Qualitative Probabilistic Networks

(the Binary Case)

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Thanks!

Contents

1 Introduction.

In the mid-eighties the framework of probabilistic networks (also known as causal or belief networks [9]) for reasoning with uncertainty was developed. The framework is characterised by a powerful formalism for representing a probability distribution over a set of variables and by algorithms for probabilistic inference. As the increasing number of successful applications in differing problem domains demonstrate, probabilistic networks have by now established their position of valuable representations of uncertainty in Artificial Intelligence research.

The development of a probabilistic network for a given problem domain involves various stages. In the first stage, the variables and values that play a role in the problem domain are identified. Then, the independencies between these variables are identified and modelled in an acyclic directed graph. In the final stage, the digraph is quantified by conditional probabilities for every variable. It turns out that especially this final stage is very complicated. Often not enough data is available to allow for reliable probability estimation. Available information may not be directly amenable for encoding in the network. Domain experts may be reluctant to provide numerical probabilities, since they are often not familiar enough with the concept of probability. Moreover, it has been shown that people are not capable of giving reliable numbers. These difficulties stand in the way of probabilistic networks being widely applied.

In the beginning of the nineties, the framework of *qualitative* probabilistic networks [12] started to develop. In qualitative probabilistic networks, the problems with elicitating probabilities mentioned above are partly solved. A qualitative probabilistic network is similar to a probabilistic network in the sense that it equally consists of a digraph modelling independencies. Instead of quantifying the digraph with conditional probabilities for the variables, however, the relationships between variables are expressed by qualitative constraints on the probability distribution over them.

In the context of qualitative probabilistic networks, considerable research has been done into the different kinds of qualitative probabilistic information and the mathematical interpretation of such qualitative information. In this thesis, we review this research. By providing a new formal underpinning for qualitative probabilistic networks, we are able to extend on former research and prove several new properties with regard to the relationships among qualitative information.

The remainder of this thesis is organised as follows. To understand the concepts from qualitative probabilistic networks, some knowledge of graph theory, probability theory and probabilistic networks is needed; these preliminaries are presented in section 2. In section 3 we present the formal definitions of a qualitative probabilistic network and the different types of qualitative information. Relations among the various types of qualitative information are proven in section 4. Section 5 presents some conclusions.

2 Preliminaries.

In this section we present some notions from probability theory, from graph theory and from the theory of probabilistic networks (see [6]).

2.1 Probability Theory.

We will review some basic notions and theorems from probability theory that we will use in the sequel.

We will use the following convention for referring to variables and their values. The set of all variables discerned will be denoted by U. We will assume each variable from U to be binary. Variables will be denoted by capital letters. For a variable C, its values will be denoted by c and $\bar{c}$; we take $c > \bar{c}$.

The set of variables U may be looked upon as spanning a *Boolean algebra* $\mathcal{B}$, that is, a set of propositions with two binary operations $\wedge$ (conjunction) and $\vee$ (disjunction), a unary operator $\neg$ (negation) and two constants *false* and *true* which behave according to logical truth tables. For any $a, b \in \mathcal{B}$ we will often write ab instead of $a \wedge b$, and $\bar{a}$ instead of $\neg a$. We now define a probability distribution as a function on the set of variables U.

Definition 2.1 (Probability Distribution) *Let $\mathcal{B}$ be the Boolean algebra of propositions spanned by U. Let $Pr : \mathcal{B} \to [0,1]$ be a function such that*

- *for all $a \in \mathcal{B}$, we have $Pr(a) \geq 0$, and furthermore $Pr(false) = 0$,*
- *$Pr(true) = 1$, and*
- *for all $a, b \in \mathcal{B}$, if $a \wedge b = false$ then $Pr(a \vee b) = Pr(a) + Pr(b)$.*

Then, Pr is called a probability distribution *on U.*

In the sequel, we will want to consider a probability distribution on a subset of the set of all variables.

Lemma 2.2 *Let Pr be a probability distribution on U. Let $A \in U$ and $X \subseteq U \backslash \{A\}$. Then, the probabilities*

$$Pr(x) = Pr(x \wedge a) + Pr(x \wedge \bar{a})$$

for all value assignments x to X define a probability distribution on X.

We call the probability distribution as defined in the lemma above, the *marginal distribution* on X.

In the sequel, we will often have to deal with *conditional probabilities*.

Definition 2.3 (Conditional Probability) *Let Pr be a probability distribution on U and let $X, Y \subseteq U$. For any value assignment x to X and y to Y, with $Pr(y) > 0$, the* conditional probability *of x given y, denoted as $Pr(x|y)$, is defined as*

$$Pr(x|y) = \frac{Pr(x \wedge y)}{Pr(y)}.$$

Conditional probabilities also define a probability distribution, so properties that hold for probability distributions, also hold for conditional probability distributions. We will assume that all conditional probabilities we use are well-defined.

The following theorems state two other useful properties.

Lemma 2.4 *Let Pr be a probability distribution on U. Let $A \in U$ and let $X = U \backslash \{A\}$. Then,*

$$Pr(x) = Pr(x|a)Pr(a) + Pr(x|\bar{a})Pr(\bar{a})$$

for all value assignments x to X.

Theorem 2.5 (Bayes) *Let Pr be a probability distribution on U. Let $A \in U$ and let $X = U \backslash \{A\}$. Then,*

$$Pr(x|a_0) = \frac{Pr(a_0|x)Pr(x)}{Pr(a_0)}.$$

for all value assignments x to X and a_0 to A.

Note that Bayes' Theorem may be used to reverse the 'direction' of conditional probabilities.

We present one more definition and another theorem about conditional independencies.

Definition 2.6 (Conditional Independence) *Let Pr be a probability distribution on U. Let $X, Y, Z \subseteq U$. The set of variables X is said to be* conditionally independent *of Y given Z, denoted as $I_{Pr}(X, Z, Y)$, if $Pr(x|y \wedge z) = Pr(x|z)$, for all value assignments x to X, y to Y and z to Z, respectively.*

We will see that variables that are not independent of each other, may have qualitative relations among them. These qualitative influential relations among variables are based on the concept of *stochastic dominance* [15]. Stochastic dominance captures the notion that higher values for a variable make higher values for some other variable more likely.

Definition 2.7 *Let $B \in U$ be a variable and let F_B and $F_B^{'}$ be cumulative probability distribution functions (CDFs) for B. Then, F_B is said to* dominate $F_B^{'}$ *by* first order stochastic dominance *(FSD) if and only if for any* *value* $b_i \in \{b, \bar{b}\}$ *of* B*, the cumulative probability for* b_i *is smaller for* F_B *than for* $F_B^{'}$*. That is,*

$$F_B \, FSD \, F_B^{'} \quad \Leftrightarrow \quad \forall\, b_i[F_B(b_i) \leq F_B^{'}(b_i)].$$

FSD is defined analogously for cumulative *conditional* distributions. Note that $Pr(B < b_i|a) \leq Pr(B < b_i|\bar{a})$, and $Pr(B \geq b_i|a) \geq Pr(B \geq b_i|\bar{a})$ are equivalent for both values b_i of B and the values $a, \bar{a}$ of variable A. So, $F_B(b_i|a)$ FSD $F_B(b_i|\bar{a}) \Leftrightarrow Pr(B \geq b_i|a) \geq Pr(B \geq b_i|\bar{a})$.

Milgrom [8] has proven the following property for stochastic dominance, which will be used later on.

Theorem 2.8 (Binary Milgrom) *Let* Pr *be a probability distribution on* U*. Let* A*,* $B \in U$*. Then,*

$$Pr(b|ax) \geq Pr(b|\bar{a}x) \quad \Leftrightarrow \quad \frac{Pr(a|bx)}{Pr(a|\bar{b}x)} \geq \frac{Pr(\bar{a}|bx)}{Pr(\bar{a}|\bar{b}x)}$$

for all value assignments x *to the variables other than* A*, that influence* B*.*

Proof: Assume that

$$Pr(b|ax) \geq Pr(b|\bar{a}x)$$

By employing the definition of conditional probability, we find

$$Pr(b|ax) \geq Pr(b|\bar{a}x) \Leftrightarrow Pr(abx)Pr(\bar{a}x) \geq Pr(\bar{a}bx)Pr(ax)$$

Now, by marginalisation, we find

$$\begin{aligned}
Pr(abx)Pr(\bar{a}x) &\geq Pr(\bar{a}bx)Pr(ax) && \Leftrightarrow \\
Pr(abx)\Big(Pr(\bar{a}bx) + Pr(\bar{a}\bar{b}x)\Big) &\geq Pr(\bar{a}bx)\Big(Pr(abx) + Pr(a\bar{b}x)\Big) && \Leftrightarrow \\
Pr(abx)Pr(\bar{a}\bar{b}x) &\geq Pr(\bar{a}bx)Pr(a\bar{b}x) && \Leftrightarrow \\
\frac{Pr(abx)Pr(\bar{a}\bar{b}x)}{Pr(\bar{a}bx)Pr(a\bar{b}x)} &\geq 1 && \Leftrightarrow \\
\frac{Pr(abx)Pr(\bar{a}\bar{b}x)}{Pr(\bar{a}bx)Pr(a\bar{b}x)}\frac{Pr(bx)}{Pr(bx)}\frac{Pr(\bar{b}x)}{Pr(\bar{b}x)} &\geq 1
\end{aligned}$$

By once more employing the definition of conditional probability, we find

$$\begin{aligned}
\frac{Pr(abx)Pr(\bar{a}\bar{b}x)}{Pr(\bar{a}bx)Pr(a\bar{b}x)}\frac{Pr(bx)}{Pr(bx)}\frac{Pr(\bar{b}x)}{Pr(\bar{b}x)} \geq 1 \quad &\Leftrightarrow \quad \frac{Pr(a|bx)}{Pr(a|\bar{b}x)}\frac{Pr(\bar{a}|\bar{b}x)}{Pr(\bar{a}|bx)} \geq 1 \\
&\Leftrightarrow \quad \frac{Pr(a|bx)}{Pr(a|\bar{b}x)} \geq \frac{Pr(\bar{a}|bx)}{Pr(\bar{a}|\bar{b}x)}
\end{aligned}$$

□

2.2 Graph Theory.

We review some basic notions from graph theory.

Definition 2.9 (Directed Graph) *A* directed graph *G is an ordered pair $G = (V, R)$, where $V = \{V_1, \ldots, V_n\}, n \geq 1$, is a finite set of vertices and R is a set of ordered pairs $(V_i, V_j), V_i, V_j \in V$, called arcs.*

Definition 2.10 (Successor and Predecessor) *Let $G = (V, R)$ be a directed graph. Let $V_i, V_j \in V$. Then, vertex V_j is called a* successor *of vertex V_i if there is an arc $(V_i, V_j) \in R$; the set of all successors of V_i in G is denoted by $\sigma(V_i)$. Similarly, vertex V_i is called a* predecessor *of vertex V_j if there is an arc $(V_i, V_j) \in R$; the set of all predecessors of V_i in G is denoted by $\pi(V_i)$.*

Definition 2.11 (Path, Length of Path in Directed Graph) *Let $G = (V, R)$ be a directed graph. Let $V_0, V_k \in V$. A* path *from V_0 to V_k in G is a sequence of vertices $V_0, V_1, \ldots, V_k$ such that $(V_{i-1}, V_i) \in R, i = 1, \ldots, k, k \geq 0$; k is called the* length *of the path.*

Definition 2.12 ((A)cyclic Graph) *Let $G = (V, R)$ be a directed graph. Let $V_0 \in V$. A* cycle *is a path of length at least one from V_0 to V_0. G is called a* cyclic *graph if it contains at least one cycle; a graph without cycles is called* acyclic.

We will now define the notion of a trail. A trail in a directed graph can be looked upon as a path in the graph's underlying undirected graph.

Definition 2.13 (Trail of variables) *Let $G = (V, R)$ be a directed graph. A* trail *T is a, possibly empty, sequence of variables $V_i \in V$, $i = 1, \ldots, n$, with $(V_i, V_{i+1}) \in R$ or $(V_{i+1}, V_i) \in R$ for every two successive variables $V_i, V_{i+1} \in T$. We will write $T = \{V_j, T', V_k\}$ to denote the trail from V_j to V_k over the sequence of variables T'.*

We will also define the *inversion* of a trail.

Definition 2.14 (Inversing trails) *Let $G = (V, R)$ be a directed graph and let $T = \{V_i, \{V_{i+1}, \ldots, V_{j-1}\}, V_j\}$ be a trail from V_i to V_j in G, $V_k \in V$, $k = i, \ldots, j$, $i, j \geq 1$. Then, the trail $T^{-1} = \{V_j, \{V_{j-1}, \ldots, V_{i+1}\}, V_i\}$ from V_j to V_i is the* inverse *of T.*

2.3 Probabilistic Networks.

The theory of probabilistic networks provides formalism for representing a problem domain. A probabilistic network consists of two parts: a *qualitative* *representation* of the problem domain and an associated *quantitative representation.* The qualitative part of a probabilistic network takes the form of an acyclic directed graph. Each vertex represents a domain variable. An arc between two vertices represents the direct influence of one variable on the other. Absence of an arc between two vertices means that the corresponding variables do not influence each other directly.

Associated with the graphical part of a probabilistic network is a set of numerical functions. These functions express (conditional) probabilities, which describe the influence of values of predecessors of a vertex in the graph, on the values of the vertex itself. More formally, a probabilistic network is defined as in the following definition.

Definition 2.15 (Probabilistic Network) *A* probabilistic network *is a tuple $B = (G, \Gamma)$ such that*

- *$G = (V, R)$ is an acyclic directed graph with vertices $V = \{V_1, \ldots, V_n\}$, $n \geq 1$, and arcs R, and*
- *$\Gamma = \{\gamma_{V_i} | V_i \in V\}$ is a set of real-valued nonnegative functions γ_{V_i} : $\{v_i, \neg v_i\} \times \{x_i\} \rightarrow [0, 1]$, called (conditional probability)* assessment functions*, such that for each value assignment x_i to $X_i = \pi(V_i)$, we have $\gamma_{V_i}(\neg v_i | x_i) = 1 - \gamma_{V_i}(v_i | x_i), i = 1, \ldots, n$.*

From here on, when referring to a probabilistic network, we will use the term *quantitative* probabilistic network.

In the sequel, we will need to use the graph of the marginal distribution over a subset of all variables.

Definition 2.16 (Marginal Network) *Let $G = (V, R)$ be a directed graph. Let $B, V_i, V_j \in V$, $i, j = 1, \ldots, n$. The* marginal network of G, omitting variable B *is the graph $G^{'} = (V^{'}, R^{'})$, where*

- *$V^{'} = V \backslash \{B\}$ and*
- *for all $V_i \in \pi(B)$ and $V_j \in \sigma(B)$: if (V_i, B), $(B, V_j) \in R$ then $(V_i, V_j) \in R^{'}$.*

There exist several relationships between directed graphs and probability distributions. An important relationship is given by the (directed) I-map. In order to define an I-map, we need the notion of d-separation.

