{a}))$$

This formula contains parameters from K. Replacing those by their Σ_n-definitions we get a Σ_n-formula with parameters in A, defining an element c of K; then

$$K \models c < t(\vec{a}) \wedge \varphi(c, \vec{a}) \wedge \forall y < c \neg\varphi(y, \vec{a})$$

by the assumption on φ and what we have proved about quantifier-free formulas, so $K \models \exists x < t(\vec{a})\varphi(x, \vec{a})$. The converse is, again, easy, so $K \prec_{\Delta_0} \mathcal{M}$.

We now prove for $0 \leq k < n$ that $K \prec_{\Sigma_k} \mathcal{M}$ implies $K \prec_{\Sigma_{k+1}} \mathcal{M}$. Since the bijection $j^m : \mathcal{M}^m \to \mathcal{M}$ is Δ_0-definable and has Δ_0-definable inverses j^m_i $(1 \leq i \leq m)$, it restricts to a bijection $K^m \to K$; hence for a Σ_{k+1}-formula φ we may assume that $\varphi \equiv \exists y \psi$ with $\psi \in \Pi_k$. If $\mathcal{M} \models \varphi$ then again by LNP, $\mathcal{M} \models \exists y(\psi(y) \wedge \forall w < y \neg\psi(w))$. This formula contains parameters from K; replacing those by their Σ_n-definitions we get

$$\mathcal{M} \models \exists y \exists v_1 \cdots v_r(\bigwedge_{i=1}^{r} \theta_j(v_j) \wedge \psi(y, \vec{v}) \wedge \forall w < y \neg\psi(w, \vec{v}))$$

The part following $\exists y$ is Σ_n in parameters from A so defines an element c of K. Since $K \prec_{\Sigma_k} \mathcal{M}$, $K \models \psi(c)$, and hence $K \models \varphi$. Using proposition 3.8, we conclude that $K \prec_{\Sigma_{k+1}} \mathcal{M}$, which concludes the induction step and therefore the proof. ∎

Proposition 4.7 *Let $\mathcal{M}$ be a model of* PA*, A a finite subset of $\mathcal{M}$ and $n \geq 1$. If $K^n(\mathcal{M}; A)$ contains nonstandard elements, it is not a model of* PA*.*

Proof. Since A is finite, $A = \{a_1, \ldots, a_k\}$ for some k. There is, in $K = K^n(\mathcal{M}; A)$, a function $c \mapsto [\vec{a}, c]$ where $[\vec{a}, c]$ is such that

$$\forall i < k(([\vec{a}, c])_i = a_{i+1} \wedge ([\vec{a}, c])_k = c)$$

(This is Σ_1-definable in $\mathcal{M}$, and $K \prec_{\Sigma_n} \mathcal{M}$) Since every $c \in K$ is Σ_n-definable in $a_1, \ldots, a_k$, there is for each $c \in K$ an $e \in \mathbb{N}$ such that

$$\mathcal{M} \models \mathrm{Tr}_n(e, [\vec{a}, c]) \wedge \forall y(\mathrm{Tr}_n(e, [\vec{a}, y]) \rightarrow y = c)$$

This is a conjunction of a Σ_n and a Π_n-formula, so it holds in K too. Therefore, for each nonstandard $d \in K$ we have

$$K \models \forall c \exists e < d(\mathrm{Tr}_n(e, [\vec{a}, c]) \wedge \forall y(\mathrm{Tr}_n(e, [\vec{a}, y]) \rightarrow y = c))$$

Were K a model of PA, it would satisfy the Underspill Principle; then there would be a *standard* d for which this formula would hold. But it is not hard to see that in that case, K would be finite. This is impossible for models of PA. ∎

Exercise 57. Show that even $K^1(\mathcal{M}; \emptyset)$ may contain nonstandard elements.

Theorem 4.8 (Ryll-Nardzewski) *No consistent extension of* PA *is finitely axiomatized.*

Proof. Suppose T is a finitely axiomatized, consistent extension of PA. Let $\mathcal{M}$ be a nonstandard model of T and pick $a \in \mathcal{M}$ nonstandard. Since T is finitely axiomatized, all axioms of T are Σ_n for some n. But then $K^n(\mathcal{M}; \{a\}) \prec_{\Sigma_n} \mathcal{M}$, so $K^n(\mathcal{M}; \{a\})$, containing the nonstandard element a, is a model of T. This contradicts proposition 4.7. ∎

4.3 Coded Sets

An important tool for the study of models of PA is the theory of *coded sets*. Let $\mathcal{M}$ be a model of PA. A subset $S \subseteq \mathbb{N}$ is said to be *coded in $\mathcal{M}$* if there is $c \in \mathcal{M}$ such that

$$S = \{n \in \mathbb{N} \mid \mathcal{M} \models (c)_n = 0\}$$

For each $S \subseteq \mathbb{N}$, let $p_S(x)$ be the type $\{(x)_i = 0 \mid i \in S\} \cup \{(x)_i \neq 0 \mid i \notin S\}$. So S is coded in $\mathcal{M}$ if and only if $\mathcal{M}$ realizes p_S.

We call $\{S \subseteq \mathbb{N} \mid S \text{ is coded in } \mathcal{M}\}$ the *standard system* of $\mathcal{M}$, and denote it by $\mathrm{SSy}(\mathcal{M})$.

Clearly, for the standard model $\mathcal{N}$, $\mathrm{SSy}(\mathcal{N})$ consists of precisely the finite subsets of $\mathbb{N}$, but for nonstandard models $\mathcal{M}$, $\mathrm{SSy}(\mathcal{M})$ turns out to have interesting structure.

Proposition 4.9 *For nonstandard $\mathcal{M}$,* SSy($\mathcal{M}$) *contains every recursive subset of* $\mathbb{N}$.

