Algebraicity and moniodal algebraicity of chromatic homotopy theory

Sven van Nigtevecht

26 October 2023

These are (somewhat expanded) notes for a talk I gave in the *Freudenthal topology seminar* in Utrecht. The talk was intended as an introduction to algebraicity results of Pstrągowski [Pst21], Patchkoria and Pstrągowski [PP21], and more recently Barkan [Bar23b]. I will give some pointers to these papers for readers who want to dive deeper, but I will not give a reference for every statement. When I omit explanations however, these sources do provide detailed proofs. If you find mistakes in these notes (big or small), please email me!

Remark. It should be mentioned that the papers [PP21] and [Bar23b] do much more than prove chromatic algebraicity. Patchkoria and Pstrągowski set up a very general framework for derived ∞-categories of stable ∞-categories, leading to a very general algebraicity theorem. Barkan sets up a categorical language (in terms of modules over the ∞-category of filtered spectra) for thinking about these things, and then proves a general monoidal algebraicity theorem using this. In these notes, I have tried to extract a few essential ideas and focus only on the case of chromatic homotopy theory. The reader who wishes to delve into these papers can expect to find a much larger treasure trove than this map suggests them to be.

Warning. My notation for the indices differs slightly from the cited sources.

1 Motivation

If E is a ring spectrum (with some additional conditions), one can ask how much E “remembers” about the stable homotopy groups of spheres. More formally, one writes down a resolution of the sphere in terms of smash powers of E, and considers the difference between the sphere and the limit of this resolution. Often this limit is the E-localisation of the sphere. The resolution itself gives rise to a spectral sequence computing its limit: the *E-based Adams spectral sequence*. It is of signature

$$E^2_{s,t} \cong \operatorname{Ext}^{s,t}_{E_*E}(E_*, E_*) \implies \pi_{t-s}(\mathbf{S}_E).$$

The left-hand side denotes Ext groups in the abelian category of (graded) E_*E-*comodules*, which we denote by Comod_{E_*E}. One likes to think of the E^2-page as algebraic, because it can be defined purely in terms of the homological algebra of E_*E-comodules. The abutment $\pi_*\mathbf{S}_E$ on the other hand we think of as topological. The spectral sequence moves one from algebra to topology, and the difference between these two realms is recorded by the differentials.

One gets different spectral sequences for varying E. Adams' original spectral sequence is the case $E = \mathrm{H}\mathbf{F}_p$, which converges to $\pi_*(\mathbf{S}_p^\wedge)$. The Adams–Novikov spectral sequence is the case $E = \mathrm{MU}$, which converges to $\pi_*\mathbf{S}$. Both of these are difficult in that they look at all of the sphere at once (even at just one prime, the sphere is still extremely complicated). The philosophy of chromatic homotopy theory is that, at every prime, a spectrum consists of different pieces of different 'heights'. There is an Adams spectral sequence that computes the "height $\leqslant n$" part of the p-local sphere: this is obtained by letting E be Morava E-theory at height n. The resulting spectral sequence converges to the homotopy groups of $L_n\mathbf{S}_{(p)}$, the "height $\leqslant n$ version" of the p-local sphere.

In this talk we focus on this spectral sequence. For reasons that will become more apparent later, we want to work with Johnson-Wilson theory $\mathrm{E}(n)$ rather than Morava E-theory E_n. Recall that $\mathrm{E}(n)$ is a ring spectrum with homotopy groups

$$\pi_*\mathrm{E}(n) \cong \mathbf{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}], \qquad \text{where } |v_i| = 2(p^i - 1).$$

Observe that $\pi_*\mathrm{E}(n)$ is concentrated in degrees that are multiples of $2(p-1)$. It turns out that we make this change without any issues: the abelian category Comod_{E_*E} and the resulting Adams spectral sequence do not change if we take E to be Morava E-theory or Johnson–Wilson theory.