Definition 2.17 *Let $G = (V, R)$ be an acyclic directed graph where $V = \{V_1, \ldots, V_n\}$. A trail T from V_i to V_j, $V_i, V_j \in V$, in G is* blocked *by a set of vertices $W \subseteq V$ if (at least) one of the following conditions holds:*

1. *The trail T contains the vertices $X_1, X_2, X_3 \in V$ such that $X_2 \in W$ and $(X_2, X_1), (X_2, X_3) \in R$;*

2. *The trail T contains the vertices $X_1, X_2, X_3 \in V$ such that $X_2 \in W$ and $(X_1, X_2), (X_2, X_3) \in R$;*

3. *The trail T contains the vertices $X_1, X_2, X_3 \in V$ such that (X_1, X_2), $(X_3, X_2) \in R$ and $\sigma^*(X_2) \cap W = \varnothing$.*

Definition 2.18 (d-separation) *Let $G = (V, R)$ be an acyclic directed graph. Let $X, Y, Z \subseteq V$ be sets of vertices in G. The set Z is said to* d-separate *X from Y in G, written $< X|Z|Y >_G^d$, if every trail in G from a vertex in X to a vertex in Y is blocked by Z.*

Definition 2.19 (I-map) *Let $G = (V, A)$ be an acyclic directed graph. For $X, Y, Z \subseteq V$, let $I(X, Y, Z)$ be an independence relation as defined in definition 2.6. Then, G is called an* independency map *(I-map) for I if*

$$< X|Z|Y >_G^d \quad \Rightarrow \quad I(X, Z, Y),$$

for every $X, Y, Z \subseteq V$.

The graph of a probabilistic network is taken as an I-map of the (conditional) independencies that exist in the problem domain. Vertices that are non-adjacent in an I-map of Pr correspond to independent variables; those that are adjacent, however, need not necessarily be dependent.

The assessment functions provide all information necessary for uniquely defining a joint probability distribution on the variables discerned, that respects the independency relationships portrayed by the graphical part of the network.

3 Defining a Qualitative Probabilistic Network.

Qualitative probabilistic networks are an abstraction of quantitative probabilistic networks, replacing quantitative relations among variables by qualitative relations. In this section we will present the definition of a qualitative probabilistic network and its types of qualitative relations.

3.1 Qualitative Probabilistic Networks.

A qualitative probabilistic network has the same graphical basis as a quantitative probabilistic network. Note that therefore its graph is also taken to be an I-map.

We will now give the definition of a qualitative probabilistic network and illustrate it with an example.

Definition 3.1 (Qualitative Probabilistic Network) *A* qualitative probabilistic network *(QPN) is a tuple $Q = (V, R, H)$, where*

- *$G = (V, R)$ is an acyclic, directed graph with vertices V and arcs R, and*
- *H is a set of directed hyperedges $(\{V_i, \dots, V_j\}, V_k)$, where $\{V_i, \dots V_j\} \subseteq V$, $\{V_i, \dots V_j\} \neq \varnothing$, and $V_k \in V$, $k \neq i, \dots, j$.*

The arcs R and directed hyperedges H of a qualitative probabilistic network with each other represent *qualitative relationships* expressing dominance properties among the variables discerned in a joint probability distribution. The arcs R with each other represent the independencies among the variables in V by means of the d-separation criterion. Each arc corresponds to a directed *qualitative influence* between two variables. *However*, not every qualitative influence corresponds to an arc. The set of directed hyperedges H represents all qualitative influences and *qualitative synergies*, describing interactions of influences among the variables discerned. In the following sections, we will further elaborate on these influences and synergies.

QPNs are generally depicted as digraphs; vertices are represented by circles, with arrows (arcs) between them. In the following, some figures will contain squares, which stand for the set of all predecessors of a certain vertex that aren't shown explicitly.

We give an example, which we will use in the sequel to explain some of the qualitative relations that are possible in a QPN.

Example 3.2 *We consider the game of Baseball and address the question what the chances are of a batter running four bases.*

- *A batter can run four bases, represented by the variable $4B$ with the values $yes4$ and $no4$, whenever*

- *he hits a home-run, which usually means he has to hit the ball outside the fences. The larger the distance to the fences (variable: DF, values: $far\_away, close\_by$), the less likely a batter will hit the ball over the fence. Yet, the better the batter, (variable: QB, values: $good, bad$) the more likely he will hit a home-run.*
- *he doesn't hit a home-run, but the field-players make mistakes (variable: FE, values: $error, no\_error$) which enable him to run four bases.*

- *When a batter gives effect to the ball while hitting it (variable: EB, values: yes, no), the change of direction of the ball may surprise a field-player who was going to field the ball, causing him to let it slip or even miss it.*

- *When it rains (variable: RN, values: wet, dry), players are less manoeuvrable and the ball will be heavy and slippery, which makes it harder to catch it.*

The causal relationships among the six variables discerned are represented in the digraph shown in Figure 1.

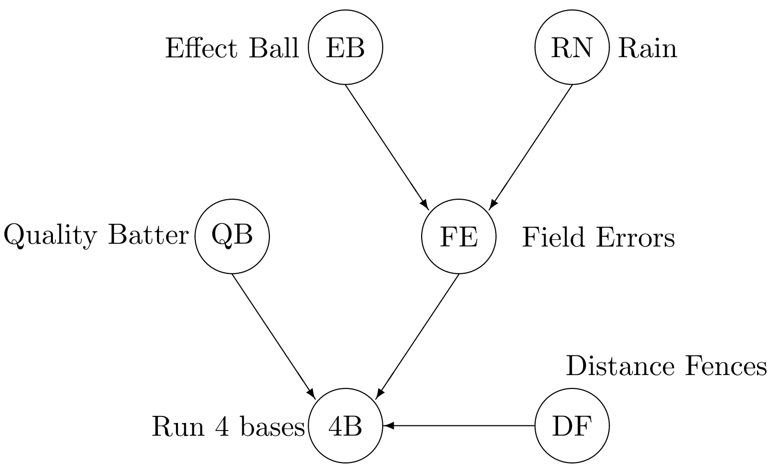


Figure 1: Baseball example

3.2 Qualitative Influences.

A qualitative influence expresses a monotonic property of the probabilistic influence between two variables [11]. This is, it expresses how the values of one variable relate to values of an other variable, where this relationship is a monotone one. A qualitative influence of a variable A on a variable B is denoted by $S^\delta(A, B)$, where δ is one of $+$, $-$, 0 or ?, denoting the direction of influence. If, for example, $\delta = +$, then the qualitative influence $S^+(A, B)$ expresses that higher values for A make higher values for B more likely, that is, $Pr(b|a) \geq Pr(b|\bar{a})$.

Definition 3.3 (Binary Qualitative Influence) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B \in V$ such that $(A, B) \in R$ and let $X = \pi(B)\backslash\{A\}$. Then, variable A* positively influences *variable B, written $S^+(A, B) \in H$, iff for the value b of B and the values a, $\bar{a}$ of A, we have*

$$Pr(b|ax) \geq Pr(b|\bar{a}x)$$

for all possible value assignments x to X. Negative qualitative influence and zero qualitative influence are defined analogously, where $\geq$ is replaced by $\leq$ and $=$, respectively. When the relationship between the variables A and B is neither $+, -$ nor 0, then we write $S^?(A, B)$.

In Figure 2 the hyperedge from A to B expresses a qualitative influence $S^\delta(A, B)$. As every arc in the digraph of a qualitative probabilistic network corresponds to a qualitative influence, we will, from now on, omit the hyperedges expressing these influences and only draw the arcs.

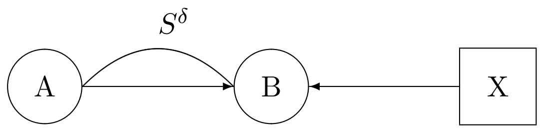


Figure 2: Qualitative influence

Example 3.4 *We consider once more the Baseball example. In this example we have, among others, the following qualitative influences:*

- $S^+(QB, 4B)$. *QB has a positive influence on $4B$ since a higher value for QB* increases *the chance of a higher value for $4B$: the chance that a batter can run four bases is higher if he is a good batter.*

- $S^-(DF, 4B)$. *DF has a negative influence on $4B$ since a higher value for DF* decreases *the chance of a higher value for $4B$: the chance that a batter can hit the ball behind the fences and can therefore run four bases, is lower for far-away-fences than it is for close-by-fences.*

We can also define an influence on a *trail* of variables. Informally, the influence $\hat{S}^{\delta}(A, B, T)$ on trail T from variable A to variable B has the same meaning as $S^{\delta}(A, B)$ if, in the graph, you replace the trail T from A to B by the arc (A, B). Note that this also means that when we encounter a $S^{\delta}(A, B)$ in a theorem in the next section, this theorem also holds for an influence on one of the trails from A to B.

Definition 3.5 (Influence on a Trail) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B \in V$ and let $T = \{A, T^{'}, B\}$ be a trail from A to B in Q, where $T^{'} \in T$ is the subtrail of T. Then $\hat{S}^{\delta}(A, B, T)$ holds in Q if and only if $S^{\delta}(A, B)$ holds in the marginal network of Q omitting every vertex on $T^{'}$.*

If there is no arc between every two successive variables on trail T, then $\delta = 0$.

We will also encounter the notation $\hat{S}^{\delta}(A, B, T \cup T^{'})$. Informally, this has the same meaning as $S^{\delta}(A, B)$ where the two parallel trails T and $T^{'}$ are replaced by one influence arc.

3.3 Qualitative Synergy.

A qualitative *synergy* describes the interaction between three or more variables. There are two types of qualitative *synergy*: *additive* synergy and *product* synergy. An additive synergy represents the *joint* influence of two or more causes on a common effect. A negative additive synergy of two variables with respect to a third variable expresses that the joint influence of those two variables is smaller than the sum of their individual influences. Thus, the effect of variation of one variable can be weakened by simultaneous variation of the other variable.

Definition 3.6 (Binary Additive Synergy) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $X = \pi(C) \backslash \{A, B\}$. Then, variables A and B exhibit* positive additive synergy *with respect to variable C, written $Y^{+}(\{A, B\}, C) \in H$, iff for the value c of C, the values a, $\bar{a}$ of A and the values b, $\bar{b}$ of B, we have*

$$Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$$

for all possible value assignments x to X. Negative additive synergy and zero additive synergy are defined analogously, where $\geq$ is replaced by $\leq$ and $=$, respectively. When the relationship is neither $+, -$ nor 0, then we write $Y^{?}(\{A, B\}, C)$.

In the sequel, we will often encounter the situation where we have trails, instead of arcs, from the two causes to the common effect.

Definition 3.7 (Additive Synergy on Trails) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, D \in V$, let $T_1 = \{A, T_1^{'}, D\}$ be a trail from A to D over subtrail $T_1^{'} \in T_1$ in Q and let $T_2 = \{B, T_2^{'}, D\}$ be a trail from B to D over subtrail $T_2^{'} \in T_2$ in Q. Then $\hat{Y}^{\delta}(\{A, B\}, D, T_1, T_2)$ holds in Q if and only if $Y^{\delta}(\{A, B\}, D)$ holds in the marginal network of Q omitting every vertex on $T_1^{'}$ and $T_2^{'}$.*

We will also encounter the notation $\hat{Y}^{\delta}(\{A, B\}, C, T \cup T^{'}, W \cup W^{'})$. Informally, this has the same meaning as $Y^{\delta}(\{A, B\}, C)$ where the two parallel trails T and $T^{'}$ are replaced by one influence arc from A to C and the two parallel trails W and $W^{'}$ are replaced by one influence arc from B to C.

In Figure 3, the hyperedge from the variables A and B to the variable C expresses the additive synergy $Y^{\delta}(\{A, B\}, C)$.

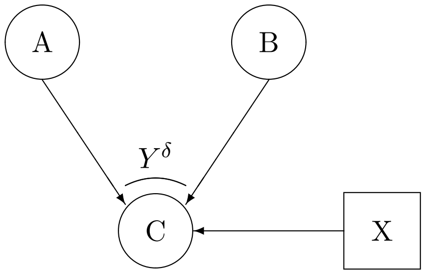


Figure 3: Additive Synergy

Example 3.8 *We consider once more the Baseball example. In this example we have, among others, the following additive synergy:*

- $Y^{+}(\{EB, RN\}, FE)$. *The joint influence of higher values for EB and RN on FE is* greater *than the sum of their individual effects. For these three variables, we may for example have the following probabilities:*
  - $Pr(fe|eb, rn) = 0.7$
  - $Pr(fe|\bar{eb}, \bar{rn}) = 0.05$
  - $Pr(fe|eb, \bar{rn}) = 0.2$
  - $Pr(fe|\bar{eb}, rn) = 0.3$

  Both the presence of rain and an effect of the ball can cause a field-player to make an error. When neither of these causes is present then the chance of the player making an error is rather small. However, *when it's raining, the grass and the ball are slippery, and when the ball also has an effect, it is* extra *hard for a player to catch the ball. Therefore the joint influence of eb and rn on fe is greater than the sum of their individual effects:*

$$Pr(fe|eb, rn) + Pr(fe|\bar{eb}, \bar{rn}) \geq Pr(fe|eb, \bar{rn}) + Pr(fe|\bar{eb}, rn).$$

A product synergy [5] [7] [13] captures the sign of the causal interaction between two causes of a common effect that has been observed (or that has indirect evidential support). Recall that two causes of a common effect may become dependent of each other once we observe a value for their common effect. A product synergy now captures the sign of the induced qualitative influence between these causes.

An example of a causal interaction is the negative intercausal interaction known as *explaining away*, which models negative influence of the presence of one cause on the likelihood of another cause being present, given an observed common effect.

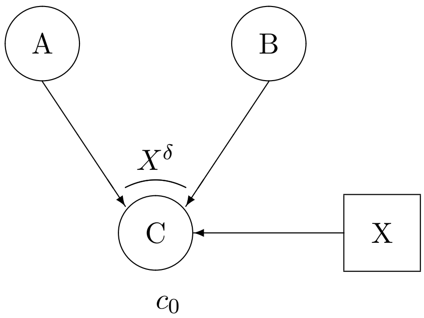


Figure 4: Product Synergy

Definition 3.9 (Binary Product Synergy) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $X = \pi(C) \backslash \{A, B\}$. Then, variables A and B exhibit* positive product synergy *with respect to the value c_0 of C (ranging over c, $\bar{c}$), written $X^+(\{A, B\}, c_0) \in H$, iff for the values a, $\bar{a}$ of A and the values b, $\bar{b}$ of B, we have*

$$Pr(c_0|abx)Pr(c_0|\bar{a}\bar{b}x) \geq Pr(c_0|a\bar{b}x)Pr(c_0|\bar{a}bx)$$

for all possible value assignments x to X. Negative product synergy and zero product synergy are defined analogously, by replacing $\geq$ with $\leq$ and $=$, respectively. When the relationship is neither $+, -$ nor 0 then we write $X^?(\{A, B\}, c_0)$.