Proof. Let $S \subseteq \mathbb{N}$ be recursive. By theorem 1.14, there is a Σ_1-formula $\theta(x)$ such that:

$$n \in S \Rightarrow \mathrm{PA} \vdash \theta(\overline{n})$$
$$n \notin S \Rightarrow \mathrm{PA} \vdash \neg\theta(\overline{n})$$

In $\mathcal{M}$, the formula $\exists x \forall i < y((x)_i = 0 \leftrightarrow \theta(i))$ is true for every standard y. By Overspill, there is a nonstandard c for which it holds. Since $\mathcal{M}$ is a model of PA, we have

$$n \in S \Leftrightarrow \mathcal{M} \models (c)_n = 0$$

■

The following converse shows that the property of being coded in every nonstandard model is in fact equivalent to being recursive:

Proposition 4.10 *For every nonrecursive set S there is a nonstandard model $\mathcal{M}$ in which S is not coded.*

Proof. Let T be the theory $\mathrm{PA} \cup \{c > \overline{n} \mid n \in \mathbb{N}\}$. We wish to find a model of T which omits p_S. By the Omitting Types theorem, it suffices to show that T locally omits p_S. Suppose for the contrary that $\varphi(c, x)$ is a formula, consistent with T, such that for all $i \in \mathbb{N}$:

$$i \in S \Rightarrow T \vdash \forall x(\varphi(c,x) \rightarrow (x)_i = 0)$$
$$i \notin S \Rightarrow T \vdash \forall x(\varphi(c,x) \rightarrow (x)_i \neq 0)$$

It follows that, in fact,

$$i \in S \Leftrightarrow T \vdash \forall x(\varphi(c,x) \rightarrow (x)_i = 0)$$
$$i \notin S \Leftrightarrow T \vdash \forall x(\varphi(c,x) \rightarrow (x)_i \neq 0)$$

since $\varphi(c, x)$ is consistent with T. Therefore to decide whether $i \in S$, we can look for the shortest proof in T (which is a recursively axiomatized theory) of either $\forall x(\varphi(c,x) \rightarrow (x)_i = 0)$ or $\forall x(\varphi(c,x) \rightarrow (x)_i \neq 0)$. So S is recursive after all. ■

Our next theorem says that no standard system can consist of exactly the recursive sets.

Proposition 4.11 *For every nonstandard model $\mathcal{M}$ there is a nonrecursive set which is coded in $\mathcal{M}$.*

Proof. By a similar Overspill argument as in the proof of 4.9, there is a nonstandard $c \in \mathcal{M}$ such that for all $i \in \mathbb{N}$,

$$\mathcal{M} \models (c)_i = 0 \leftrightarrow \Pi_1\mathrm{Form}(c) \wedge V(c) = 0 \wedge \mathrm{Tr}_1^c(i, [0])$$

so the set S coded by c is the set of codes of Π_1-formulas $\varphi(v_0)$ with at most v_0 free, such that $\varphi(0)$ is true in $\mathcal{M}$. Were S recursive, the theory

$$T = \mathrm{PA} \cup \{\varphi \,|\, \ulcorner\varphi\urcorner \in S\} \cup \{\neg\varphi \,|\, \varphi \in \Pi_1 \wedge V(\ulcorner\varphi\urcorner) = 0 \wedge \ulcorner\varphi\urcorner \notin S\}$$

would be a consistent, recursively axiomatized extension of PA and by Gödel's First Incompleteness Theorem there is a Π_1-sentence ψ which is independent of T; but this is impossible since either $\ulcorner\psi\urcorner \in S$ or $\neg\psi \in T$. ∎

The following proposition characterizes $\mathrm{SSy}(\mathcal{M})$ in terms of the $\mathcal{M}$-definable subsets of $\mathbb{N}$:

Proposition 4.12 *Let $\mathcal{M}$ be a nonstandard model of* PA. *Then $S \in \mathrm{SSy}(\mathcal{M})$ if and only if for some formula $\varphi(x, y_1, \ldots, y_k)$ and parameters $a_1, \ldots, a_k \in \mathcal{M}$:*

$$S = \{n \in \mathbb{N} \,|\, \mathcal{M} \models \varphi(n, a_1, \ldots, a_k)\}$$

Proof. Clearly, if S is coded by $c \in \mathcal{M}$, the formula $(c)_x = 0$ defines S in the parameter c. The converse uses a similar Overspill argument as in the proof of proposition 4.9. For any standard x,

$$\mathcal{M} \models \exists y \forall i < x((y)_i = 0 \leftrightarrow \varphi(i, a_1, \ldots, a_k))$$

so by Overspill this holds for some nonstandard $b \in \mathcal{M}$; but then for $n \in \mathbb{N}$ we have $\mathcal{M} \models \varphi(n, a_1, \ldots, a_k)$ if and only if $\mathcal{M} \models (b)_n = 0$, so the set $\{n \in \mathbb{N} \,|\, \mathcal{M} \models \varphi(n, a_1, \ldots, a_k)\}$ is coded in $\mathcal{M}$. ∎

Exercise 58.

a) If $\mathcal{M}_1 \prec_{\Delta_0} \mathcal{M}_2$ then $\mathrm{SSy}(\mathcal{M}_1) \subseteq \mathrm{SSy}(\mathcal{M}_2)$;

b) if $\mathcal{M}_1 \subseteq_e \mathcal{M}_2$ and $\mathcal{M}_1$ is nonstandard, then $\mathrm{SSy}(\mathcal{M}_1) = \mathrm{SSy}(\mathcal{M}_2)$.

Exercise 59. Let $\mathcal{M}$ be a nonstandard model of PA. Prove that if $S \in \mathrm{SSy}(\mathcal{M})$, there is $a \in \mathcal{M}$ such that $n \in S$ iff $\mathcal{M} \models p_n | a$, where p_n is the n-th prime number.

The following famous theorem applies proposition 4.11. To some extent, it explains why it is hard to give "concrete" nonstandard models of PA. It asserts that "nonstandard models cannot be recursive". A countable model of PA is called *recursive* if it is of the form $(\mathbb{N}; \oplus, \otimes, \prec, n_0, n_1)$ with $\oplus, \otimes$ recursive functions and $\prec$ a recursive relation.

Theorem 4.13 (Tennenbaum) *No countable nonstandard model of* PA *is recursive.*

Proof. Let $\mathcal{M} = (\mathbb{N}; \oplus, \otimes, \prec, n_0, n_1)$ be a countable nonstandard model. We show that $\oplus$ is not recursive.

By proposition 4.11, $\mathcal{M}$ codes a nonrecursive set S; and by the exercise above we may assume that for some $a \in \mathcal{M}$, $S = \{n \in \mathbb{N} \,|\, \mathcal{M} \models p_n | a\}$. The function $n \mapsto p_n$ is recursive, and so $\mathcal{M} \models p_{\bar{n}} = \overline{p_n}$, which is

$$\underbrace{n_1 \oplus \cdots \oplus n_1}_{p_n \text{ times}}$$

If $\mathcal{M}$ is a model of PA, it satisfies division with remainder, so for each n there are $k \in \mathbb{N}$ and $i < p_n$, such that

$$a = \underbrace{k \oplus \cdots \oplus k}_{p_n \text{ times}} \oplus \underbrace{n_1 \oplus \cdots \oplus n_1}_{i \text{ times}}$$

Were $\oplus$ recursive, we could, recursively in n, find k and i (simply by enumerating and computing the terms in question) and hence, by checking whether $i = 0$, decide the question $n \in S$?, so S is recursive; contradiction. ∎

Exercise 60. If $a \in \mathcal{M}$ is such that $S = \{n \in \mathbb{N} \mid \mathcal{M} \models p_n | a\}$, then $b = 2^a$ satisfies $S = \{n \in \mathbb{N} \mid \mathcal{M} \models \exists x(x^{p_n} = b)\}$. Use this for an alternative proof of theorem 4.13, now showing that $\otimes$ is not recursive.