Henceforth, we will fix a prime p and height n, and E will denote Johnson-Wilson theory $\mathrm{E}(n)$ *at the prime p.*

The E-based Adams spectral sequence has some interesting features: it becomes simpler as the prime gets larger.

(i) If $p > n+1$, then this spectral sequence has a horizontal vanishing line on the E^2-page. As a consequence, there are only finitely many differentials $d_1, \ldots, d_r$, and collapses after that.

(ii) If $2(p-1) > n^2 + n$, then the spectral sequence even collapses on the E^2-page (for degree reasons), i.e., there are *no differentials*.

This second point suggests that there is no difference between the algebraic situation and the topological one if p is sufficiently large with respect to n. This is indeed the case.

Theorem (Crude version). *If $2(p-1) > n^2+n$, then there is an equivalence of categories*

$$h\mathrm{Sp}_E \simeq h\mathscr{D}(E_*E).$$

Here $h\mathrm{Sp}_E$ is the homotopy category of E-local spectra, which we think of as p-local spectra of height $\leqslant n$. By $\mathscr{D}(E_*E)$, we mean the derived ∞-category of *differential E_*E-comodules*. I will not define this category further, but suffice it to say that it is the derived ∞-category of an *abelian* category (namely, that of comodules equipped with a differential; do note this is not the same as the derived category of Comod_{E_*E}).

The above equivalence becomes stronger as the prime gets larger. To make this precise, we think about how strict the inequality $2(p-1) > n^2 + n$ is: we insert an $\alpha \geqslant 1$ and write $2(p-1) \geqslant n^2 + n + \alpha$.

Theorem A (Patchkoria–Pstragowski). *If $2(p-1) \geqslant n^2 + n + \alpha$, then there is an equivalence*

$$h_\alpha \mathrm{Sp}_E \simeq h_\alpha \mathscr{D}(E_*E).$$

Here $h_\alpha \mathscr{C}$ denotes the *homotopy α-category* of $\mathscr{C}$, i.e., the ∞-category obtained by $(\alpha-1)$-truncating the mapping spaces in $\mathscr{C}$. For instance, the usual homotopy category of $\mathscr{C}$ is $h_1\mathscr{C}$. In general, $h_\alpha \mathscr{C}$ is an α-category.

Remark 1.1. To see more concretely that this is indeed a strengthening, suppose that $\alpha \geqslant 2$ and that the inequality from Theorem A holds, so that we get an equivalence on homotopy 2-categories. Recall that the homotopy category of a stable ∞-category has a natural triangulated structure. It turns out that this triangulated structure is only an invariant of the homotopy 2-category. Hence, if $\alpha \geqslant 2$, we find that the equivalence $h\mathrm{Sp}_E \simeq h\mathscr{D}(E_*E)$ from the crude version can even be made into a triangulated equivalence.

Remark 1.2. Theorem A has a rich history. Bousfield [Bou85] considered the height 1 case at odd primes. Franke announced a proof of Theorem A, but Patchkoria later found a mistake in this proof. Pstrągowski [Pst21] proved Theorem A in his thesis with the more restrictive bound $p-1 \geqslant n^2+n+\alpha$ (i.e., allowing half the number of primes). Later Patchkoria and Pstrągowski [PP21] gave an improved version of Pstrągowski's original proof, and thereby proved Theorem A with the bound as above.

More recently, Barkan proved a symmetric monoidal version.

Theorem B (Barkan). *If $2(p-1) \geqslant n^2 + (\alpha+3)n + \alpha$, then there is a symmetric monoidal equivalence*

$$h_\alpha \mathrm{Sp}_E \simeq h_\alpha \mathscr{D}(E_*E).$$

I'll give an outline of the proof of Theorem A, and then discuss the modifications and further technology needed to prove Theorem B.