In the sequel, we will encounter the situation where we have trails, instead of arcs, from the two causes to the common effect.

Definition 3.10 (Product Synergy on Trails) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, D \in V$, let $T_1 = \{A, T_1^{'}, D\}$ be a trail from A to D over subtrail $T_1^{'} \in T_1$ in Q and let $T_2 = \{B, T_2^{'}, D\}$ be a trail from B to D over subtrail $T_2^{'} \in T_2$ in Q. Then $\hat{X}^{\delta}(\{A, B\}, D, T_1, T_2)$ holds in Q if and only if $X^{\delta}(\{A, B\}, D)$ holds in the marginal network of Q omitting every vertex on $T_1^{'}$ and $T_2^{'}$.*

In Figure 4, the hyperedge from the variables A and B to the variable C with value c_0 expresses the product synergy $X^{\delta}(\{A, B\}, c_0)$.

Example 3.11 *We consider once more the Baseball example. In this example we have, among others, the following product synergy:*

- $X^{-}(\{QB, FE\}, yes4)$. *If the value $yes4$ is known with certainty for variable $4B$, then a higher value for QB makes a higher value for FE* less *likely: if someone has just run four bases and he is known to be a good batter, then it is less likely that he came home on field errors.*

Note that the difference between additive synergy and product synergy is not just the operator in their respective definitions. While additive synergy is defined for *all* values of a common successor, product synergy is defined for a *single* value of such a successor. Therefore, we have as many product synergies as there are values for the common successor.

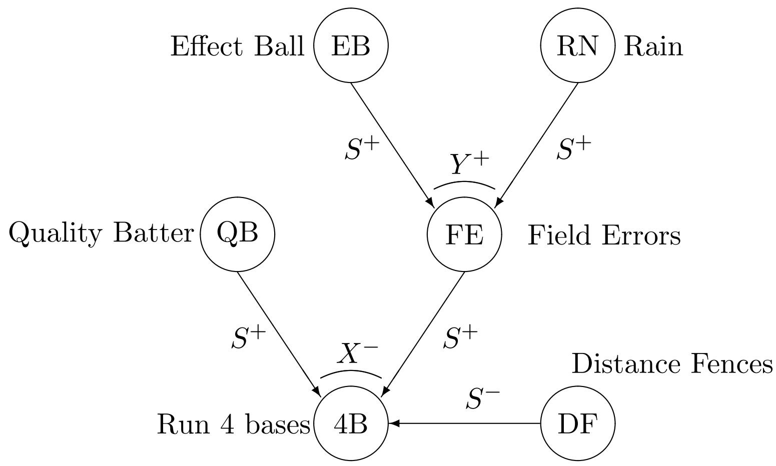


Figure 5: Baseball example with qualitative information

Example 3.12 *Once again, we consider the Baseball example. Figure 5 shows the digraph of the qualitative probabilistic network for the example, now extended with some of the qualitative influences and synergies we described above.*

4 The set of hyperedges of a QPN.

Symmetry	S	Theorem 4.1
	Y	Theorem 4.2
	X	Theorem 4.3
Transitivity	S	Theorem 4.4
	Y	Theorem 4.5
Composition	S	Theorem 4.6
	Y	Theorem 4.7
$S \Rightarrow Y$		Theorem 4.8
$S \Rightarrow X$		Theorem 4.9
Intercausal Reasoning	Common effect observed	Theorem 4.13
	Indirect evidence observed	Theorem 4.14
$X \Rightarrow Y$	$\delta = +$	Theorem 4.15
	$\delta = -$	Theorem 4.16
$Y \Rightarrow X$	$\delta = -$	Theorem 4.15
	$\delta = +$	Theorem 4.16
$X[c] \Rightarrow X[\bar{c}]$		Theorem 4.19

Table 1: Theorems in this section

In this section we will prove a number of properties of the set of hyperedges H of a qualitative probabilistic network. We will show that some of the influences and synergies can be derived from a small set of qualitative influences and synergies from H. From these properties we have that only a subset of H will be sufficient for the representation of the QPN. Table 1 gives a survey of the structure we have brought upon the list of theorems.

4.1 Symmetry.

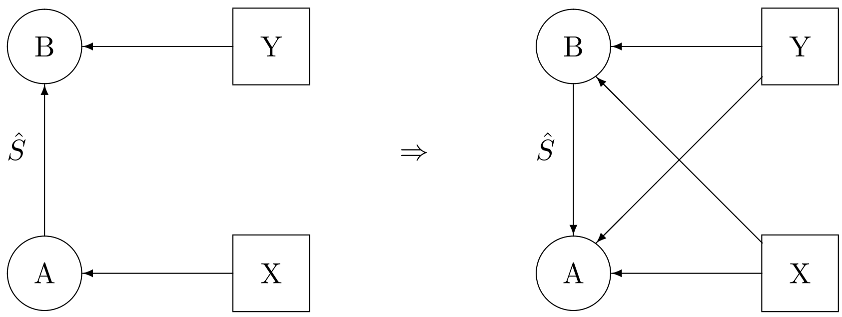
For reasoning with a qualitative probabilistic network, we need to be able to propagate qualitative changes in all directions. For this purpose, qualitative influences and synergies have to be specified for *two* directions. However, it suffices to specify them in only *one* direction due to the property of *symmetry* [1, Th. 6.4].

Theorem 4.1 (Symmetry of Binary Qualitative Influence) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B \in V$ such that $(A, B) \in R$. Then,*

$$\hat{S}^{\delta}(A, B, \{A, \varnothing, B\}) \Leftrightarrow \hat{S}^{\delta}(B, A, \{B, \varnothing, A\}),$$

for all $\delta \in \{+, -, 0, ?\}$.

Proof: We will prove the theorem for $\delta = +$. Proofs for $\delta = -, 0, ?$ are analogous. The relations between the variables involved in the proof are as depicted in the figure below.



By definition, we have that

$$\hat{S}^+(A, B, \{A, \varnothing, B\}) \quad \Leftrightarrow \quad Pr(b|ay) \geq Pr(b|\bar{a}y)$$

for all value assignments y to Y. Using the fact that the variable B is independent of the set of variables X given A and applying theorem 2.8, we get:

$$\hat{S}^+(A, B, \{A, \varnothing, B\}) \quad \Leftrightarrow \quad \frac{Pr(a|bxy)}{Pr(a|\bar{b}xy)} \geq \frac{Pr(\bar{a}|bxy)}{Pr(\bar{a}|\bar{b}xy)}$$

Now, using Bayes' theorem and simplifying, we find:

$$\begin{aligned} \hat{S}^+(A, B, \{A, \varnothing, B\}) \quad &\Leftrightarrow \quad \frac{Pr(b|axy)Pr(axy)Pr(\bar{b}xy)}{Pr(bxy)Pr(\bar{b}|axy)Pr(axy)} \geq \\ &\quad \frac{Pr(b|\bar{a}xy)Pr(\bar{a}xy)Pr(\bar{b}xy)}{Pr(bxy)Pr(\bar{b}|\bar{a}xy)Pr(\bar{a}xy)} \\ &\Leftrightarrow \quad \frac{Pr(b|axy)}{Pr(b|\bar{a}xy)} \geq \frac{Pr(\bar{b}|axy)}{Pr(\bar{b}|\bar{a}xy)} \end{aligned}$$

Again, we use theorem 2.8 and find:

$$\hat{S}^+(A, B, \{A, \varnothing, B\}) \Leftrightarrow Pr(a|bxy) \geq Pr(a|\bar{b}xy)$$

Since by replacing the arc (A, B) by the arc (B, A), the nodes A and B inherit each others parents (the arc-reversal property [10]), we get:

$$\hat{S}^+(A, B, \{A, \varnothing, B\}) \Leftrightarrow \hat{S}^+(B, A, \{B, \varnothing, A\})$$

by definition.

□

Note that the above theorem only shows that the *sign* of influence is symmetric. The *magnitude* of the influence in one direction can differ considerably from the magnitude of the influence in the other direction. Also note that symmetry holds for *trails* as well, since on a trail we can reverse every arc necessary. Thus,

$$\hat{S}^{\delta}(A, B, T) \Leftrightarrow \hat{S}^{\delta}(B, A, T^{-1}),$$

where T^{-1} is the "inversed" trail of trail T.

Theorem 4.2 (Symmetry of Binary Additive Synergy) *Let* $Q = (V, R, H)$ *be a qualitative probabilistic network. Let* $A, B, C \in V$ *such that* (A, C), $(B, C) \in R$. *Then,*

$$Y^{\delta}(\{A, B\}, C) \quad \Leftrightarrow \quad Y^{\delta}(\{B, A\}, C),$$

for all $\delta \in \{+, -, 0, ?\}$.

Proof: We will prove the theorem for $\delta = +$. Proofs for $\delta = -, 0, ?$ are analogous.

By definition, we have that

$$Y^{+}(\{A, B\}, C) \Leftrightarrow Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$$

By rearranging values for A and B, we get

$$\begin{aligned} Y^{+}(\{A, B\}, C) &\Leftrightarrow Pr(c|bax) + Pr(c|\bar{b}\bar{a}x) \geq Pr(c|b\bar{a}x) + Pr(c|\bar{b}ax) \\ &\Leftrightarrow Y^{+}(\{B, A\}, C) \end{aligned}$$

by definition.

□

Theorem 4.3 (Symmetry of Binary Product Synergy) *Let* $Q = (V, R, H)$ *be a qualitative probabilistic network. Let* $A, B, C \in V$ *such that* (A, C), $(B, C) \in R$ *and let* c_0 *be a value of* C. *Then,*

$$X^{\delta}(\{A, B\}, c_0) \quad \Leftrightarrow \quad X^{\delta}(\{B, A\}, c_0),$$

for all $\delta \in \{+, -, 0, ?\}$.

Proof: The proof is analogous to the proof of theorem 4.2.

□

Note that symmetry of additive synergy and product synergy also hold for trails. Suppose $T_1 = \{A, \ldots, C\}$ and $T_2 = \{B, \ldots, C\}$ are trails, then

$$\hat{Y}^{\delta}(\{A, B\}, C, T_1, T_2) \Leftrightarrow \hat{Y}^{\delta}(\{B, A\}, C, T_2, T_1)$$

and

$$\hat{X}^{\delta}(\{A, B\}, C, T_1, T_2) \Leftrightarrow \hat{X}^{\delta}(\{B, A\}, C, T_2, T_1).$$

4.2 Transitivity.

The definition of qualitative influence implies *transitivity*. Transitivity of qualitative influence is an essential property for reasoning with a qualitative network. For the Baseball example, we can deduce that when a ball has an effect, it increases the possibility of running four bases, even though the network does not include an arc $(EB, 4B)$. Again this shows that only a subset of H needs to be defined.

In the following we will need operators to combine the influences on a trail and to combine parallel converging influences. These operators are defined in table 2 and were first introduced to compute the sign of influence upon network reduction, rather than along a trail [12]. Note that when we combine the influences on a trail into one influence, then this influence corresponds to an arc in the network of the *marginal* probability distribution over all variables except the ones on the trail.

$\otimes$	$+$	$-$	0	?
$+$	$+$	$-$	0	?
$-$	$-$	$+$	0	?
0	0	0	0	0
?	?	?	0	?

$\oplus$	$+$	$-$	0	?
$+$	$+$	?	$+$	?
$-$	?	$-$	$-$	?
0	$+$	$-$	0	?
?	?	?	?	?

Table 2: The $\otimes$ and $\oplus$ operators for combining signs.

The following two theorems generalise existing transitivity properties [12, Th. 4.2 & Th. 7.2] and rephrase them to apply trails.

Theorem 4.4 (Transitivity of Binary Qualitative Influence) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, B), (B, C) \in R$ and let $T = \{A, \{B\}, C\}$ be the trail from A via B to C in Q. Let $\otimes$ be the operator as defined in table 2. Then,*

$$\hat{S}^{\delta_1}(A, B, \{A, \varnothing, B\}) \wedge \hat{S}^{\delta_2}(B, C, \{B, \varnothing, C\}) \quad \Rightarrow \quad \hat{S}^{\delta_1 \otimes \delta_2}(A, C, T),$$

for all $\delta_i \in \{+, -, 0, ?\}, i = 1, 2$.

Proof: We will prove the theorem for $\delta_1 = \delta_2 = +$; proofs for other combinations of δ_1 and δ_2 are analogous. The relations between the variables involved in the proof are shown in the figure below.

We assume that $\hat{S}^+(A, B, \{A, \varnothing, B\})$ and $\hat{S}^+(B, C, \{B, \varnothing, C\})$ hold. By definition these are equivalent to $S^+(A, B)$ and $S^+(B, C)$.

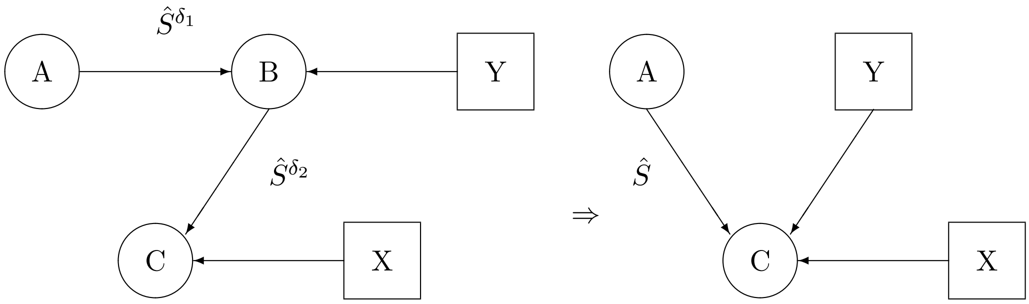
From $S^+(A, B)$ we find

$$\begin{aligned} S^+(A, B) \quad \Leftrightarrow \quad & Pr(b|ay) \geq Pr(b|\bar{a}y) \Leftrightarrow \\ & Pr(b|ay) - Pr(b|\bar{a}y) \geq 0 \end{aligned}$$

for all possible value assignments y to Y, by definition. Analogously, we find

$$S^+(B,C) \Leftrightarrow Pr(c|bx) \geq Pr(c|\bar{b}x) \Leftrightarrow$$
$$Pr(c|bx) - Pr(c|\bar{b}x) \geq 0$$

for all possible value assignments x to X, by definition.