Since the proof of theorem 4.13 (and the exercise you have just done) in fact shows that for any countable model $\mathcal{M} = (\mathbb{N}; \oplus, \otimes, \prec, n_0, n_1)$, every set $S \in \mathrm{SSy}(\mathcal{M})$ is recursive in each of $\oplus, \otimes$, we have the following corollary, stated as exercise:

Exercise 61. Let $\mathcal{M} = (\mathbb{N}; \oplus, \otimes, \prec, n_0, n_1)$ be a countable nonstandard model of PA. If $\mathcal{N} \prec \mathcal{M}$, then neither of $\oplus, \otimes$ is arithmetical.

4.4 Scott sets; Theorems of Scott and Friedman

A *Scott set* (or completion closed, or c-closed set) is a subset $\mathcal{X}$ of $\mathcal{P}(\mathbb{N})$ such that the following conditions hold:

i) $\emptyset \in \mathcal{X}$ and $\mathcal{X}$ is closed under binary intersections and complements;

ii) $\mathcal{X}$ is closed under 'recursive in': if $Y \in \mathcal{X}$ and $X \leq_T Y$, then $X \in \mathcal{X}$;

iii) if $\mathcal{X}$ contains an infinite binary tree T, then $\mathcal{X}$ contains an infinite path in T.

To explain requirement iii): here we consider every natural number as the code of a unique finite sequence of natural numbers, as in section 2.1. We write $x \sqsubseteq y$ if $\mathrm{lh}(x) \leq \mathrm{lh}(y) \wedge \forall i < \mathrm{lh}(x)((x)_i = (y)_i)$. A subset T of $\mathbb{N}$ is a *binary tree* if $\forall x \in T \forall i < \mathrm{lh}(x)((x)_i \leq 1)$ and $\forall xy(y \in T \wedge x \sqsubseteq y \rightarrow x \in T)$.

X is a *branch* of T if X is a subtree of T and $\forall xy \in X(x \sqsubseteq y \vee y \sqsubseteq x)$.

Exercise 62. Show the following consequence of the definition of Scott sets: if $X_1, \ldots, X_n$ are elements of a Scott set $\mathcal{X}$ and Y is recursive in $X_1, \ldots, X_n$, then $Y \in \mathcal{X}$.

König's Lemma says that every infinite binary tree has an infinite branch. One defines an infinite sequence of elements x_n of T, such that $\mathrm{lh}(x_n) = n$ and $\{y \in T \mid x_n \subseteq y\}$ is infinite: $x_0 = \langle\rangle$, and if x_n is defined satisfying the requirements, then let $x_{n+1} = x_n * \langle 0 \rangle$ if $\{y \in T \mid x_n * \langle 0 \rangle \sqsubseteq y\}$ is infinite; otherwise, let $x_{n+1} = x_n * \langle 1 \rangle$.

This result fails if one relativizes everything to recursive sets:

Lemma 4.14 (Kleene) *There is an infinite, primitive recursive binary tree which does not have a recursive infinite branch. Therefore every Scott set contains nonrecursive sets.*

Proof. Recursion theory tells us that there are infinite partial recursive functions, taking values in $\{0,1\}$, which cannot be extended to total recursive functions (e.g., the function $x \mapsto \mathrm{sg}(\{x\}(x))$ is such a function). Let f be the code of such a function and let

$$T = \{x \mid \forall i < \mathrm{lh}(x)((x)_i \leq 1 \wedge \forall u < \mathrm{lh}(x)(T(f,i,u) \to U(u) = (x)_i))\}$$

T is primitive recursive and infinite, since the function coded by f is infinite; but every infinite branch through T is a total function $\mathbb{N} \to \{0,1\}$ which extends the function coded by f, and is therefore nonrecursive.

T is in every Scott set, because $T \leq_T \emptyset$, so by requirement iii) of Scott sets, every Scott set contains a nonrecursive set. ■

Scott sets are intimately related to standard systems of nonstandard models of PA.

Proposition 4.15 *Let $\mathcal{M}$ be a nonstandard model of* PA. *Then* $\mathrm{SSy}(\mathcal{M})$ *is a Scott set.*

Proof. We check the conditions for a Scott set.

i): Since PA $\vdash \forall x \exists z \forall i < x((z)_i \neq 0)$, there is $d \in \mathcal{M}$ such that $\mathcal{M} \models (d)_i \neq 0$ for all standard i; so d codes the empty set.

If b codes S and c codes T then there is (using Overspill) a d such that for all standard i, $\mathcal{M} \models (d)_i = (b)_i^2 + (c)_i^2$; so d codes $S \cap T$. The case of complement is left to you.

ii): Suppose Y is coded by b and $X \leq_T Y$. One can show, in a similar way as we showed the representability of recursive functions, that there is a Σ_1-formula $\varphi(v_0, v_1)$ such that

$$X = \{n \in \mathbb{N} \mid \mathcal{M} \models \varphi(\overline{n}, b)\}$$

So X is parametrically definable in $\mathcal{M}$, hence in $\mathrm{SSy}(\mathcal{M})$ by 4.12.

iii): Suppose T is an infinite binary tree, coded by $b \in \mathcal{M}$. Then for all standard m,

$$\mathcal{M} \models \exists x \forall i < m(\mathrm{lh}((x)_i) = i \wedge \forall j < i((x)_j \sqsubseteq (x)_i) \wedge (b)_{(x)_i} = 0)$$

(I apologize for the use of the same notation for two different ways of coding, in the same formula!)