2 Proof outline

The proof is a detailed analysis of what it means to "pass from algebra to topology". Going forward, we will write $\mathscr{C}$ for either Sp_E or $\mathscr{D}(E_*E)$. We will decompose $\mathscr{C}$ as a tower

$$\mathscr{C} \longrightarrow \cdots \longrightarrow \mathscr{M}_k(\mathscr{C}) \longrightarrow \mathscr{M}_{k-1}(\mathscr{C}) \longrightarrow \cdots \longrightarrow \mathscr{M}_0(\mathscr{C}) \simeq \mathrm{Comod}_{E_*E}.$$

This is a limit diagram, where the bottom is equivalent to the algebraic category Comod_{E_*E}. The other $\mathscr{M}_k(\mathscr{C})$ can be thought of as intermediate stages in passing from algebra to topology.

This tower has two special properties (having fixed an $\alpha \geqslant 1$):

(i) for k large enough, we have an equivalence $h_\alpha\mathscr{C} \simeq h_\alpha\mathscr{M}_k(\mathscr{C})$;

(ii) for k small enough, we have an equivalence $\mathscr{M}_k(\mathrm{Sp}_E) \simeq \mathscr{M}_k(\mathscr{D}(E_*E))$.

When these two ranges overlap, we get an algebraicity result. We will determine these two ranges explicitly, and thereby prove the bounds stated in Theorem A and Theorem B.

3 Construction of the tower

To construct the tower, we need a 'derived ∞-category of $\mathscr{C}$'. This might sound strange at first: $\mathscr{C}$ is a stable ∞-category, which we like to think of as 'already derived'. By a derived category of $\mathscr{C}$, we mean deriving it with respect to the E-homology functor $E_*(-)$. Objects in this derived category can be thought of as formal E-based Adams resolutions of objects of $\mathscr{C}$. The precise definition for these purposes can get technical however, so I will omit any specific details going forward. We will write $\mathscr{D}(\mathscr{C})$ for this derived ∞-category of $\mathscr{C}$. It can be made symmetric monoidal, and it has a natural t-structure that is compatible with the monoidal structure.

Example 3.1. For those who know what the following words mean: one incarnation of $\mathscr{D}(\mathscr{C})$ one can take is *hypercomplete E-based synthetic spectra*, as defined by Pstragowski [Pst22]. This is what Pstragowski used in [Pst21], and in the end it proves a version of Theorem A with the suboptimal bound $p-1 \geqslant n^2+n+\alpha$. A large part of Patchkoria–Pstrągowski [PP21] is devoted to a different definition of $\mathscr{D}(\mathscr{C})$, based on injectives rather than projectives, that yields a better range for this theorem. ▲

Write $\mathscr{D}_{\geqslant 0}(\mathscr{C})$ for the connective part of the t-structure on $\mathscr{D}(\mathscr{C})$. Write $\mathbf{1}^{\leqslant k}$ for the k-truncation of the monoidal unit. This is naturally a commutative algebra object of $\mathscr{D}_{\geqslant 0}(\mathscr{C})$, so we can consider module objects over it.

Definition 3.2. Let $0 \leqslant k \leqslant \infty$. The ∞-category of **potential *k*-stages** in $\mathscr{C}$, denoted by $\mathscr{M}_k(\mathscr{C})$, is the full subcategory of

$$\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C}))$$

on those X such that

$$X \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0}$$

is a discrete object (i.e., lives in the heart of $\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C}))$).

Proposition 3.3.

- *We have equivalences* $\mathscr{M}_0(\mathscr{C}) \simeq \mathrm{Comod}_{E_*E}$ *and* $\mathscr{M}_\infty(\mathscr{C}) \simeq \mathscr{C}$.
- *For every* k, *the* ∞*-category* $\mathscr{M}_k(\mathscr{C})$ *is a* $(k+1)$*-category.*

Warning 3.4. The ∞-category $\mathscr{M}_k(\mathscr{C})$ is generally *not* closed under the relative tensor product $\otimes_{\mathbf{1}^{\leqslant k}}$. This is similar to the usual difference between the tensor product and the derived tensor product: the derived tensor product can have nonzero higher homology groups. The smash product of spectra is not 'derived' from the perspective of E-homology, so the same can happen in this case. The category $\mathscr{M}_k(\mathscr{C})$ was defined using a discreteness condition, and as such will in general not be closed under the tensor product. As such, it is not a symmetric monoidal ∞-category.