From these observations, we have

$$S^+(A,B) \wedge S^+(B,C) \Rightarrow$$

$$\Big(Pr(c|bx) - Pr(c|\bar{b}x)\Big)\Big(Pr(b|ay) - Pr(b|\bar{a}y)\Big) \geq 0 \Leftrightarrow$$

$$Pr(c|bx)\Big(Pr(b|ay) - Pr(b|\bar{a}y)\Big) - Pr(c|\bar{b}x)\Big(Pr(b|ay) - Pr(b|\bar{a}y)\Big) \geq 0 \Leftrightarrow$$

$$Pr(c|bx)\Big(Pr(b|ay) - Pr(b|\bar{a}y)\Big) + Pr(c|\bar{b}x)\Big(Pr(\bar{b}|ay) - Pr(\bar{b}|\bar{a}y)\Big) \geq 0$$

By rearranging terms, we find

$$Pr(c|bx)\Big(Pr(b|ay) - Pr(b|\bar{a}y)\Big) + Pr(c|\bar{b}x)\Big(Pr(\bar{b}|ay) - Pr(\bar{b}|\bar{a}y)\Big) \geq 0 \Leftrightarrow$$

$$Pr(c|bx)Pr(b|ay) + Pr(c|\bar{b}x)Pr(\bar{b}|ay) -$$
$$Pr(c|bx)Pr(b|\bar{a}y) - Pr(c|\bar{b}x)Pr(\bar{b}|\bar{a}y) \geq 0$$

Now observe that B is independent of X and that C is independent of Y

and A given B. From these observations we find

$$\begin{aligned}
&Pr(c|bx)Pr(b|ay) + Pr(c|\bar{b}x)Pr(\bar{b}|ay) - \\
&Pr(c|bx)Pr(b|\bar{a}y) - Pr(c|\bar{b}x)Pr(\bar{b}|\bar{a}y) \geq 0 && \Leftrightarrow \\
&Pr(c|abxy)Pr(b|axy) + Pr(c|a\bar{b}xy)Pr(\bar{b}|axy) \\
&-Pr(c|\bar{a}bxy)Pr(b|\bar{a}xy) - Pr(c|\bar{a}\bar{b}xy)Pr(\bar{b}|\bar{a}xy) \geq 0 && \Leftrightarrow \\
&Pr(c|axy) - Pr(c|\bar{a}xy) \geq 0 && \Leftrightarrow \\
&\hat{S}^{+}(A, C, T)
\end{aligned}$$

by definition.

□

Note that the above theorem also holds for *trails* from the variable A to the variable B and from B to the variable C. We would then get:

$$\hat{S}^{\delta_1}(A, B, T_1) \wedge \hat{S}^{\delta_2}(B, C, T_2) \Rightarrow \hat{S}^{\delta_1 \otimes \delta_2}(A, C, T'),$$

where T_1 is a trail from A to B, T_2 is a trail from B to C and T' is the concatenation of the trails T_1 and T_2.

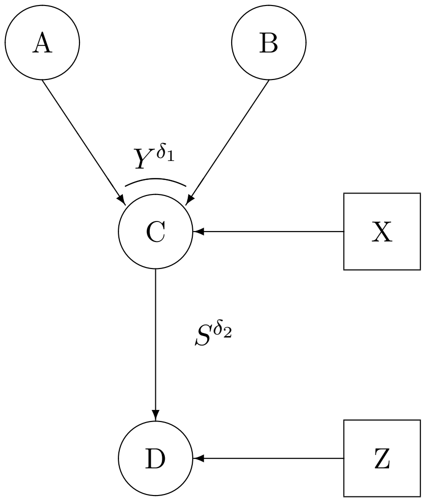
The property of transitivity also holds for additive synergy, when looked upon as extending qualitative synergy through qualitative influences. So, if certain variables exhibit a synergy on a variable V_i and we have $S^{\delta}(V_i, V_j)$, then we can "push down" that synergy to apply to the variable V_j.

Theorem 4.5 (Transitivity of Binary Additive Synergy) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C, D \in V$ such that (A, C), (B, C), $(C, D) \in R$ and let $T_1 = \{A, \{C\}, D\}$ and $T_2 = \{B, \{C\}, D\}$ be the trails from A and B to D in Q. Let $\otimes$ be the operator as defined in table 2. Then,*

$$Y^{\delta_1}(\{A, B\}, C) \wedge S^{\delta_2}(C, D) \Rightarrow \hat{Y}^{\delta_1 \otimes \delta_2}(\{A, B\}, D, T_1, T_2),$$

for all $\delta_i \in \{+, -, 0, ?\}, i = 1, 2$.

Proof: We will prove the theorem for $\delta_1 = \delta_2 = +$. Proofs for the other combinations of signs of δ_i are analogous. The relations between the variables used in the proof are shown in the figure below.



We assume that $Y^{\delta_1}(\{A,B\},C)$ and $S^{\delta_2}(C,D)$ hold.

From $Y^{\delta_1}(\{A,B\},C)$ we find

$$\begin{aligned} Y^{\delta_1}(\{A,B\},C) &\Leftrightarrow Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|\bar{a}bx) + Pr(c|a|nietbx) \\ &\Leftrightarrow Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - \\ &\quad\;\; Pr(c|\bar{a}bx) - Pr(c|a|nietbx) \geq 0 \end{aligned}$$

for all possible value assignments x to X, by definition. Analogously, we find

$$\begin{aligned} S^{+}(C,D) &\Leftrightarrow Pr(d|cz) \geq Pr(d|\bar{c}z) \\ &\Leftrightarrow Pr(d|cz) - Pr(d|\bar{c}z) \geq 0 \end{aligned}$$

for all possible value assignments z to Z, by definition. From these observations, we have

$$Y^{+}(\{A,B\},C) \wedge S^{+}(C,D) \quad \Rightarrow$$

$$\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - Pr(c|\bar{a}bx) - Pr(c|a\bar{b}x)\Big)$$
$$\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big) \geq 0 \quad \Leftrightarrow$$

$$Pr(d|cz)\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x) - Pr(c|\bar{a}bx)\Big) -$$
$$Pr(d|\bar{c}z)\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x) - Pr(c|\bar{a}bx)\Big) \geq 0 \quad \Leftrightarrow$$

$$Pr(d|cz)\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x) - Pr(c|\bar{a}bx)\Big) +$$
$$Pr(d|\bar{c}z)\Big(Pr(\bar{c}|abx) + Pr(\bar{c}|\bar{a}\bar{b}x) - Pr(\bar{c}|a\bar{b}x) - Pr(\bar{c}|\bar{a}bx)\Big) \geq 0$$

By rearranging terms, we find

$$Pr(d|cz)\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x) - Pr(c|\bar{a}bx)\Big) +$$
$$Pr(d|\bar{c}z)\Big(Pr(\bar{c}|abx) + Pr(\bar{c}|\bar{a}\bar{b}x) - Pr(\bar{c}|a\bar{b}x) - Pr(\bar{c}|\bar{a}bx)\Big) \geq 0 \quad \Leftrightarrow$$

$$\begin{array}{l} Pr(d|cz)Pr(c|abx) + Pr(d|\bar{c}z)Pr(\bar{c}|abx) + \\ Pr(d|cz)Pr(c|\bar{a}\bar{b}x) + Pr(d|\bar{c}z)Pr(\bar{c}|\bar{a}\bar{b}x) - \\ Pr(d|cz)Pr(c|a\bar{b}x) - Pr(d|\bar{c}z)Pr(\bar{c}|a\bar{b}x) - \\ Pr(d|cz)Pr(c|\bar{a}bx) - Pr(d|\bar{c}z)Pr(\bar{c}|\bar{a}bx) \geq 0 \end{array}$$

Now observe that D is independent of A, B and X given C and that C is independent of Z. From these observations we find

$$\begin{array}{l} Pr(d|abcxz)Pr(c|abxz) + Pr(d|ab\bar{c}xz)Pr(\bar{c}|abxz) + \\ Pr(d|\bar{a}\bar{b}cxz)Pr(c|\bar{a}\bar{b}xz) + Pr(d|\bar{a}\bar{b}\bar{c}xz)Pr(\bar{c}|\bar{a}\bar{b}xz) - \\ Pr(d|a\bar{b}cxz)Pr(c|a\bar{b}xz) - Pr(d|a\bar{b}\bar{c}xz)Pr(\bar{c}|a\bar{b}xz) - \\ Pr(d|\bar{a}bcxz)Pr(c|\bar{a}bxz) - Pr(d|\bar{a}b\bar{c}xz)Pr(\bar{c}|\bar{a}bxz) \geq 0 \quad \Leftrightarrow \end{array}$$

$$Pr(d|abxz) + Pr(d|\bar{a}\bar{b}xz) - Pr(d|a\bar{b}xz) - Pr(d|\bar{a}bxz) \geq 0 \quad \Leftrightarrow$$

$$\hat{Y}^{+}(\{A, B\}, D, T_1, T_2)$$

by definition.

□

Note that transitivity also holds in the situation where there is an edge *above* a variable B that has a synergy with variable C on variable D. This is,

$$S^{\delta_1}(A, B) \wedge Y^{\delta_2}(\{B, C\}, D) \Rightarrow \hat{Y}^{\delta_1 \otimes \delta_2}(\{A, C\}, D, \{A, \{B\}, D\}, \{C, \varnothing, D\}).$$

4.3 Composition.

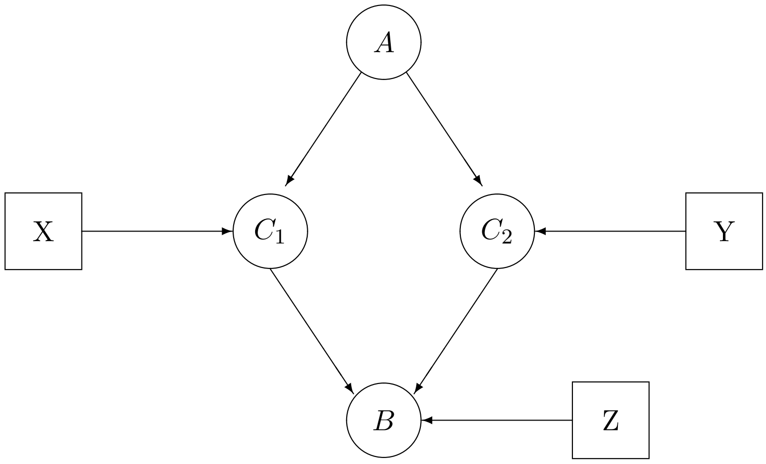
Besides combining influences *on* trails, we can also combine influences among *several parallel trails* into a single influence. The following two theorems generalise existing composition properties [12, Th. 4.3 & Th. 7.3] and rephrase them to apply trails.

Theorem 4.6 (Combining Influences) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B \in V$ and let T_1 and T_2 be two different trails from A to B. Let $\otimes$ and $\oplus$ be the operators as defined in table 2. Then,*

$$\hat{S}^{\delta_1}(A, B, T_1) \wedge \hat{S}^{\delta_2}(A, B, T_2) \quad \Rightarrow \quad \hat{S}^{\delta_1 \oplus \delta_2}(A, B, T_1 \cup T_2),$$

for all $\delta_i \in \{+, -, 0, ?\}, i = 1, 2$.

Proof: We will prove the theorem for $\delta_1 = \delta_2 = +$. Proofs for the other combinations of the signs of δ_i are analogous. The relations between the variables used in the proof are shown in the figure below.



We assume that $\hat{S}^+(A, B, T_1)$ and $\hat{S}^+(A, B, T_2)$ hold. From $\hat{S}^+(A, B, T_1)$ we find

$$\begin{aligned} \hat{S}^+(A, B, T_1) &\Leftrightarrow & Pr(b|ac_2xz) \geq Pr(b|\bar{a}c_2xz) \\ &\wedge & Pr(b|a\bar{c_2}xz) \geq Pr(b|\bar{a}\bar{c_2}xz) \end{aligned}$$

for all possible value assignments x, z to X and Z, respectively, by definition. Analogously, from $\hat{S}^+(A, B, T_2)$ we find

$$\begin{aligned} \hat{S}^+(A, B, T_2) &\Leftrightarrow & Pr(b|ac_1yz) \geq Pr(b|\bar{a}c_1yz) \\ &\wedge & Pr(b|a\bar{c_1}yz) \geq Pr(b|\bar{a}\bar{c_1}yz) \end{aligned}$$

for all possible value assignments y, z to Y and Z, respectively, by definition. We will now prove these both imply

$$Pr(b|axyz) - Pr(b|\bar{a}xyz) \geq 0 \quad \Leftrightarrow \quad \hat{S}^+(A, B, T_1 \cup T_2),$$

so their summation implies

$$2(Pr(b|axyz) - Pr(b|\bar{a}xyz)) \geq 0 \quad \Leftrightarrow \quad \hat{S}^+(A, B, T_1 \cup T_2),$$

which will prove the theorem. Since the proof using $\hat{S}^+(A, B, T_1)$ is analogous to the proof using $\hat{S}^+(A, B, T_2)$, we will only show the latter.

If the resulting sign on a trail is positive then either the first and last part of the trail both have a positive sign, or both have a negative sign.

$$\begin{aligned} \hat{S}^{+}(A,B,T_2) \;\Leftrightarrow\; & S^{+}(A,C_1) \wedge S^{+}(C_1,B) \\ \vee\; & S^{-}(A,C_1) \wedge S^{-}(C_1,B) \end{aligned}$$

We will only consider the case where both signs are positive; the case where both signs are negative is analogous.

$$\begin{aligned} S^{+}(A,C_1) \wedge S^{+}(C_1,B) \;\Leftrightarrow\; & Pr(c_1|ax) - Pr(c_1|\bar{a}x) \geq 0 \\ \wedge\; & Pr(b|ac_1yz) - Pr(b|a\bar{c_1}yz) \geq 0 \\ \wedge\; & Pr(b|\bar{a}c_1yz) - Pr(b|\bar{a}\bar{c_1}yz) \geq 0 \end{aligned}$$

Variable C_1 is independent of the variables Y and Z; variable B is independent of variable X given variable C_1. From the observations above, it can be shown that

$$\begin{aligned} & S^{+}(A,C_1) \wedge S^{+}(C_1,B) \quad \Rightarrow \\ & Pr(c_1|axyz)\Big(Pr(b|ac_1xyz) - Pr(b|a\bar{c_1}xyz)\Big) + Pr(b|a\bar{c_1}xyz) - \\ & \Big(Pr(c_1|\bar{a}xyz)\Big(Pr(b|\bar{a}c_1xyz) - Pr(b|\bar{a}\bar{c_1}xyz)\Big) + Pr(b|\bar{a}\bar{c_1}xyz)\Big) \geq 0 \end{aligned}$$

Rearranging, we find

$$S^{+}(A,C_1) \wedge S^{+}(C_1,B) \quad \Rightarrow$$

$$\begin{aligned} & Pr(c_1|axyz)\Big(Pr(b|ac_1xyz) - Pr(b|a\bar{c_1}xyz)\Big) + Pr(b|a\bar{c_1}xyz) - \\ & \Big(Pr(c_1|\bar{a}xyz)\Big(Pr(b|\bar{a}c_1xyz) - Pr(b|\bar{a}\bar{c_1}xyz)\Big) + Pr(b|\bar{a}\bar{c_1}xyz)\Big) \geq 0 \quad \Leftrightarrow \end{aligned}$$

$$\begin{aligned} & Pr(b|ac_1xyz)Pr(c_1|axyz) + Pr(b|a\bar{c_1}xyz) - \\ & Pr(b|a\bar{c_1}xyz)Pr(c_1|axyz) - Pr(b|\bar{a}c_1xyz)Pr(c_1|\bar{a}xyz) - \\ & Pr(b|\bar{a}\bar{c_1}xyz) + Pr(b|\bar{a}\bar{c_1}xyz)Pr(c_1|\bar{a}xyz) \geq 0 \quad \Leftrightarrow \end{aligned}$$

$$\begin{aligned} & Pr(b|ac_1xyz)Pr(c_1|axyz) + Pr(b|a\bar{c_1}xyz)Pr(\bar{c_1}|axyz) - \\ & Pr(b|\bar{a}c_1xyz)Pr(c_1|\bar{a}xyz) - Pr(b|\bar{a}\bar{c_1}xyz)Pr(\bar{c_1}|\bar{a}xyz) \geq 0 \quad \Leftrightarrow \end{aligned}$$

$$Pr(b|axyz) - Pr(b|\bar{a}xyz) \geq 0 \quad \Leftrightarrow$$

$$\hat{S}^{+}(A,B,T_1 \cup T_2)$$

by definition.