By Overspill, there is a nonstandard m satisfying this formula; but then for any x doing it for m, x codes an infinite path in T. ■

For the following lemma, we need the notion of a *recursive language*. A first order language $\mathcal{L}$ is *recursive* if there are recursive subsets $R_{\mathcal{L}}$, $F_{\mathcal{L}}$ and $C_{\mathcal{L}}$ of $\mathbb{N}$, bijections between $R_{\mathcal{L}}$ and the set of relation symbols of $\mathcal{L}$, $F_{\mathcal{L}}$ and the set of function symbols of $\mathcal{L}$, and $C_{\mathcal{L}}$ and the set of constants of $\mathcal{L}$, such that the functions $\mathrm{ar}_R : R_\mathcal{L} \to \mathbb{N}$ and $\mathrm{ar}_F : F_\mathcal{L} \to \mathbb{N}$, which give, modulo these bijections, the arity of a relation and function symbol, are recursive.

Don't get confused: *all* interesting languages are recursive. The point is, that we have, just as for $\mathcal{L}_{\mathrm{PA}}$, an effective coding of all $\mathcal{L}$-formulas, sentences, proofs ...

Let $\mathcal{L}$ be a recursive language. By this effective coding, we can say that $X \subseteq \mathbb{N}$ *codes an* $\mathcal{L}$*-theory* T: for some axiomatization A of T, $X = \{\ulcorner\varphi\urcorner \mid \varphi \in A\}$. Suppose X codes the theory T. We have, just as is section 2.2, a predicate $\mathrm{Prf}_T(x, y)$: x codes a proof of the formula coded by y, and all undischarged assumptions of this proof have codes in X. Clearly, the predicate $\mathrm{Prf}_T(x, y)$ is recursive in X.

Lemma 4.16 *Let T be a consistent theory in a recursive language $\mathcal{L}$, and $\mathcal{X}$ a Scott set. If T is coded by some $X \in \mathcal{X}$, then there is a complete consistent extension of T coded by some $X' \in \mathcal{X}$.*

Proof. Fix an effective enumeration $\phi_0, \phi_1, \ldots$ of all $\mathcal{L}$-sentences.

With every finite 01-sequence x we associate a sentence ϕ_x: if $x = \langle\rangle$ then $\phi_x = \exists v(v = v)$, and if $\mathrm{lh}(x) = n + 1$ then $\phi_x = \phi_{x'} \wedge \phi_n$ if $x = x' * \langle 0\rangle$, and $\phi_x = \phi_{x'} \wedge \neg\phi_n$ if $x = x' * \langle 1\rangle$. The map $x \mapsto \ulcorner\phi_x\urcorner$ is clearly recursive. Let Y be the binary tree

$$\{x \mid \forall i < \mathrm{lh}(x)((x)_i \leq 1) \wedge \forall k < \mathrm{lh}(x)\neg\mathrm{Prf}_T(k, \ulcorner\neg\phi_x\urcorner)\}$$

Since T is consistent, Y is infinite; moreover, Y is recursive in X. So $Y \in \mathcal{X}$. Since $\mathcal{X}$ is a Scott set, $\mathcal{X}$ contains an infinite path P through Y. But then $\{\phi_x \mid x \in P\}$ axiomatizes a complete consistent extension of T, and $X' = \{\ulcorner\phi_x\urcorner \mid x \in P\}$ is recursive in P, so an element of $\mathcal{X}$. ∎

Theorem 4.17 (Scott) *Let $\mathcal{X}$ be a countable Scott set. Then $\mathcal{X} = \mathrm{SSy}(\mathcal{M})$ for some model $\mathcal{M}$ of* PA.

Proof. Enumerate $\mathcal{X}$ as $X_0, X_1, \ldots$

Fix a set $C = \{c_0, c_1, \ldots\}$ of new constants. Let $\mathcal{L}_n$ be the language $\mathcal{L}_{\mathrm{PA}} \cup \{c_0, \ldots, c_{n-1}\}$. Every $\mathcal{L}_n$ is recursive. Let $\mathcal{L} = \bigcup_n \mathcal{L}_n$. We build a complete $\mathcal{L}$-theory T in stages.

Stage 0. Since $\mathcal{L}_{\mathrm{PA}}$ is recursive and PA a recursively axiomatized theory, hence coded by an element of $\mathcal{X}$, we apply Lemma 4.16 to pick a complete consistent extension T_0 of PA in $\mathcal{L}_{\mathrm{PA}}$, which is coded by some element of $\mathcal{X}$.

Stage $2n + 1$. Let

$$T_{2n+1} = T_{2n} \cup \{(c_n)_{\bar{m}} = 0 \mid m \in X_n\} \cup \{(c_n)_{\bar{m}} \neq 0 \mid m \notin X_n\}$$

So T_{2n+1} makes sure that c_n codes X_n. Note that T_{2n+1} is recursive in T_{2n} and X_n, hence in $\mathcal{X}$.

Stage $2n+2$. Since T_{2n+1} is coded in $\mathcal{X}$, we apply Lemma 4.16 again, to obtain a complete consistent extension of T_{2n+1} in $\mathcal{L}_{n+1}$, which is coded in $\mathcal{X}$. We let this be T_{2n+2}.

Let $T = \bigcup_n T_n$. Then T is consistent since every T_n is, and T is a complete $\mathcal{L}$-theory since every $\mathcal{L}$-sentence is already an $\mathcal{L}_n$-sentence for some n, so provable or refutable in T_{2n+2}.

Let $\mathcal{M}$ be a model of T and $A \subseteq \mathcal{M}$ be the set of interpretations of the constants from C. Let $\mathcal{K} = K(\mathcal{M}; A)$. $\mathcal{K}$ is a model of T, hence of PA, and we claim that $\mathcal{X} = \mathrm{SSy}(\mathcal{K})$.

Since $c_n^{\mathcal{M}} \in \mathcal{K}$ and $c_n^{\mathcal{M}}$ codes X_n, clearly $\mathcal{X} \subseteq \mathrm{SSy}(\mathcal{K})$. For the converse, using 4.12, let $X \in \mathrm{SSy}(\mathcal{K})$ so for some $\varphi(x, k_1, \ldots, k_r)$,

$$X = \{n \in \mathbb{N} \mid \mathcal{K} \models \varphi(\bar{n}, k_1, \ldots, k_r)\}$$

Here the $k_1, \ldots, k_r$ are parameters from $\mathcal{K}$, so they are $\mathcal{M}$-definable in elements from A. Replacing the k_i by their definitions and reminding ourselves that $\mathcal{M}$ models the complete theory T, we see that there is an $\mathcal{L}$-formula $\varphi^*(v, c_0, \ldots, c_m)$ such that

$$X = \{n \in \mathbb{N} \mid T \vdash \varphi^*(\bar{n}, c_0, \ldots, c_m)\}$$

But $T \vdash \varphi^*(\bar{n}, c_0, \ldots, c_m)$ if and only if $T_{2m+2} \vdash \varphi^*(\bar{n}, c_0, \ldots, c_m)$. We conclude that X is recursive in T_{2m+2} (not just r.e., since T_{2m+2} is complete), which is coded in $\mathcal{X}$; hence $X \in \mathcal{X}$ since $\mathcal{X}$ is a Scott set. ∎

It is possible to strengthen theorem 4.17 to Scott sets of cardinality at most $\aleph_1$. The consequence is:

Corollary 4.18 *If the Continuum Hypothesis holds, then for every $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$: $\mathcal{X}$ is a Scott set if and only if $\mathcal{X} = \mathrm{SSy}(\mathcal{M})$ for some nonstandard model $\mathcal{M}$ of* PA.