For every k, truncation defines a functor $\mathscr{M}_{k+1}(\mathscr{C}) \to \mathscr{M}_k(\mathscr{C})$, resulting in a tower as claimed. For these specific $\mathscr{C}$, this tower is actually a limit diagram, i.e., it identifies $\mathscr{C}$ as the limit

$$\mathscr{C} \simeq \lim_k \mathscr{M}_k(\mathscr{C}).$$

This is not a formal consequence of the setup: it is related to the convergence of the Adams spectral sequence.

This tower is an important one: it is the tower used to set up *Goerss–Hopkins obstruction theory* from an ∞-categorical perspective; see [PV22]. We will use this obstruction theory to prove the existence of the two ranges discussed above.

4 The first range

We need the following fact.

Proposition 4.1. *If* $p > n+1$, *the abelian category* Comod_{E_*E} *has cohomological dimension* n^2+n.

Meaning, if $M, N \in \mathrm{Comod}_{E_*E}$ are comodules, then if $s > n^2+n$, we have for all t that

$$\mathrm{Ext}^{s,t}_{E_*E}(M,N) = 0.$$

(Bear in mind this does not hold if M and N are merely objects in the derived category of Comod_{E_*E}; they need to be 'honest' comodules.)

To prove the existence of the first range, note that because the tower is a limit diagram, it suffices to prove that $\mathscr{M}_{k+1}(\mathscr{C}) \to \mathscr{M}_k(\mathscr{C})$ induces an equivalence on homotopy α-categories for k large enough. We will even determine the explicit bound needed for this.

The proof uses (linear) Goerss–Hopkins obstruction theory, which says the following. For every $k \geqslant 0$ and every object $X \in \mathscr{M}_k(\mathscr{C})$, there is an obstruction to lifting X to an object of $\mathscr{M}_{k+1}(\mathscr{C})$ that lives in

$$\operatorname{Ext}^{k+3,k+1}_{E_*E}(X \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0},\ X \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0}).$$

Since $X \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0}$ is discrete, it lives in $\mathscr{M}_0(\mathscr{C})$, which can be identified with Comod_{E_*E}. It follows that these obstructions vanish if $k+3 \geqslant n^2+n+1$, in which case we see that the functor $\mathscr{M}_{k+1}(\mathscr{C}) \to \mathscr{M}_k(\mathscr{C})$ is essentially surjective.

We want more however: we want an equivalence on (truncated) mapping spaces too. Supposing that $X, Y \in \mathscr{M}_k(\mathscr{C})$ are objects which lift to $\mathscr{M}_{k+1}(\mathscr{C})$, there is also an obstruction theory to lifting a map $X \to Y$ to $\mathscr{M}_{k+1}(\mathscr{C})$, and this lives in

$$\operatorname{Ext}^{k+2,k+1}_{E_*E}(X \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0},\ Y \otimes_{\mathbf{1}^{\leqslant k}} \mathbf{1}^{\leqslant 0}).$$

This lift is unique if the same Ext group of bidegree $(k+1,\ k+1)$ vanishes. We learn that $h\mathscr{M}_{k+1}(\mathscr{C}) \to h\mathscr{M}_k(\mathscr{C})$ is an equivalence whenever $k+1 \geqslant n^2+n+1$.