□

When a variable C can be reached by different trails from two of C's predecessors, then an additive synergy exists for each pair of these trails (providing these trails leave from different predecessors). All these additive synergies can be combined into one.

Theorem 4.7 (Combining Additive Synergies) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C, D \in V$. Let T_1 and W_1 be two different trails from A to D and let T_2, W_2 be two different trails from B to D. Let $\otimes$ and $\oplus$ be operators as defined by table 2. Then,*

$$\hat{Y}^{\delta_1}(\{A,B\}, D, T_1, T_2) \wedge \hat{Y}^{\delta_2}(\{A,B\}, D, W_1, W_2) \wedge$$
$$\hat{Y}^{\delta_3}(\{A,B\}, D, T_1, W_2) \wedge \hat{Y}^{\delta_4}(\{A,B\}, D, T_2, W_1) \quad \Rightarrow$$
$$\hat{Y}^{\delta_1 \oplus \delta_2 \oplus \delta_3 \oplus \delta_4}(\{A,B\}, D, T_1 \cup W_1, T_2 \cup W_2),$$

for all $\delta_i \in \{+, -, 0, ?\}, i = 1, \ldots, 7$.

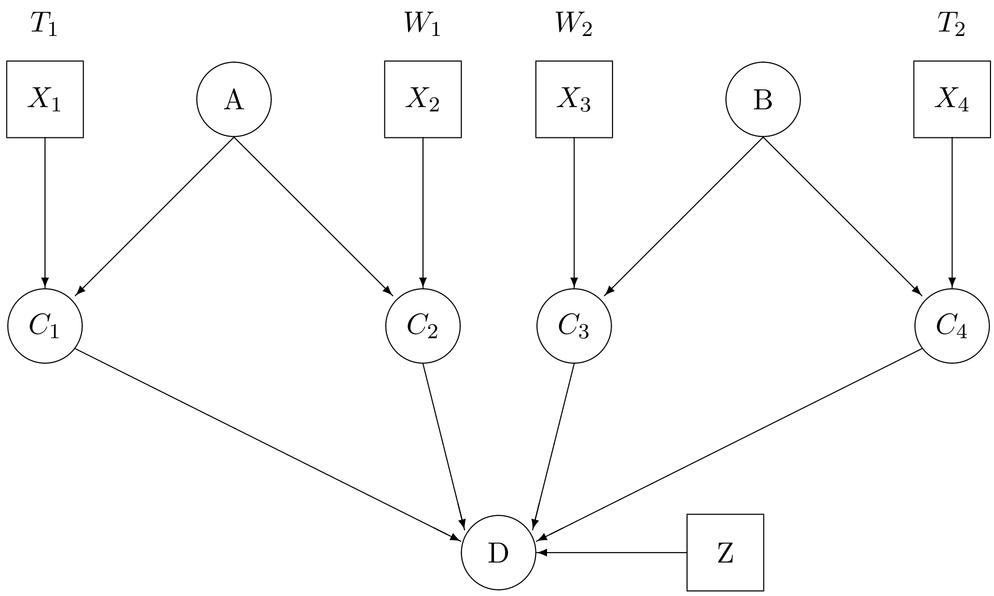
Proof: We will prove the theorem for $\delta_i = +$. Proofs for the other combinations of signs for δ_i are analogous. The relations between the variables used in the proof are shown in the figure below.

We assume that $\hat{Y}^{+}(\{A,B\}, D, T_1, T_2)$, $\hat{Y}^{+}(\{A,B\}, D, W_1, W_2)$, $\hat{Y}^{+}(\{A,B\}, D, T_1, W_2)$ and $\hat{Y}^{+}(\{A,B\}, D, T_2, W_1)$ hold.

From $\hat{Y}^{+}(\{A,B\}, D, T_1, T_2)$ we find

$$\hat{Y}^{+}(\{A,B\}, D, T_1, T_2) \quad \Leftrightarrow \quad Pr(d|abC_2C_3x_1x_4z) + Pr(d|\bar{a}\bar{b}C_2C_3x_1x_4z) \geq Pr(d|\bar{a}bC_2C_3x_1x_4z) + Pr(d|a\bar{b}C_2C_3x_1x_4z)$$

for all possible value assignments x_1, x_4, z to X_1, X_4 and Z, respectively, and for all possible value assignments to C_2 and C_3, by definition.



Analogously, from $\hat{Y}^+(\{A,B\},D,W_1,W_2)$ we find

$$\hat{Y}^+(\{A,B\},D,W_1,W_2) \Leftrightarrow Pr(d|abC_1C_4x_2x_3z) + Pr(d|\bar{a}\bar{b}C_1C_4x_2x_3z) \geq Pr(d|\bar{a}bC_1C_4x_2x_3z) + Pr(d|a\bar{b}C_1C_4x_2x_3z)$$

for all possible value assignments x_2, x_3, z to X_2, X_3 and Z, respectively, and for all possible value assignments to C_1 and C_4, by definition. Analogously, from $\hat{Y}^+(\{A,B\},D,T_1,W_2)$ we find

$$\hat{Y}^+(\{A,B\},D,T_1,W_2) \Leftrightarrow Pr(d|abC_2C_4x_1x_3z) + Pr(d|\bar{a}\bar{b}C_2C_4x_1x_3z) \geq Pr(d|\bar{a}bC_2C_4x_1x_3z) + Pr(d|a\bar{b}C_2C_4x_1x_3z)$$

for all possible value assignments x_1, x_3, z to X_1, X_3 and Z, respectively, and for all possible value assignments to C_2 and C_4, by definition. Analogously, from $\hat{Y}^+(\{A,B\},D,T_2,W_1)$ we find

$$\hat{Y}^+(\{A,B\},D,T_2,W_1) \Leftrightarrow Pr(d|abC_1C_3x_2x_4z) + Pr(d|\bar{a}\bar{b}C_1C_3x_2x_4z) \geq Pr(d|\bar{a}bC_1C_3x_2x_4z) + Pr(d|a\bar{b}C_1C_3x_2x_4z)$$

for all possible value assignments x_2, x_4, z to X_2, X_4 and Z, respectively, and for all possible value assignments to C_1 and C_3, by definition.

Now consider $\hat{Y}^+(\{A,B\},D,W_1,W_2)$. Using transitivity of additive synergy, the following should hold.

$$\begin{aligned}\hat{Y}^+(\{A,B\},D,W_1,W_2) \Leftrightarrow\ & Y^+(\{C_2,C_3\},D) \wedge S^+(A,C_2) \wedge S^+(B,C_3)\\ \vee\ & Y^+(\{C_2,C_3\},D) \wedge S^-(A,C_2) \wedge S^-(B,C_3)\\ \vee\ & Y^-(\{C_2,C_3\},D) \wedge S^+(A,C_2) \wedge S^-(B,C_3)\\ \vee\ & Y^-(\{C_2,C_3\},D) \wedge S^-(A,C_2) \wedge S^+(B,C_3)\end{aligned}$$

We will now consider the case where all signs are positive.

$$Y^+(\{C_2,C_3\},D) \wedge S^+(A,C_2) \wedge S^+(B,C_3) \quad \Leftrightarrow$$

$$\begin{aligned}& Pr(d|ABc_2c_3x_1x_4) + Pr(d|AB\bar{c_2}\bar{c_3}x_1x_4) -\\ & Pr(d|AB\bar{c_2}c_3x_1x_4) - Pr(d|ABc_2\bar{c_3}x_1x_4) \geq 0\\ \wedge\ & Pr(c_2|ax_2) - Pr(c_2|\bar{a}x_2) \geq 0\\ \wedge\ & Pr(c_3|bx_3) - Pr(c_3|\bar{b}x_3) \geq 0,\end{aligned}$$

for all value assignments $c_2, c_3, x_1, x_2, x_3, x_4$ to C_2, C_3, X_1, X_2, X_3 and X_4 and for all possible value assignments to A and B, by definition.

C_2 is independent of B, X_1, X_3 and X_4; C_3 is independent of A, X_1, X_2 and X_4 and D is independent of X_2, X_3 given C_2 and C_3, respectively. From all the above observations, it can be shown that

$$\begin{aligned}
&Pr(c_2|abx_1x_2x_3x_4)\Big(Pr(c_3|abc_2x_1x_2x_3x_4)[\\
&Pr(d|abc_2c_3x_1x_2x_3x_4) - Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4)]+\\
&Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)[\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4) - Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)]+\\
&Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4) - Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)\Big)+\\
&Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)-\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4) \qquad \geq 0
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&Pr(d|abc_2c_3x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|abc_2x_1x_2x_3x_4)+\\
&Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)-\\
&Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|abc_2x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)-\\
&Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4) - Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)-\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4) \geq 0 \qquad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|abc_2c_3x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|abc_2x_1x_2x_3x_4)+\\
&Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)\Big(1 - Pr(c_3|abc_2x_1x_2x_3x_4)\Big)+\\
&Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)\Big(1 - Pr(c_2|abx_1x_2x_3x_4)\Big)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)\Big(1 - Pr(c_2|abx_1x_2x_3x_4)\Big)\\
&\Big(1 - Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)\Big) \geq 0 \qquad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|abc_2c_3x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(c_3|abc_2x_1x_2x_3x_4)+\\
&Pr(d|abc_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)Pr(\bar{c_3}|abc_2x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}c_3x_1x_2x_3x_4)Pr(\bar{c_2}|abx_1x_2x_3x_4)Pr(c_3|ab\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(\bar{c_2}|abx_1x_2x_3x_4)Pr(\bar{c_3}|ab\bar{c_2}x_1x_2x_3x_4) \geq 0 \qquad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|abc_2x_1x_2x_3x_4)Pr(c_2|abx_1x_2x_3x_4)+\\
&Pr(d|ab\bar{c_2}x_1x_2x_3x_4)Pr(\bar{c_2}|abx_1x_2x_3x_4) \geq 0 \qquad \Leftrightarrow
\end{aligned}$$

$$Pr(d|abx_1x_2x_3x_4) \geq 0$$

In a similar way, we find

$$Y^+(\{C_2, C_3\}, D) \wedge S^-(A, C_2) \wedge S^-(B, C_3) \Rightarrow$$

$$\begin{aligned}
&Pr(d|\bar{a}\bar{b}c_2c_3x_1x_2x_3x_4)Pr(c_2|\bar{a}\bar{b}x_1x_2x_3x_4)Pr(c_3|\bar{a}\bar{b}c_2x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}\bar{b}c_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|\bar{a}\bar{b}x_1x_2x_3x_4)Pr(\bar{c_3}|\bar{a}\bar{b}c_2x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}\bar{b}\bar{c_2}c_3x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}\bar{b}x_1x_2x_3x_4)Pr(c_3|\bar{a}\bar{b}\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}\bar{b}\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}\bar{b}x_1x_2x_3x_4)Pr(\bar{c_3}|\bar{a}\bar{b}\bar{c_2}x_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|\bar{a}\bar{b}c_2x_1x_2x_3x_4)Pr(c_2|\bar{a}\bar{b}x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}\bar{b}\bar{c_2}x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}\bar{b}x_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$Pr(d|\bar{a}\bar{b}x_1x_2x_3x_4) \geq 0$$

and

$$Y^-(\{C_2, C_3\}, D) \wedge S^-(A, C_2) \wedge S^+(B, C_3) \Rightarrow$$

$$\begin{aligned}
&Pr(d|\bar{a}bc_2c_3x_1x_2x_3x_4)Pr(c_2|\bar{a}bx_1x_2x_3x_4)Pr(c_3|\bar{a}bc_2x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}bc_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|\bar{a}bx_1x_2x_3x_4)Pr(\bar{c_3}|\bar{a}bc_2x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}b\bar{c_2}c_3x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}bx_1x_2x_3x_4)Pr(c_3|\bar{a}b\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|\bar{a}b\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}bx_1x_2x_3x_4)Pr(\bar{c_3}|\bar{a}b\bar{c_2}x_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|\bar{a}bc_2x_1x_2x_3x_4)Pr(c_2|\bar{a}bx_1x_2x_3x_4)+\\
&Pr(d|\bar{a}b\bar{c_2}x_1x_2x_3x_4)Pr(\bar{c_2}|\bar{a}bx_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$Pr(d|\bar{a}bx_1x_2x_3x_4) \geq 0$$

and

$$Y^-(\{C_2, C_3\}, D) \wedge S^+(A, C_2) \wedge S^-(B, C_3) \Rightarrow$$

$$\begin{aligned}
&Pr(d|a\bar{b}c_2c_3x_1x_2x_3x_4)Pr(c_2|a\bar{b}x_1x_2x_3x_4)Pr(c_3|a\bar{b}c_2x_1x_2x_3x_4)+\\
&Pr(d|a\bar{b}c_2\bar{c_3}x_1x_2x_3x_4)Pr(c_2|a\bar{b}x_1x_2x_3x_4)Pr(\bar{c_3}|a\bar{b}c_2x_1x_2x_3x_4)+\\
&Pr(d|a\bar{b}\bar{c_2}c_3x_1x_2x_3x_4)Pr(\bar{c_2}|a\bar{b}x_1x_2x_3x_4)Pr(c_3|a\bar{b}\bar{c_2}x_1x_2x_3x_4)+\\
&Pr(d|a\bar{b}\bar{c_2}\bar{c_3}x_1x_2x_3x_4)Pr(\bar{c_2}|a\bar{b}x_1x_2x_3x_4)Pr(\bar{c_3}|a\bar{b}\bar{c_2}x_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&Pr(d|a\bar{b}c_2x_1x_2x_3x_4)Pr(c_2|a\bar{b}x_1x_2x_3x_4)+\\
&Pr(d|a\bar{b}\bar{c_2}x_1x_2x_3x_4)Pr(\bar{c_2}|a\bar{b}x_1x_2x_3x_4) \geq 0 \quad \Leftrightarrow
\end{aligned}$$

$$Pr(d|a\bar{b}x_1x_2x_3x_4) \geq 0$$

It can be shown that summing the first two and subtracting the last two still results in a positive result, therefore

$$\begin{aligned}
&Pr(d|abx_1x_2x_3x_4) + Pr(d|\bar{a}\bar{b}x_1x_2x_3x_4)-\\
&Pr(d|\bar{a}bx_1x_2x_3x_4) - Pr(d|a\bar{b}x_1x_2x_3x_4) \quad \geq 0
\end{aligned}$$

The other conditions of the theorem give this same result, so summing them we get

$$4\Big(Pr(d|abx_1x_2x_3x_4) + Pr(d|\bar{a}\bar{b}x_1x_2x_3x_4) - Pr(d|\bar{a}bx_1x_2x_3x_4) - Pr(d|a\bar{b}x_1x_2x_3x_4)\Big) \geq 0 \quad \Leftrightarrow$$

$$\hat{Y}^{+}(\{A,B\}, D, T_1 \cup W_1, T_2 \cup W_2)$$

by definition.