But as far as I know, it is still an open problem whether the Continuum Hypothesis can be eliminated from this result.

The following lemma is another application of the partial truth predicates Tr_n. We shall need it for the proof of Friedman's Theorem that every countable nonstandard model of PA is isomorphic to a proper initial segment of itself. But the Lemma is interesting in its own right. It states a *saturation* property for nonstandard models of PA.

Lemma 4.19 *Let $\mathcal{M}$ be a nonstandard model of* PA.

a) *For any n-tuple $a_0, \ldots, a_{n-1}$ of elements of $\mathcal{M}$, the set*

$$\{\ulcorner\theta(v_0, \ldots, v_{n-1})\urcorner \mid \theta \in \Sigma_k, \mathcal{M} \models \theta(a_0, \ldots, a_{n-1})\}$$

is in $\mathrm{SSy}(\mathcal{M})$*;*

b) *for any type $\Theta(v_0, \ldots, v_{n+m-1})$ consisting of Σ_k-formulas, and any m-tuple $b_0, \ldots, b_{m-1} \in \mathcal{M}$, if $\{\ulcorner\theta\urcorner \mid \theta \in \Theta\} \in \mathrm{SSy}(\mathcal{M})$ and the type*

$$\{\theta(v_0, \ldots, v_{n-1}, b_0, \ldots, b_{m-1}) \mid \theta \in \Theta\}$$

is consistent with $\mathcal{M}$, it is realized in $\mathcal{M}$.

The same results hold with Π_k *instead of* Σ_k*.*

Proof. a) We have for $\theta(v_0, \ldots, v_{n-1}) \in \Sigma_k$:

$$\mathcal{M} \models \theta(a_0, \ldots, a_{n-1}) \Leftrightarrow \mathcal{M} \models \mathrm{Tr}_k(\overline{\ulcorner\theta\urcorner}, [a_0, \ldots, a_{n-1}])$$

so the statement follows from proposition 4.12.

b) Let $d \in \mathcal{M}$ code the set $\{\ulcorner\theta\urcorner \,|\, \theta \in \Theta\}$. Let $x \mapsto [x, \vec{b}]$ be a definable function such that

$$\forall i < n(([x, \vec{b}])_i = (x)_i) \wedge \forall i < n + m(n \leq i \rightarrow ([x, \vec{b}])_i = b_{i-n})$$

Then if $\{\theta(v_0, \ldots, v_{n-1}, b_0, \ldots, b_{m-1}) \,|\, \theta \in \Theta\}$ is consistent with $\mathcal{M}$, we have for each standard number y, that

$$\exists x \forall i < y((d)_i = 0 \rightarrow \mathrm{Tr}_k(i, [x, \vec{b}]))$$

is true in $\mathcal{M}$. By Overspill, there is a nonstandard y for which this sentence is true. Suppose $x \in \mathcal{M}$ satisfies this for nonstandard y. Then for $a_i = (x)_i$ we have

$$\mathcal{M} \models \theta(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1})$$

for all $\theta \in \Theta$.

The statements for Π_k follow simply from replacing Tr_k by Tr_k^c. ∎

Theorem 4.20 *Let* $\mathcal{M}, \mathcal{M}'$ *be countable nonstandard models of* PA*. Then the following two statements are equivalent:*

i) $\mathcal{M}$ *is isomorphic to an initial segment of* $\mathcal{M}'$

ii) $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{M}')$ *and* $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}')$

where $\mathrm{Th}_{\Sigma_1}(\mathcal{M})$ *is the set of* Σ_1*-sentences true in* $\mathcal{M}$*.*

Proof. We do the implication ii) $\Rightarrow$ i), leaving the other direction as an exercise.

Suppose $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{M}')$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}')$. We are going to construct an isomorphism between $\mathcal{M}$ and an initial segment of $\mathcal{M}'$ by a back-and-forth construction.

Fix enumerations $\alpha = (a'_0, a'_1, \ldots)$ of $\mathcal{M}$ and $\beta = (b'_0, b'_1, \ldots)$ of $\mathcal{M}'$. At each stage n, we assume we have defined a partial embedding

$$\{a_0, \ldots, a_{i_n - 1}\} \rightarrow \{b_0, \ldots, b_{i_n - 1}\}$$

of $\mathcal{M}$ into $\mathcal{M}'$, satisfying

$$(*) \qquad \mathrm{Th}_{\Sigma_1}(\mathcal{M}, a_0, \ldots, a_{i_n - 1}) \subseteq \mathrm{Th}(\mathcal{M}', b_0, \ldots, b_{i_n - 1})$$

For $n = 0$ we let $i_0 = 0$, and we use the assumption that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}')$.

Now suppose $(a_0, \ldots, a_{i_n - 1}) \rightarrow (b_0, \ldots, b_{i_n - 1})$ is defined, satisfying $(*)$. Let a_{i_n} be the first a' in the enumeration α that is not among $a_0, \ldots, a_{i_n - 1}$, and consider the type

$$\tau_n = \{\theta(v_{i_n}, v_0, \ldots, v_{i_n - 1}) \in \Sigma_1 \,|\, \mathcal{M} \models \theta(a_{i_n}, a_0, \ldots, a_{i_n - 1})\}$$

By Lemma 4.19 a), τ_n is coded in $\mathrm{SSy}(\mathcal{M})$, hence also in $\mathrm{SSy}(\mathcal{M}')$. Moreover, the type $\{\theta(v_{i_n}, b_0, \ldots, b_{i_n-1}) \mid \theta \in \tau_n\}$ is consistent with $\mathcal{M}'$ since for any finite $\theta_1, \ldots, \theta_r \in \tau_n$ we have

$$\exists v_{i_n}(\bigwedge_{j=1}^{r} \theta_j(v_{i_n}, a_0, \ldots, a_{i_n-1})) \in \mathrm{Th}_{\Sigma_1}(\mathcal{M}, a_0, \ldots, a_{i_n-1})$$

so by $(*)$, $\exists v_{i_n}(\bigwedge_{j=1}^{r} \theta_j(v_{i_n}, b_0, \ldots, b_{i_n-1}))$ holds in $\mathcal{M}'$.