More generally, the obstruction to lifting an α-morphism from $\mathscr{M}_k$ to $\mathscr{M}_{k+1}$ lives in bidegree $(k+3-\alpha,\ k+1)$, and this lift is unique if the Ext group of bidegree $(k+2-\alpha,\ k+1)$ vanishes. We learn that $h_\alpha\mathscr{M}_{k+1}(\mathscr{C}) \to h_\alpha\mathscr{M}_k(\mathscr{C})$ is an equivalence whenever

$$k+2-\alpha \geqslant n^2+n+1 \iff k \geqslant n^2+n+\alpha-1.$$

This is the advertised first range.

5 The second range

This is the more technical part of the argument, so for time reasons I will be more brief. The methods are inspired by the original height 1 work by Bousfield [Bou85].

The crucial observation is that $\pi_*\mathrm{E}(n)$ are concentrated in degrees that are multiples of $2(p-1)$. This allows one to define a functor (where the left-hand side denotes the full subcategory of *injective* comodules)

$$\beta\colon \mathrm{Comod}^{\mathrm{inj}}_{E_*E} \longrightarrow h_k\mathscr{C}$$

which is an inverse to homology: there is a natural isomorphism $E_*(\beta(M)) \cong M$. This is done using obstruction theory again. At some point we can no longer guarantee that obstructions vanish, so that we can only define a functor to $h_k\mathscr{C}$ instead of $\mathscr{C}$ itself. For this, we need $k \leqslant 2(p-1)-1$; this will be our second range.

This functor induces an adjunction

$$\mathscr{D}_{\geqslant 0}(\mathrm{Comod}_{E_*E}) \underset{\beta_*}{\overset{\beta^*}{\rightleftarrows}} \mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C})).$$

One proves that this is monadic, i.e., it witnesses that $\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C}))$ is the category of algebras over the monad $\beta_*\beta^*$:

$$\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C})) \simeq \mathrm{Alg}_{\beta_*\beta^*}(\mathscr{D}_{\geqslant 0}(\mathrm{Comod}_{E_*E})).$$

The upshot of this is that the left-hand side turns out to be independent of $\mathscr{C}$. This equivalence also respects the condition defining the subcategory $\mathscr{M}_k(\mathscr{C})$, so that this category is also independent of $\mathscr{C}$:

$$\mathscr{M}_k(\mathrm{Sp}_E) \simeq \mathscr{M}_k(\mathscr{D}(E_*E)) \qquad \text{whenever } k \leqslant 2(p-1)-1.$$

Having found the two ranges, we deduce an algebraicity result precisely when they overlap. This happens when

$$2(p-1)-1 \geqslant n^2+n+\alpha-1 \iff 2(p-1) \geqslant n^2+n+\alpha,$$

which is the bound stated in Theorem A.

6 The monoidal version

Our next goal is to discuss Barkan's adaptations of the above proof strategy to the symmetric monoidal case.

The first hurdle to overcome is that $\mathscr{M}_k(\mathscr{C})$ is not closed under the tensor product; see Warning 3.4. The solution is to not consider it as a symmetric monoidal ∞-category, but as an *∞-operad*.

As an informal refresher on ∞-operads: an (coloured) **∞-operad** is an ∞-category $\mathscr{O}$ equipped with *multi-mapping spaces*: for each $r \geqslant 0$ and for all objects $X_1, \ldots, X_r, Y$, there is a space

$$\mathrm{Map}_{\mathscr{O}}(X_1, \ldots, X_r;\ Y)$$

of **morphisms of arity r**. These should be compatible in natural ways: one should be able to compose multi-maps with ordinary morphisms, but also plug in multi-mappings into other multi-mappings (of different arities), and these should satisfy the conditions one expects them to have.