□

4.4 Influences and Synergies.

When some of the influences are zero or unknown, you can say the same thing for the synergies involved. This is rather obvious, since a zero influence between variable A and variable B means that the values of A and B are independent of each other, so the *joint* influence of A and another variable on B will also be zero. We include the following theorems for the sake of completeness.

δ	$\hat{S}^0(A,C,T_1)$	$\hat{S}^+(A,C,T_1)$	$\hat{S}^-(A,C,T_1)$	$\hat{S}^?(A,C,T_1)$
$\hat{S}^0(B,C,T_2)$	0	0	0	0
$\hat{S}^+(B,C,T_2)$	0	$0,+,-,?$	$0,+,-,?$	?
$\hat{S}^-(B,C,T_2)$	0	$0,+,-,?$	$0,+,-,?$	?
$\hat{S}^?(B,C,T_2)$	0	?	?	?

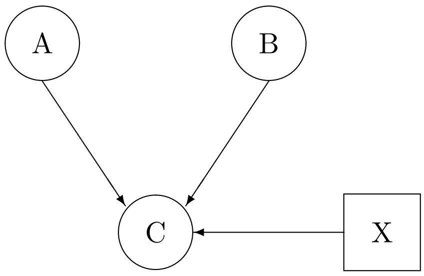
Table 3: The possible signs of δ in $\hat{Y}^{\delta}(\{A,B\},C,T_1,T_2)$ and $\hat{X}^{\delta}(\{A,B\},c,T_1,T_2)$, given the signs of the influences $\hat{S}(A,C,T_1)$ and $\hat{S}(B,C,T_2)$.

Theorem 4.8 (Influences and Additive Synergy) *Let $Q = (V,R,H)$ be a qualitative probabilistic network. Let $A,B,C \in V$ and let $T_1 = \{A,\varnothing,C\}$ and $T_2 = \{B,\varnothing,C\}$ be the trails from A to C and B to C, respectively. Then,*

$$\hat{S}^{\delta_1}(A,C,T_1) \wedge \hat{S}^{\delta_2}(B,C,T_2) \Rightarrow \hat{Y}^{\delta_3}(\{A,B\},C,T_1,T_2),$$

with the possible combinations of $\delta_i, i = 1,2,3$ as in Table 3.

Proof: We will prove the theorem for $\delta_1 = +$ and $\delta_2 = 0$. Proofs for the other combinations of δ_1 and δ_2 are analogous. The relations between the variables used in the proof are shown in the figure below.



We assume that $\hat{S}^+(A, C, T_1)$ and $\hat{S}^0(B, C, T_2)$ hold. By definition, we have

$$\begin{array}{rcl} \hat{S}^+(A, C, T_1) & \Leftrightarrow & Pr(c|abx) \geq Pr(c|\bar{a}bx) \\ & \wedge & Pr(c|a\bar{b}x) \geq Pr(c|\bar{a}\bar{b}x) \end{array}$$

and

$$\begin{array}{rcl} \hat{S}^0(B, C, T_2) & \Leftrightarrow & Pr(c|abx) = Pr(c|a\bar{b}x) \\ & \wedge & Pr(c|\bar{a}bx) = Pr(c|\bar{a}\bar{b}x) \end{array}$$

for all value assignments x to X. From this we can conclude

$$Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) = Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx) \Leftrightarrow \hat{Y}^0(\{A, B\}, C, T_1, T_2)$$

by definition.

$\square$

Note that the above theorem can be generalised to all sorts of trails from variable A and variable B to variable C, using the theorems 4.4 and 4.5.

Theorem 4.9 (Influences and Product Synergy) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ and let $T_1 = \{A, \varnothing, C\}$ and $T_2 = \{B, \varnothing, C\}$ be the trails from A to C and B to C, respectively. Then,*

$$\hat{S}^{\delta_1}(A, C, T_1) \wedge \hat{S}^{\delta_2}(B, C, T_2) \Rightarrow \hat{X}^{\delta_3}(\{A, B\}, c, T_1, T_2),$$

with the possible combinations of $\delta_i, i = 1, 2, 3$ as in Table 3.

Proof: We will prove the theorem for $\delta_1 = +$ and $\delta_2 = 0$. Proofs for the other combinations of δ_1 and δ_2 are analogous. The relations between the variables used in the proof are shown in the figure by the proof of theorem 4.8.

We assume that $\hat{S}^+(A, C, T_1)$ and $\hat{S}^0(B, C, T_2)$ hold. By definition, we have

$$\begin{aligned} \hat{S}^+(A, C, T_1) &\Leftrightarrow Pr(c|abx) \geq Pr(c|\bar{a}bx) \\ &\wedge \; Pr(c|a\bar{b}x) \geq Pr(c|\bar{a}\bar{b}x) \end{aligned}$$

and

$$\begin{aligned} \hat{S}^0(B, C, T_2) &\Leftrightarrow Pr(c|abx) = Pr(c|a\bar{b}x) \\ &\wedge \; Pr(c|\bar{a}bx) = Pr(c|\bar{a}\bar{b}x) \end{aligned}$$

for all value assignments x to X. From this we can conclude

$$Pr(c|abx)Pr(c|\bar{a}\bar{b}x) = Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) \Leftrightarrow \hat{X}^0(\{A, B\}, c, T_1, T_2)$$

by definition.

□

Note that the above theorem can be generalised to all sorts of trails from variable A and variable B to variable C, using theorem 4.4.

4.5 Intercausal Reasoning.

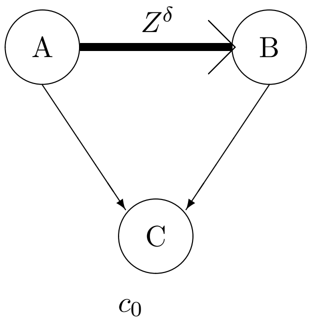


Figure 6: Intercausal Inference

Observing a common effect of two causes can induce a dependence between those causes and therefore a qualitative influence. This is shown in figure 6 and explained in the following example.

Example 4.10 *Let's consider once more the baseball example and the following situation: you observe a player running four bases. Now, suppose that you know he is an extremely good batter. Then, he has probably hit the* *ball beyond the reach of any fieldplayer. The other cause of him running four bases, that is, by field errors, becomes less likely. Thus, knowing the player is a good batter, the probability of field errors allowing a homerun becomes less than the a-priori probability of field errors allowing a homerun.*

This common and intuitively compelling pattern of reasoning is called *explaining away* [14], because one cause explains the observed effect and reduces the need to invoke other causes. Explaining away is an example of *intercausal inference*, that is, reasoning between two causes with a common effect, in contrast with pure causal reasoning.

To capture the influence between two causes, we introduce a new type of qualitative influence, also called *intercausal influence*, which will be denoted by $Z^\delta(\{A,B\},c_0)$. This qualitative influence basically has the same meaning as $S^\delta(A,B)$ *on observation of the value* c_0 for the variable C. In case of explaining away, the influence on this hyperedge is negative.

Definition 4.11 (Binary Intercausal Influence) *Let* $Q=(V,R,H)$ *be a qualitative probabilistic network. Let* $A,B,C \in V$ *such that* (A,C), $(B,C) \in R$ *and let* $X = \pi(B)\backslash\{A\} \cup \pi(C)\backslash\{A,B\}$. *Then, the* intercausal influence *between variable* A *and variable* B *upon observation of the value* c_0 *for variable* C is positive, *written* $Z^+(\{A,B\},c_0) \in H$, *iff for the value* b *of* B *and the values* a, $\bar{a}$ *of* A, *we have*

$$Pr(b|ac_0x) \geq Pr(b|\bar{a}c_0x)$$

for all possible value assignments x *to* X. *Negative intercausal influence and zero intercausal influence are defined analogously, where* $\geq$ *is replaced by* $\leq$ *and* $=$, *respectively. When the relationship between the variables* A *and* B *is neither* $+,-$ *nor* 0, *then we write* $Z^?(\{A,B\},c_0)$.

In Figure 6 the hyperedge from A to B expresses an intercausal influence $Z^\delta(\{A,B\},c_0)$.

As for all the other qualitative relations, we can define an intercausal influence on a trail.

Definition 4.12 (Intercausal Influence on a Trail) *Let* $Q=(V,R,H)$ *be a qualitative probabilistic network. Let* $A,B \in V$ *and let* $T=\{A,T',B\}$ *be a trail from* A *to* B *in* Q, *where* $T' \in T$ *is the subtrail of* T. *Then* $\hat{Z}^\delta(\{A,B\},c_0,T)$ *holds in* Q *if and only if* $Z^\delta(\{A,B\},c_0)$ *holds in the marginal network of* Q *omitting every vertex on* T'.

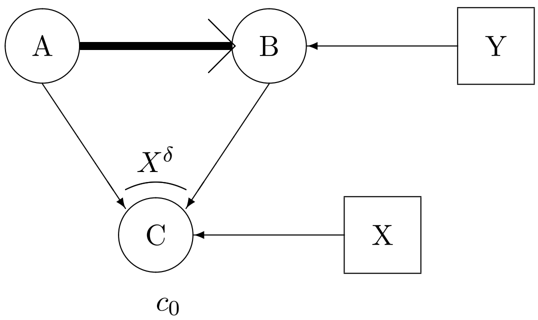
Next, we will present two theorems that show the situations in which an intercausal influence arises. The first theorem rephrases a known result [1, Th. 6.1][5, Th. 3][14, Th. 1] in terms of our newly defined notation for intercausal influence.

Theorem 4.13 (Intercausal Reasoning I) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $T_1 = \{A, \varnothing, C\}$, $T_2 = \{B, \varnothing, C\}$ be trails from variable A to variable C and variable B to variable C, respectively, in Q. Then, for a given value c_0 of the variable C,*

$$\hat{X}^{\delta}(\{A, B\}, c_0, T_1.T_2) \quad \Leftrightarrow \quad Z^{\delta}(\{A, B\}, c_0),$$

for all $\delta \in \{+, -, 0, ?\}$.

Proof: We will prove the theorem for $\delta = +$. Proofs for $\delta = -, 0, ?$ are analogous. The relations between the variables used in the proof are shown in the figure below.



We assume that $\hat{X}^{+}(\{A, B\}, c_0, T_1, T_2)$ holds. From $\hat{X}^{+}(\{A, B\}, c_0, T_1, T_2)$ we find

$$\hat{X}^{+}(\{A, B\}, c_0, T_1, T_2) \Leftrightarrow Pr(c_0|abx)Pr(c_0|\bar{a}\bar{b}x) - Pr(c_0|\bar{a}bx)Pr(c_0|a\bar{b}x) \geq 0$$

for all possible value assignments x to X, by definition. $Pr(\bar{b}|y)$ is always positive for all value assignments y to Y. C is independent of Y, given B. Using these observations, we get

$$Pr(c_0|abx)Pr(c_0|\bar{a}\bar{b}x) - Pr(c_0|\bar{a}bx)Pr(c_0|a\bar{b}x) \geq 0 \qquad \Leftrightarrow$$

$$Pr(\bar{b}|y)\Big(Pr(c_0|abxy)Pr(c_0|\bar{a}\bar{b}xy) - Pr(c_0|\bar{a}bxy)Pr(c_0|a\bar{b}xy)\Big) \geq 0 \quad \Leftrightarrow$$

$$\begin{aligned}&Pr(c_0|abxy)Pr(c_0|\bar{a}\bar{b}xy)Pr(\bar{b}|y)- \\ &Pr(c_0|\bar{a}bxy)Pr(c_0|a\bar{b}xy)Pr(\bar{b}|y)+ \\ &Pr(c_0|abxy)Pr(c_0|\bar{a}bxy)Pr(b|\bar{a}xy)- \\ &Pr(c_0|abxy)Pr(c_0|\bar{a}bxy)Pr(b|\bar{a}xy) \geq 0 \qquad \Leftrightarrow\end{aligned}$$

B is independent of A and X, therefore $Pr(\bar{b}|y) = Pr(\bar{b}|axy) = Pr(\bar{b}|\bar{a}xy)$.