By Lemma 4.19 b), $\{\theta(v_{i_n}, b_0, \ldots, b_{i_n-1}) \mid \theta \in \tau_n\}$ is realized by some $b_{i_n} \in \mathcal{M}'$. Clearly now,

$$\mathrm{Th}_{\Sigma_1}(\mathcal{M}, a_0, \ldots, a_{i_n}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}', b_0, \ldots, b_{i_n})$$

Now, if there is no $b \in \mathcal{M}' \setminus \{b_0, \ldots, b_{i_n}\}$ such that $b < b_k$ for some $k \leq i_n$, we put $i_{n+1} = i_n + 1$ and we proceed to the next stage.

Otherwise, we pick the first such b in the enumeration β, fix k, and consider the type

$$\sigma_n = \{\theta(v_{i_n+1}, v_0, \ldots, v_{i_n}) \in \Pi_1 \mid \mathcal{M}' \models \theta(b, b_0, \ldots, b_{i_n})\}$$

Again, σ_n is coded in $\mathrm{SSy}(\mathcal{M}')$, hence in $\mathrm{SSy}(\mathcal{M})$.

Moreover, $\{\theta(v_{i_n+1}, a_0, \ldots, a_{i_n}) \mid \theta \in \sigma_n\}$ is a Π_1-type consistent with $\mathcal{M}$ for the following reason: for any finite $\theta_1, \ldots, \theta_r \in \sigma_1$ we have

$$\mathcal{M}' \models \exists v_{i_n+1} < b_k \bigwedge_{j+1}^{r} \theta_j(v_{i_n+1}, b_0, \ldots, b_{i_n})$$

which is a Π_1-sentence, and since $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, a_0, \ldots, a_{i_n}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}', b_0, \ldots, b_{i_n})$ we have $\mathrm{Th}_{\Pi_1}(\mathcal{M}', b_0, \ldots, b_{i_n}) \subseteq \mathrm{Th}_{\Pi_1}(\mathcal{M}, a_0, \ldots, a_{i_n})$ (check!). So

$$\mathcal{M} \models \exists v_{i_n+1} < a_k \bigwedge_{j+1}^{r} \theta_j(v_{i_n+1}, a_0, \ldots, a_{i_n})$$

By Lemma 4.19 b), let $a \in \mathcal{M}$ realize $\{\theta(v_{i_n+1}, a_0, \ldots, a_{i_n}) \mid \theta \in \sigma_n\}$.

Put $a_{i_n+1} = a$, $b_{i_n+1} = b$. Check that

$$\mathrm{Th}_{\Sigma_1}(\mathcal{M}, a_0, \ldots, a_{i_n+1}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}', b_0, \ldots, b_{i_n+1})$$

We put $i_{n+1} = i_n + 2$, and proceed to the next stage.

The second part of each stage (when applied) will eventually make sure that we map onto an initial segment of $\mathcal{M}'$. ∎

Exercise 63. Prove yourself the direction i) $\Rightarrow$ ii) of Theorem 4.20.

Let us see how Theorem 4.20 easily implies (a simple form of) Friedman's Theorem:

Theorem 4.21 (Friedman) *Let $\mathcal{M}$ be a countable nonstandard model of* PA. *Then $\mathcal{M}$ is isomorphic to a propser initial segment of itself.*

Proof. By the MacDowell-Specker Theorem, or rather the simple Omitting Types argument at the beginning of section 3.5 (bearing in mind that the Omitting Types Theorem produces countable models), $\mathcal{M}$ has a countable proper elementary end-extension $\mathcal{M}'$.

We have seen that for $\mathcal{M} \subseteq_e \mathcal{M}'$, $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{M}')$. Also, since $\mathcal{M} \prec \mathcal{M}'$, $\mathrm{Th}_{\Sigma_1}(\mathcal{M}') \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M})$. By Theorem 4.20, $\mathcal{M}'$ is isomorphic to an initial segment of $\mathcal{M}$. But $\mathcal{M}$ was also a proper initial segment of $\mathcal{M}'$. Composing the two embeddings, we obtain the statement of the theorem. ∎

Appendix

In this chapter I put two, unrelated, results which I find interesting. One is Skolem's original construction of a nonstandard model for PA; the other is a theorem about the residue rings of infinite (nonstandard) primes in nonstandard models.

Skolem's Construction

Up to now, we haven't really seen a *concrete* nonstandard model of PA: all our existence theorems rely on the Completeness Theorem (or ultraproducts). In the first paper where nonstandard models were introduced, by Skolem in 1934, he gave a construction which is rather different.

Let $\mathcal{F}$ be the set of arithmetically definable functions from $\mathbb{N}$ to $\mathbb{N}$. Using the denumerability of $\mathcal{F}$, we construct a function $G : \mathbb{N} \to \mathbb{N}$ such that for all $f, g \in \mathcal{F}$:

$$f(G(x)) < g(G(x)) \text{ a.e., or } f(G(x)) = g(G(x)) \text{ a.e., or } f(G(x)) > g(G(x)) \text{ a.e.}$$

where "a.e." means *almost everywhere*, i.e. from a certain $n \in \mathbb{N}$ on.

The function G is defined as follows: enumerate $\mathcal{F}$ as $f_0, f_1, \ldots$ We define a sequence $A_0 \supseteq A_1 \supseteq \cdots$ of infinite subsets of $\mathbb{N}$, with the property that for all $k, l \leq n$,

$$(*) \qquad \begin{array}{l} \forall x \in A_n(f_k(x) < f_l(x)) \text{ or } \forall x \in A_n(f_k(x) = f_l(x)) \\ \text{or } \forall x \in A_n(f_k(x) > f_l(x)) \end{array}$$

Then we can define G as follows: let $G(0)$ be the least element of A_0, and $G(n+1)$ the least element of A_{n+1} which is above $G(n)$.

Put $A_0 = \mathbb{N}$. Suppose A_n is defined satisfying $(*)$, and infinite. The restrictions of $f_0, \ldots, f_n$ to A_n form, by pointwise ordering, a linearly ordered set $g_0 < \cdots < g_k$ for some $k \leq n$. Then

$$\begin{aligned} A_n = & \textstyle\bigcup_{i=0}^{k}\{x \in A_n \mid f_{n+1}(x) = g_i(x)\} \\ & \cup \{x \in A_n \mid f_{n+1}(x) < g_0(x)\} \\ & \cup \textstyle\bigcup_{i=0}^{k-1}\{x \in A_n \mid g_i(x) < f_{n+1}(x) < g_{i+1}(x)\} \\ & \cup \{x \in A_n \mid g_k(x) < f_{n+1}(x)\} \end{aligned}$$

This is a finite union of sets, so since A_n is infinite, one of these sets is; pick an infinite member of this union, and call it A_{n+1}. Clearly, A_{n+1} satisfies $(*)$. This completes the definition of the sets A_n, and hence the definition of G.

Now define an equivalence relation on $\mathcal{F}$: $f \equiv g$ iff $f(G(x)) = g(G(x))$ a.e. Let $\mathcal{M} = \mathcal{F}/\equiv$. The operations of pointwise addition and multiplication on $\mathcal{F}$ are well-defined on $\mathcal{M}$ too. Letting $0^{\mathcal{M}} = [\lambda x.0]$, $1^{\mathcal{M}} = [\lambda x.1]$ (we write $[f]$ for the $\equiv$-class of f), and $[f] < [g]$ iff $f(G(x)) < g(G(x))$ a.e. (this is well-defined on equivalence classes), we have that $\mathcal{M}$ is an $\mathcal{L}_{\mathrm{PA}}$-structure.

Theorem 4.22 *$\mathcal{M}$ is a proper elementary extension of $\mathcal{N}$.*

Proof. One proves by induction, that for formulas $\varphi(v_1, \ldots, v_k)$ and $[f_1], \ldots, [f_k] \in \mathcal{M}$,

$$\mathcal{M} \models \varphi([f_1], \ldots, [f_k]) \text{ if and only if } \mathcal{N} \models \varphi(f_1(G(n)), \ldots, f_k(G(n))) \text{ a.e.}$$

This is immediate for atomic formulas, and the induction steps for the propositional connectives are easy. The step for $\exists$ goes as follows:

If $\mathcal{M} \models \exists y \varphi([f_1], \ldots, [f_k])$ so for some $g \in \mathcal{F}$, $\mathcal{M} \models \varphi([g], [f_1], \ldots, [f_k])$, then by induction hypothesis $\mathcal{N} \models \varphi(g(G(n)), f_1(G(n)), \ldots, f_k(G(n)))$ a.e. so certainly $\mathcal{N} \models \exists y \varphi(f_1(G(n)), \ldots, f_k(G(n)))$ a.e.

For the converse, if $\mathcal{N} \models \exists y \varphi(f_1(G(n)), \ldots, f_k(G(n)))$ a.e., let h be the arithmetically definable function such that $h(m)$ is the least a satisfying $\varphi(a, f_1(m), \ldots, f_k(m))$ (and put $h(m) = 0$ if no such a exists). By assumption then,

$$\mathcal{N} \models \varphi(h(G(n)), f_1(G(n)), \ldots, f_k(G(n))) \text{ a.e.}$$

so by induction hypothesis $\mathcal{M} \models \varphi([h], [f_1], \ldots, [f_k])$ whence $\mathcal{M} \models \exists y \varphi([f_1], \ldots, [f_k])$.

Now if we have parameters from $\mathcal{N}$, and $\mathcal{M} \models \exists y \varphi(\overline{n_1}, \ldots, \overline{n_k})$, then $\mathcal{N} \models \varphi(\overline{m}, \overline{n_1}, \ldots, \overline{n_k})$ for some $n \in \mathbb{N}$. So $\mathcal{M} \models \varphi(\overline{m}, \overline{n_1}, \ldots, \overline{n_k})$ (remember that $\overline{n}^{\mathcal{M}} = [\lambda x.n]$). By the Tarski-Vaught test, $\mathcal{M}$ is an elementary extension of $\mathcal{N}$. ∎

Residue Fields in Nonstandard Models

Here we treat an easy fact which belongs to the folklore of the subject: it was never written down by anyone, but certainly known. Nevertheless, I feel it is interesting enough to include it here.

Let $\mathcal{M}$ be a nonstandard model of PA, and p a nonstandard prime number in $\mathcal{M}$. By Euclidean division and Bézout's Theorem in $\mathcal{M}$, the set of elements $< p$ in $\mathcal{M}$ has the structure of a field, which we denote by $\mathbb{F}_p$. Since p is nonstandard, none of the elements $1, 1+1, 1+1+1, \ldots$ is divisible by p, so the characteristic of $\mathbb{F}_p$ is 0 and $\mathbb{F}_p$ contains the field $\mathbb{Q}$ of rational numbers as a subfield.

What is the relation between $\mathbb{Q}$ and $\mathbb{F}_p$? We recall a few definitions from elementary algebra. We say for fields $K \subseteq L$ that L is *algebraic* over K if for each $x \in L$ there is a polynomial $P \in K[X]$ such that $P(x) = 0$. Otherwise, L is *transcendent* over K. A *transcendence basis* of L over K is a minimal subset A of L such that L is algebraic over $K(A)$ (the least subfield of L which contains K and A). The *transcendence degree* of L over K is the cardinality of a transcendence basis of L over K. We can now state:

Theorem 4.23 *Let $\mathcal{M}$ be a nonstandard model of* PA*, and $p \in \mathcal{M}$ a nonstandard prime number. Then $\mathbb{F}_p$ is a field of infinite transcendence degree over $\mathbb{Q}$.*

Proof. We show that for any *finite* number of elements $x_1, \ldots, x_k$ of $\mathbb{F}_p$, $\mathbb{F}_p$ is not algebraic over $\mathbb{Q}(x_1, \ldots, x_k)$. Clearly, and element x of $\mathbb{F}_p$ satisfies $P(x) = 0$

in $\mathbb{F}_p$ for a polynomial P with coefficients in $\mathbb{Q}(x_1, \dots, x_k)$, if and only if there are polynomials P_1, P_2 with coefficients in $\mathbb{N}[x_1, \dots, x_k]$ (the set of polynomials in $x_1, \dots, x_k$ with coefficients in $\mathbb{N}$) such that $P_1(x) = P_2(x)$ in $\mathbb{F}_p$, that is: $\mathcal{L}_{\mathrm{PA}}$-terms t_1, t_2 in parameters $x_1, \dots, x_k$ and free variable v, such that

$$\mathcal{M} \models \mathrm{rm}(t_1(x_1, \dots, x_k, x), p) = \mathrm{rm}(t_2(x_1, \dots, x_k, x), p)$$

Let $\tau(w_1, \dots, w_k, v, u)$ be the type of all formulas of the form:

$$\mathrm{rm}(t_1(\vec{w}, v), u) = \mathrm{rm}(t_2(\vec{w}, v), u) \to \forall z < u(\mathrm{rm}(t_1(\vec{w}, z), u) = \mathrm{rm}(t_2(\vec{w}, z), u))$$

for all pairs (t_1, t_2) of $\mathcal{L}_{\mathrm{PA}}$-terms in variables $w_1, \dots, w_k, v$.