Now use the fact that there is conditioned on B.

$$\begin{aligned}
&Pr(c_0|abxy)Pr(c_0|\bar{a}\bar{b}xy)Pr(\bar{b}|axy)- \\
&Pr(c_0|\bar{a}bxy)Pr(c_0|a\bar{b}xy)Pr(\bar{b}|\bar{a}xy)+ \\
&Pr(c_0|abxy)Pr(c_0|\bar{a}bxy)Pr(b|\bar{a}xy)- \\
&Pr(c_0|\bar{a}bxy)Pr(c_0|abxy)Pr(b|\bar{a}xy) \geq 0 \qquad \Leftrightarrow \\
\\
&Pr(c_0|abxy)Pr(c_0|\bar{a}xy) - Pr(c_0|\bar{a}bxy)Pr(c_0|axy) \geq 0
\end{aligned}$$

$Pr(b|y)$ is always positive and B is independent of A and X, therefore $Pr(b|y) = Pr(b|axy) = Pr(b|\bar{a}xy) \geq 0$. Using this, we get

$$\begin{aligned}
Pr(c_0|abxy)Pr(c_0|\bar{a}xy) - Pr(c_0|\bar{a}bxy)Pr(c_0|axy) \geq 0 \quad &\Leftrightarrow \\
\frac{Pr(c_0|abxy)Pr(c_0|\bar{a}xy)Pr(b|axy) - Pr(c_0|\bar{a}bxy)Pr(c_0|axy)Pr(b|\bar{a}xy)}{Pr(c_0|axy)Pr(c_0|\bar{a}xy)} \geq 0 \quad &\Leftrightarrow \\
\frac{Pr(c_0|abxy)Pr(b|axy)}{Pr(c_0|axy)} - \frac{Pr(c_0|\bar{a}bxy)Pr(b|\bar{a}xy)}{Pr(c_0|\bar{a}xy)} \geq 0 &
\end{aligned}$$

We now use Bayes' theorem and get

$$\begin{aligned}
\frac{Pr(c_0|abxy)Pr(b|axy)}{Pr(c_0|axy)} - \frac{Pr(c_0|\bar{a}bxy)Pr(b|\bar{a}xy)}{Pr(c_0|\bar{a}xy)} \geq 0 \quad &\Leftrightarrow \\
Pr(b|ac_0xy) - Pr(b|\bar{a}c_0xy) \geq 0 \quad &\Leftrightarrow \\
Z^{+}(\{A, B\}, c_0) &
\end{aligned}$$

by definition.

□

Again the above theorem can be generalised to all sorts of trails, using theorem 4.4.

Note that if there already exists a qualitative influence between the variable A and the variable B, we can use theorem 4.6 to combine the influences[1]. The above theorem then results in

$$\hat{S}^{\delta_1}(A, B, T_1) \wedge X^{\delta_2}(\{A, B\}, c_0) \Rightarrow \hat{Z}^{\delta_1 \oplus \delta_2}(\{A, B\}, c_0, T_1 \cup T_2),$$

where T_1 is an arbitrary trail from variable A to variable B and $T_2 = \{A, \varnothing, B\}$ is the intercausal influence that arose upon observing the value c_0 for the variable C.

[1]Remember that Z is basically the same as S

Consider the situation where the two variables A and B both have an influence on variable C and C has an influence on variable D (see the figure in the proof of theorem 4.14). Suppose we observe a value for variable D. By marginalisation of the network over variable C, this observation can cause an intercausal influence between variable A and variable B. To determine the sign of this intercausal influence we need to know the signs of the different relations A and B are involved in before marginalisation.

Theorem 4.14 (Intercausal Reasoning II) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C, D \in V$ and let $T_1 = \{A, \varnothing, C\}$, $T_2 = \{B, \varnothing, C\}$ be trails from A to C and B to C, respectively, and $T_3 = \{C, \varnothing, D\}$ a trail from C to D in Q. Let $\otimes$ and $\oplus$ be the operators as defined in table 2. Then,*

$$\hat{X}^{\delta_1}(\{A, B\}, c, T_1, T_2) \wedge \hat{X}^{\delta_2}(\{A, B\}, \bar{c}, T_1, T_2) \wedge$$

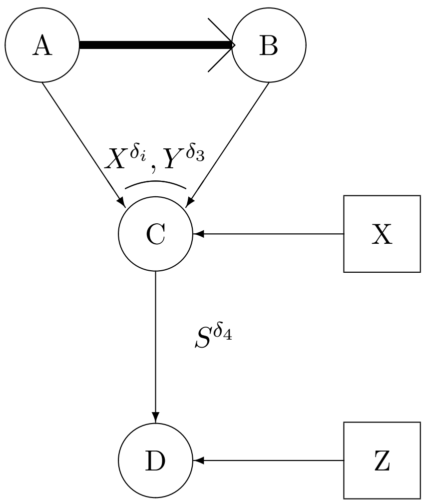
$$\hat{Y}^{\delta_3}(\{A, B\}, C, T_1, T_2) \wedge \hat{S}^{\delta_4}(C, D, T_3) \Rightarrow Z^{\delta_5}(\{A, B\}, d),$$

for all $\delta_i \in \{+, -, 0, ?\}, i = 1, \ldots, 5$, where

$$\delta_5 = \begin{cases} \delta_1 & \textit{if } (\delta_1 = \delta_3 = 0) \quad \vee \quad (\delta_1 = \delta_3 \wedge \delta_4 = +); \\ \delta_2 \oplus (- \otimes \delta_3) & \textit{if } \delta_4 = - \quad \wedge \quad \delta_2 \neq \delta_3 \quad \wedge \quad \delta_3 \neq ? \end{cases}$$

Proof: We prove the theorem for $\delta_1 = \delta_2 = \delta_3 = \delta_4 = +$. Proofs for the other combinations of the values of δ_i are analogous. The relations between the variables used in the proof are shown in the figure below.

We will prove that $\hat{X}^{+}(\{A, B\}, d, T_4, T_5)$ where $T_4 = \{A, \{C\}, D\}$ and $T_5 = \{B, \{C\}, D\}$ are trails from A, respectively B to D in Q. Using theorem 4.13 we then conclude that $Z^{+}(\{A, B\}, d)$ holds.



We assume that $\hat{X}^+(\{A,B\},c,T_1,T_2)$, $\hat{Y}^+(\{A,B\},C,T_1,T_2)$ and $\hat{S}^+(C,D,T_3)$ hold. From $\hat{X}^+(\{A,B\},c,T_1,T_2)$ we find

$$\hat{X}^+(\{A,B\},c,T_1,T_2) \Leftrightarrow Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|\bar{a}bx)Pr(c|a\bar{b}x)$$

for all possible value assignments x to X, by definition. Analogously, we find

$$\hat{Y}^+(\{A,B\},C,T_1,T_2) \Leftrightarrow Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|\bar{a}bx) + Pr(c|a\bar{b}x)$$

for all possible value assignments x to X, by definition. Analogously, we find

$$\hat{S}^+(C,D,T_3) \Leftrightarrow Pr(d|cz) \geq Pr(d|\bar{c}z)$$

for all possible value assignments z to Z, by definition. From these observations we have

$$\begin{aligned} Pr(c|abx)Pr(c|\bar{a}\bar{b}x)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 \quad &\geq \\ Pr(c|\bar{a}bx)Pr(c|a\bar{b}x)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 & \end{aligned}$$

and

$$\begin{aligned} \Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x)\Big)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(d|\bar{c}z) \quad &\geq \\ \Big(Pr(c|\bar{a}bx) + Pr(c|a\bar{b}x)\Big)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(d|\bar{c}z) & \end{aligned}$$

Combining these, we get

$$\begin{aligned} &\hat{X}^+(\{A,B\},c,T_1,T_2) \wedge \hat{Y}^+(\{A,B\},C,T_1,T_2) \wedge \hat{S}^+(C,D,T_3) \qquad \Rightarrow \\ &Pr(c|abx)Pr(c|\bar{a}\bar{b}x)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 + \\ &\Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x)\Big)Pr(d|\bar{c}z)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big) + \\ &Pr(d|\bar{c}z)^2 \qquad \geq \\ &Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 + \\ &\Big(Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)\Big)Pr(d|\bar{c}z)\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big) + \\ &Pr(d|\bar{c}z)^2 \qquad \Leftrightarrow \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 Pr(c|abx)Pr(c|\bar{a}\bar{b}x) + \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(c|abx)Pr(d|\bar{c}z) + \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(c|\bar{a}\bar{b}x)Pr(d|\bar{c}z) + Pr(d|\bar{c}z)^2 \geq \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)^2 Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) + \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(c|a\bar{b}x)Pr(d|\bar{c}z) + \\ &\Big(Pr(d|cz) - Pr(d|\bar{c}z)\Big)Pr(c|\bar{a}bx)Pr(d|\bar{c}z) + Pr(d|\bar{c}z)^2 \qquad \Leftrightarrow \end{aligned}$$

$$
\begin{aligned}
&\Big([Pr(d|cz) - Pr(d|\bar{c}z)]Pr(c|abx) + Pr(d|\bar{c}z)\Big)\\
&\Big([Pr(d|cz) - Pr(d|\bar{c}z)]Pr(c|\bar{a}\bar{b}x) + Pr(d|\bar{c}z)\Big) \geq\\
&\Big([Pr(d|cz) - Pr(d|\bar{c}z)]Pr(c|a\bar{b}x) + Pr(d|\bar{c}z)\Big)\\
&\Big([Pr(d|cz) - Pr(d|\bar{c}z)]Pr(c|\bar{a}bx) + Pr(d|\bar{c}z)\Big) \qquad \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Big(Pr(d|cz)Pr(c|abx) + Pr(d|\bar{c}z) - Pr(d|\bar{c}z)Pr(c|abx)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}\bar{b}x) + Pr(d|\bar{c}z) - Pr(d|\bar{c}z)Pr(c|\bar{a}\bar{b}x)\Big) \geq\\
&\Big(Pr(d|cz)Pr(c|a\bar{b}x) + Pr(d|\bar{c}z) - Pr(d|\bar{c}z)Pr(c|a\bar{b}x)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}bx) + Pr(d|\bar{c}z) - Pr(d|\bar{c}z)Pr(c|\bar{a}bx)\Big) \qquad \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Big(Pr(d|cz)Pr(c|abx) + Pr(d|\bar{c}z)\Big(1 - Pr(c|abx)\Big)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}\bar{b}x) + Pr(d|\bar{c}z)\Big(1 - Pr(c|\bar{a}\bar{b}x)\Big)\Big) \geq\\
&\Big(Pr(d|cz)Pr(c|a\bar{b}x) + Pr(d|\bar{c}z)\Big(1 - Pr(c|a\bar{b}x)\Big)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}bx) + Pr(d|\bar{c}z)\Big(1 - Pr(c|\bar{a}bx)\Big)
\end{aligned}
$$

Now observe that D is independent of A and B given C, x is independent of D given C and z is independent of C. We will now rewrite $Pr(d|abxz)$ by conditioning on C; we will use this rewrite for our next simplification of the above inequality.

$$
\begin{aligned}
Pr(d|abxz) &\Leftrightarrow Pr(d|abcxz)Pr(c|abxz) + Pr(d|ab\bar{c}xz)Pr(\bar{c}|abxz)\\
&\Leftrightarrow Pr(d|cz)Pr(c|abx) + Pr(d|\bar{c}z)Pr(\bar{c}|abx)
\end{aligned}
$$

Now using the rewrite of $Pr(d|abxz)$, we get

$$
\begin{aligned}
&\Big(Pr(d|cz)Pr(c|abx) + Pr(d|\bar{c}z)\Big(1 - Pr(c|abx)\Big)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}\bar{b}x) + Pr(d|\bar{c}z)\Big(1 - Pr(c|\bar{a}\bar{b}x)\Big)\Big) \geq\\
&\Big(Pr(d|cz)Pr(c|a\bar{b}x) + Pr(d|\bar{c}z)\Big(1 - Pr(c|a\bar{b}x)\Big)\Big)\\
&\Big(Pr(d|cz)Pr(c|\bar{a}bx) + Pr(d|\bar{c}z)\Big(1 - Pr(c|\bar{a}bx)\Big) \qquad \Leftrightarrow
\end{aligned}
$$

$$
Pr(d|abxz)Pr(d|\bar{a}\bar{b}xz) \geq Pr(d|a\bar{b}xz)Pr(d|\bar{a}bxz) \qquad \Leftrightarrow
$$

$$
\hat{X}^{+}(\{A, B\}, d, T_4, T_5)
$$

by definition.

□

This theorem extends a known result [1, Th. 6.3][5, Th. 4][13, Th. 3][14, Th. 6] and can be further generalised to all sorts of trails, using theorem 4.4.

4.6 Relations among synergies.

In the previous sections, we have focused on relations between influences and synergies. Product and additive synergies are also mutually related: a synergy, product or additive, that exists between two causes and a common effect, may completely describe the synergy. The following two theorems show under which conditions this occurs and extend on results presented in [14, Th. 7 & Th. 8].

Theorem 4.15 (Relating X and Y I) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $X = \pi(C) \backslash \{A, B\}$. If one of the following conditions holds:*

1. $Pr(c|abx) \geq Pr(c|a\bar{b}x)$ *and* $Pr(c|abx) \geq Pr(c|\bar{a}bx)$
2. $Pr(c|abx) \leq Pr(c|a\bar{b}x)$ *and* $Pr(c|abx) \leq Pr(c|\bar{a}bx)$
3. $Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)$ *and* $Pr(c|\bar{a}\bar{b}x) \geq Pr(c|\bar{a}bx)$
4. $Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x)$ *and* $Pr(c|\bar{a}\bar{b}x) \leq Pr(c|\bar{a}bx)$

Then

- $X^+(\{A, B\}, c) \Rightarrow Y^+(\{A, B\}, C)$,
- $Y^-(\{A, B\}, C) \Rightarrow X^-(\{A, B\}, c)$,
- $X^0(\{A, B\}, c) \Rightarrow Y^0(\{A, B\}, C)$ *and*
- $Y^0(\{A, B\}, C) \Rightarrow X^0(\{A, B\}, c)$.

Proof: We will prove the theorem for the first two properties only. The proofs for the last two properties are analogous.

We prove the first property stated in the theorem. We assume that $X^+(\{A, B\}, c)$ holds. From $X^+(\{A, B\}, c)$ we find

$$X^+(\{A, B\}, c) \Leftrightarrow Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$$

for all possible value assignments x to X, by definition. We can use this inequality to bound the sum of $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$:

$$\begin{aligned} Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \quad &\geq \quad Pr(c|abx) + \frac{Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)}{Pr(c|abx)} \\ &= \quad \Big(Pr(c|a\bar{b}x) - Pr(c|a\bar{b}x)\Big) + Pr(c|abx) + \\ &\quad\; \frac{Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)}{Pr(c|abx)} \end{aligned}$$

The right-hand side of this expression can be rewritten as

$$Pr(c|a\bar{b}x) - Pr(c|a\bar{b}x) + Pr(c|abx) + \frac{Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)}{Pr(c|abx)} +$$

$$Pr(c|\bar{a}bx) - Pr(c|\bar{a}bx)\frac{Pr(c|abx)}{Pr(c|abx)}$$

which is equivalent to

$$Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx) +$$

$$[Pr(c|abx) - Pr(c|a\bar{b}x) - \frac{Pr(c|abx) - Pr(c|a\bar{b}x)}{Pr(c|abx)} Pr(c|\bar{a}bx)].$$

The bracketed expression is guaranteed to be non-negative under either of the first two conditions of the theorem, establishing $Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$ which is by definition $Y^+(\{A, B\}, C)$.

Note that the entire argument is symmetric in $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$. Therefore, we have

$$\begin{array}{ll} Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) & \geq \\ Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx) + & \end{array}$$

$$[Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x) - \frac{Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x)}{Pr(c|\bar{a}\bar{b}x)} Pr(c|\bar{a}bx)]$$

The bracketed expression is guaranteed to be non-negative under either of the last two conditions of the theorem, establishing $Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$ which is by definition $Y^+(\{A, B\}, C)$.