The set of codes of elements of τ is recursive, hence, by 4.9, in $\mathrm{SSy}(\mathcal{M})$. Also, τ consists of Δ_0-formulas. And the type $\tau(x_1, \dots, x_k, v, p)$ is consistent with $\mathcal{M}$ since every polynomial can have at most finitely many roots, unless it is the zero polynomial, and $\mathbb{F}_p$ is infinite. So $\tau(x_1, \dots, x_k, v, p)$ is finitely satisfied in $\mathcal{M}$. By Lemma 4.19, $\tau(x_1, \dots, x_k, v, p)$ is realized by an element $a \in \mathcal{M}$. One sees that $\mathrm{rm}(a, p)$ is an element of $\mathbb{F}_p$ which is not a zero of a nontrivial polynomial with coefficients in $\mathbb{Q}(x_1, \dots, x_k)$. This holds for any k, so the theorem is proved. ∎

Miscellaneous Exercises

Exercise 64. The scheme of strong Δ_0-collection is the scheme:

$$\forall a, z \exists t \forall x \leq z \forall y(\theta(x, y, a) \to \exists w \leq t\theta(x, w, a))$$

where θ is a Δ_0-formula. Let S be the theory PA^- together with the scheme of strong Δ_0-collection. Prove that in S, the scheme of induction for Σ_1-formulas is provable.

Exercise 65. Give a formal proof in PA of the following sentence:

$$\forall xy(x < y \wedge \gcd(x, y) > 1 \to \exists vq(1 < v < y \wedge v{\cdot}x = q{\cdot}y))$$

Exercise 66. For this exercise, we assume that we have symbols for the manipulation of (coded) sequences in PA: we have functions $\mathrm{lh}(x)$ (the *length* of the sequence coded by x), $\langle x\rangle$ (the sequence with one element x), $(x)_i$ (the i-th element of the sequence coded by x), $\langle\rangle$ (the empty sequence), and $x * y$ (concatenation of sequences).

Let $R(x)$ be the formula

$$x = \langle\rangle \vee (\mathrm{lh}(x) < ((x)_0)^2 + 1)$$

Prove that PA proves the following principle of well-founded induction: for each formula $\varphi(v)$,

$$\mathrm{PA} \vdash \forall x(R(x) \wedge \forall y(R(x * \langle y\rangle) \to \varphi(x * \langle y\rangle)) \to \varphi(x)) \to \forall x(R(x) \to \varphi(x))$$

Exercise 67. Recall that we abbreviate $\Box\varphi$ for $\exists x\overline{\mathrm{Prf}}(x, \ulcorner\varphi\urcorner)$. The following "derivability conditions" hold:

D1	$\mathrm{PA} \vdash \varphi \Rightarrow \mathrm{PA} \vdash \Box\varphi$
D2	$\mathrm{PA} \vdash \Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$
D3	$\mathrm{PA} \vdash \Box\varphi \to \Box\Box\varphi$

i) Use these rules to show that $\mathrm{PA} \vdash \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$;

ii) Show that PA does *not* prove the implication

$$(\Box\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$$

for all φ and ψ [Hint: you may assume that $\mathrm{PA} \nvdash \Box\Box-$. Use D1–D3 and apply the Diagonalization Lemma].

Exercise 68. Let φ be a sentence in the language of PA. Prove that the following two statements are equivalent:

1) φ is preserved under end-extensions, that is: if $\mathcal{M} \subseteq_e \mathcal{M}'$ is an end-extension of models of PA and $\mathcal{M} \models \varphi$, then $\mathcal{M}' \models \varphi$;

2) φ is, in PA, equivalent to a Σ_1-sentence.

Exercise 69. If $\mathcal{M}$ is a model of PA and $a \in \mathcal{M}$, write $\mathcal{M}_a$ for $\{m \in \mathcal{M} \mid \mathcal{M} \models m < a\}$. $\mathcal{M}_a$ is an abelian group under addition modulo a.

Recall that an abelian group is *cyclic* if there is an element g such that every element of the group can be written as

$$\underbrace{g + \cdots + g}_{n \text{ times}} \text{ or } \underbrace{(-g) + \cdots + (-g)}_{n \text{ times}}$$

for some $n \in \mathbb{N}$. The element g is called a *generator* of the group.

i) Prove that there is no formula $\varphi(v_0, v_1)$ of $\mathcal{L}_{\mathrm{PA}}$ such that for every model $\mathcal{M}$ of PA and $a, b \in \mathcal{M}$:

$$\mathcal{M} \models \varphi(a, b) \Leftrightarrow \mathcal{M}_a \text{ is cyclic with generator } b$$

ii) Prove that in fact, $\mathcal{M}_a$ cannot be cyclic if a is nonstandard.

Exercise 70. Let $\mathcal{M}$ be a model of PA and $a, b \in \mathcal{M}$. Let us say that b is a *witness* for a if b codes the type of a in $\mathcal{M}$: that is, $(b)_n = 0$ if and only if $n = \ulcorner \varphi(v_0) \urcorner$ for some φ such that $\mathcal{M} \models \varphi(a)$.

i) Show that every model $\mathcal{M}$ of PA has an elementary extension $\mathcal{M}'$ such that every $a \in \mathcal{M}$ has a witness in $\mathcal{M}'$;

ii) Show that every model $\mathcal{M}$ of PA has an elementary extension $\mathcal{M}'$ such that every $a \in \mathcal{M}'$ has a witness in $\mathcal{M}'$;

iii) Show that the relation "b is a witness for a" is not definable in the language of PA.

Exercise 71. For this exercise we assume the theorem (due to Hilbert and Bernays) that there is a complete extension T of PA such that the axioms of T form a Δ^0_2-set. Prove that there is a model $\mathcal{M}$ of PA such that every element of $\mathrm{SSy}(\mathcal{M})$ is a Δ^0_2-set.

Exercise 72. Show that the collection of all Δ^0_2-sets is not a Scott set [Hint: relativize Lemma 4.14 to functions partial recursive in $\mathcal{K}$, the halting set].

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