We now prove the second property stated in the theorem. We assume that $Y^-(\{A, B\}, C)$ holds. From $Y^-(\{A, B\}, C)$ we find

$$Y^-(\{A, B\}, C) \Leftrightarrow Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$$

for all possible value assignments x to X, by definition. We can use this inequality to bound the product of $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$:

$$Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \leq Pr(c|abx)\Big(Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx) - Pr(c|abx)\Big)$$

The right-hand side of this expression can be rewritten as

$$Pr(c|a\bar{b}x)Pr(c|abx) + Pr(c|\bar{a}bx)Pr(c|abx) - Pr(c|abx)^2 +$$

$$Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) - Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) \Leftrightarrow$$

$$Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) + [(Pr(c|abx) - Pr(c|\bar{a}bx))(Pr(c|a\bar{b}x) - Pr(c|abx))].$$

The bracketed expression is guaranteed to be non-positive under either of the first two conditions of the theorem, establishing $Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$, which is by definition $X^-(\{A, B\}, c)$.

Note that the entire argument is symmetric in $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$. Therefore, we have

$$\begin{aligned}&Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \leq \\ &Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)+ \\ &[(Pr(c|\bar{a}\bar{b}x) - Pr(c|\bar{a}bx))(Pr(c|a\bar{b}x) - Pr(c|\bar{a}\bar{b}x))]\end{aligned}$$

The bracketed expression is guaranteed to be non-positive under either of the last two conditions of the theorem, establishing $Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$, which is by definition $X^-(\{A, B\}, c)$.

□

Theorem 4.16 (Relating X and Y II) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $X = \pi(C) \backslash \{A, B\}$. If one of the following conditions holds:*

1. $Pr(c|abx) \leq Pr(c|a\bar{b}x)$ *and* $Pr(c|abx) \geq Pr(c|\bar{a}bx)$
2. $Pr(c|abx) \geq Pr(c|a\bar{b}x)$ *and* $Pr(c|abx) \leq Pr(c|\bar{a}bx)$
3. $Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x)$ *and* $Pr(c|\bar{a}\bar{b}x) \geq Pr(c|\bar{a}bx)$
4. $Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)$ *and* $Pr(c|\bar{a}\bar{b}x) \leq Pr(c|\bar{a}bx)$

Then

- $X^-(\{A, B\}, c) \Rightarrow Y^-(\{A, B\}, C)$ *and*
- $Y^+(\{A, B\}, C) \Rightarrow X^+(\{A, B\}, c)$.

Proof: We prove the first property stated in the theorem. Suppose $X^-(\{A, B\}, c)$ holds. From $X^-(\{A, B\}, c)$ we find

$$X^-(\{A, B\}, c) \Leftrightarrow Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx).$$

As in the proof of theorem 4.15, we once more use this inequality to bound the sum of $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$,

$$Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \leq$$

$$(Pr(c|abx) - Pr(c|a\bar{b}x)) + Pr(c|a\bar{b}x) + \frac{Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)}{Pr(c|abx)}.$$

The right-hand side of this expression can be rewritten as

$$Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)+$$

$$[(Pr(c|abx) - Pr(c|a\bar{b}x)) - \frac{(Pr(c|abx) - Pr(c|a\bar{b}x))}{Pr(c|ab)} Pr(c|\bar{a}bx)].$$

The bracketed expression is guaranteed to be non-positive under either of the first two conditions of the theorem, establishing $Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$, which is by definition $Y^{-}(\{A, B\}, C)$.

Note that the entire argument is symmetric in $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$. Therefore, we get

$$\begin{aligned} Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) &\leq \\ Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx) + & \\ [(Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x)) - \frac{(Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x))}{Pr(c|\bar{a}\bar{b})} Pr(c|\bar{a}bx)] & \end{aligned}$$

The bracketed expression is guaranteed to be non-positive under either of the last two conditions of the theorem, establishing $Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$, which is by definition $Y^{-}(\{A, B\}, C)$.

We now prove the second property stated in the theorem. Suppose that $Y^{+}(\{A, B\}, C)$ holds. By a similar analysis as before, the positive additive synergy provides the following bound on the product of $Pr(c|abx)$ and $Pr(c|\bar{a}\bar{b}x)$:

$$Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)Pr(c|abx) + Pr(c|\bar{a}bx)Pr(c|abx) - Pr(c|abx)^2.$$

The right-hand side of this expression can be rewritten as

$$Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) + [(Pr(c|abx) - Pr(c|\bar{a}bx))(Pr(c|a\bar{b}x) - Pr(c|abx))].$$

The bracketed expression is guaranteed to be non-negative under either of the first two conditions, establishing $Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$, which is by definition $X^{+}(\{A, B\}, c)$.

A symmetric argument substituting $Pr(c|\bar{a}\bar{b}x)$ for $Pr(c|abx)$ results in

$$\begin{aligned} & Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq \\ & Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) + \\ & [(Pr(c|\bar{a}\bar{b}x) - Pr(c|\bar{a}bx))(Pr(c|a\bar{b}x) - Pr(c|\bar{a}\bar{b}x))] \end{aligned}$$

The bracketed expression is guaranteed to be non-negative under either of the last two conditions, establishing $Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$, which is by definition $X^{+}(\{A, B\}, c)$.

□

The conditions of theorem 4.15 and theorem 4.16 that show that a synergy that exists between two causes and a common effect may imply the other synergy, are automatically satisfied when the influences between those two causes and the common effect have the same sign, or opposite signs.

Corollary 4.17 *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$, such that $(A, C), (B, C) \in R$. Let $\hat{S}^{\delta_1}(A, C, \{A, \varnothing, C\})$ and $\hat{S}^{\delta_2}(B, C, \{B, \varnothing, C\})$ hold. For all $\delta_i \in \{+, -, 0\}$, if $\delta_1 = \delta_2$, then*

- $X^+(\{A,B\},c) \Rightarrow Y^+(\{A,B\},C)$,
- $Y^-(\{A,B\},C) \Rightarrow X^-(\{A,B\},c)$,
- $X^0(\{A,B\},c) \Rightarrow Y^0(\{A,B\},C)$, *and*
- $Y^0(\{A,B\},C) \Rightarrow X^0(\{A,B\},c)$.

Proof: Assume that $\hat{S}^+(A,C,\{A,\varnothing,C\})$ and $\hat{S}^+(B,C,\{B,\varnothing,C\})$ hold. From $\hat{S}^+(A,C,\{A,\varnothing,C\})$ we find

$$\begin{array}{rcl} \hat{S}^+(A,C,\{A,\varnothing,C\}) & \Leftrightarrow & Pr(c|abx) \geq Pr(c|\bar{a}bx) \\ & \wedge & Pr(c|a\bar{b}x) \geq Pr(c|\bar{a}\bar{b}x) \end{array}$$

for all possible value assignments x to $X = \pi(C)\backslash\{A,B\}$, by definition. Analogously, from $\hat{S}^+(B,C,\{B,\varnothing,C\})$ we find

$$\begin{array}{rcl} \hat{S}^+(B,C,\{B,\varnothing,C\}) & \Leftrightarrow & Pr(c|abx) \geq Pr(c|a\bar{b}x) \\ & \wedge & Pr(c|\bar{a}bx) \geq Pr(c|\bar{a}\bar{b}x) \end{array}$$

for all possible value assignments x to $X = \pi(C)\backslash\{A,B\}$, by definition. The property now follows directly from theorem 4.15.

□

Corollary 4.18 *Let $Q = (V,R,H)$ be a qualitative probabilistic network. Let $A,B \in V$, such that $(A,C),(B,C) \in R$ and let $X = \pi(C)\backslash\{A,B\}$. Let $\otimes$ be the operator as defined in table 2. Let $\hat{S}^{\delta_1}(A,C,\{A,\varnothing,C\})$ and $\hat{S}^{\delta_2}(B,C,\{B,\varnothing,C\})$ hold. For all $\delta_i \in \{+,-\}$, if $\delta_1 = - \otimes \delta_2$, then*

- $X^-(\{A,B\},c) \Rightarrow Y^-(\{A,B\},C)$, *and*
- $Y^+(\{A,B\},C) \Rightarrow X^+(\{A,B\},c)$.

Proof: The proof is analogous to the proof of corollary 4.17; the property now follows directly from theorem 4.15.

□

Note that again the corollaries above can be generalised to all sorts of trails, using theorem 4.4.

We will now present our last theorem, which states that the sign of the product synergy of two causes with respect to the value c of their common effect C and the additive synergy for these three variables, determine the sign of the product synergy with respect to the value $\bar{c}$ of variable C; this result extends on [14, Th. 5].

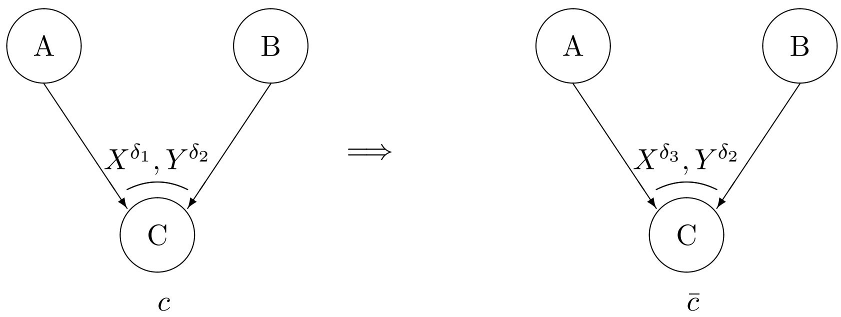
Theorem 4.19 (Relating Product Synergies for Different Values) *Let $Q = (V, R, H)$ be a qualitative probabilistic network. Let $A, B, C \in V$ such that $(A, C), (B, C) \in R$ and let $X = \pi(C) \backslash \{A, B\}$. Let $\otimes$ be the operator as defined in table 2. Then,*

$$X^{\delta_1}(\{A, B\}, c) \wedge Y^{\delta_2}(\{A, B\}, C) \quad \Rightarrow \quad X^{\delta_3}(\{A, B\}, \bar{c}),$$

where $\delta_i \in \{+, -, 0, ?\}, i = 1, 2, 3$ and

$$\delta_3 = \begin{cases} \delta_1 & \text{if } (\delta_1 = - \otimes \delta_2 \vee \delta_2 = 0) \quad \wedge \quad \delta_1 \neq 0 \\ - \otimes \delta_2 & \text{if } \delta_1 = 0 \wedge \delta_2 \neq 0 \end{cases}$$

Proof: We will prove the theorem for $\delta_1 = +$ and $\delta_2 = -$, the proofs for the other combinations of δ_1, δ_2 are analogous. The relations between the variables used in the proof are shown in the figure below.



We assume that $X^+(\{A, B\}, c)$ and $Y^-(\{A, B\}, C)$ hold. By definition, this means we have

$$Pr(c|abx)Pr(c|\bar{a}\bar{b}x) \geq Pr(c|a\bar{b}x)Pr(c|\bar{a}bx)$$

and

$$Pr(c|abx) + Pr(c|\bar{a}\bar{b}x) \leq Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)$$

respectively, for all possible value assignments x to X. Combining these, we get

$$\begin{array}{ll} Pr(c|abx)Pr(c|\bar{a}\bar{b}x) - Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) & \geq \\ \Big(Pr(c|abx) + Pr(c|\bar{a}\bar{b}x)\Big) - \Big(Pr(c|a\bar{b}x) + Pr(c|\bar{a}bx)\Big). & \end{array}$$

This is equivalent to

$$\begin{array}{ll} 1 - Pr(c|\bar{a}\bar{b}x) - Pr(c|abx) + Pr(c|abx)Pr(c|\bar{a}\bar{b}x) & \geq \\ 1 - Pr(c|\bar{a}bx) - Pr(c|a\bar{b}x) + Pr(c|a\bar{b}x)Pr(c|\bar{a}bx) & \end{array}$$

or equivalently,

$$(1 - Pr(c|abx))(1 - Pr(c|\bar{a}\bar{b}x)) \geq (1 - Pr(c|a\bar{b}x))(1 - Pr(c|\bar{a}bx))$$

which is equivalent to $X^+(\{A, B\}, \bar{c})$, by definition.

□

5 Conclusions.

In this paper we presented a new formal definition for a binary qualitative probabilistic network and its three types of qualitative information: qualitative influences, additive synergies and product synergies. In addition, we have also proven various properties that hold for qualitative influences and synergies and we have proven several relations that exist between qualitative influences and qualitative synergies. These properties and relations imply that only a subset of all qualitative influences and synergies that hold in a network, is sufficient for the representation of a QPN.

For the non-propositional case, the definitions of qualitative influence and the qualitative synergies, are slightly different. For non-binary variables, the principle of stochastic dominance builds on a total ordering on the sets of possible values of these variables. We assume that for a random variable A, the values can be indexed from the highest to the lowest in the ordering, such that $a_i \geq a_j$ for all $i \leq j$. A positive qualitative influence from variable A to variable B is then defined as

$$Pr(B \geq b_0 | a_1 x) \geq Pr(B \geq b_0 | a_2 x)$$

for any arbitrary value b_0 of B, for all values a_1, a_2 of A and for all possible value assignments x to $X = \pi(B) \backslash \{A\}$. Similar adjustments to the definitions of additive and product synergy also give definitions for the non-propositional case. Although all properties we have proven, were stated for binary variables, most of them (except for the theorems 4.14, 4.15, 4.16, 4.17, 4.18 and 4.19) can be directly generalised to the non-propositional case.

Domain experts may often be reluctant or incapable of providing reliable information, so it would be best if they need to give as less information as possible. The fact that only a subset of all qualitative influences and synergies is sufficient to define a QPN is important, since it implies that we only need little information from a domain expert. The most important property of QPNs for knowledge representation is the feasibility of constructing knowledge bases of reasonable complexity.

Though powerful in some respects, the qualitative relationships are also quite limited. They can, for example, express only monotone associations. Another draw-back is encountered when we try reasoning in a qualitative probabilistic network. The most important algorithm for reasoning in a QPN is based on the propagation of signs [3] [4], similar to message passing. During the propagation, signs are combined using the $\oplus$ and $\otimes$ operators. When a '?' sign exists in the network, then combining signs of influences and synergies will result in yet more '?' signs.

Despite these draw-backs, qualitative probabilistic networks are very useful. For reasons of modularity and precision, QPNs should be substantially easier to generate than their numeric counterparts. It is important to understand both qualitative and quantitative probabilistic networks so that the possibility arises to combine them [2]. By combining them, a domain expert can give both quantitative and qualitative information. Combining them might also solve the algorithmic problems that exist for both of them. It could be that less '?' signs are then generated. With respect to qualitative probabilistic networks, it is possible that yet other types of qualitative information exist.

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