Example 6.1. A symmetric monoidal ∞-category $\mathscr{C}$ defines an ∞-operad $\mathscr{C}^{\otimes}$ whose multi-mapping spaces are given by

$$\mathrm{Map}_{\mathscr{C}^{\otimes}}(X_1, \ldots, X_r;\ Y) = \mathrm{Map}_{\mathscr{C}}(X_1 \otimes \cdots \otimes X_r,\ Y). \qquad \blacktriangle$$

Remark 6.2. In fact, a symmetric monoidal ∞-category is usually *defined* as a particular kind of ∞-operad. Roughly speaking, an ∞-operad $\mathscr{O}$ is a symmetric monoidal ∞-category if for all r and all $X_1, \ldots, X_r$, the functor

$$\mathrm{Map}_{\mathscr{O}}(X_1, \ldots, X_r;\ -)$$

is corepresentable, say by $\bigotimes\{ X_i \}_{i\leqslant r}$, and if natural "associator" morphisms are equivalences. (This last condition is necessary because otherwise, there is no reason why $\bigotimes\{ X_1, X_2, X_3 \}$ is equivalent to $(X_1 \otimes X_2) \otimes X_3$, where we write $\otimes$ for the resulting binary tensor product.) In terms of Lurie's model [HA, Ch. 2], an ∞-operad $\mathscr{O}^{\otimes} \to \mathrm{Fin}_*$ is defined to be a symmetric monoidal ∞-category when the map is a cocartesian fibration.

We upgrade $\mathscr{M}_k(\mathscr{C})$ to an ∞-operad in the following way.

Definition 6.3. Let $\mathscr{M}_k^{\otimes}(\mathscr{C})$ denote the ∞-operad with underlying ∞-category $\mathscr{M}_k(\mathscr{C})$, equipped with multi-mapping spaces

$$\mathrm{Map}_{\mathscr{M}_k^{\otimes}(\mathscr{C})}(X_1, \ldots, X_r;\ Y) = \mathrm{Map}(X_1 \otimes_{\mathbf{1}^{\leqslant k}} \cdots \otimes_{\mathbf{1}^{\leqslant k}} X_r,\ Y),$$

where on the right-hand side we take maps in $\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C}))$. More formally: $\mathscr{M}_k^{\otimes}(\mathscr{C})$ is the full suboperad of $\mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathscr{D}_{\geqslant 0}(\mathscr{C}))^{\otimes}$ on the objects in $\mathscr{M}_k(\mathscr{C})$.

The truncation functors can be naturally upgraded to maps of ∞-operads, resulting in a limit tower of ∞-operads

$$\mathscr{C}^{\otimes} \longrightarrow \cdots \longrightarrow \mathscr{M}_k^{\otimes}(\mathscr{C}) \longrightarrow \cdots \longrightarrow \mathscr{M}_0^{\otimes}(\mathscr{C}) \simeq \mathrm{Comod}_{E_*E}^{\otimes}.$$

Even though the top and bottom of this tower are symmetric monoidal ∞-categories, the intermediate stages are not.

Our goal now is to prove a monoidal version of the two ranges, or perhaps find different ranges in which the monoidal analogues hold. It turns out that the second range does not change (as the original setup was actually already sufficiently compatible with the monoidal structure), so I will focus on the first range.

At least the desired statement is straightforward to generalise: there is also a notion of a 'homotopy α-operad' $h_\alpha \mathscr{O}$, namely where we $(\alpha - 1)$-truncate the multi-mapping spaces. To find a range in which $h_\alpha \mathscr{C}^{\otimes}$ is equivalent to $h_\alpha \mathscr{M}_k^{\otimes}(\mathscr{C})$ is not very straightforward however. Namely, we would like to find a range for k in which $h_\alpha \mathscr{M}_{k+1}^{\otimes}(\mathscr{C}) \to h_\alpha \mathscr{M}_k^{\otimes}(\mathscr{C})$ is an equivalence, i.e., it induces an equivalence on

$$\tau_{\alpha-1} \mathrm{Map}(X_1, \ldots, X_r;\ Y)$$

for *all* r. This is a problem: when we take a tensor product in this derived category, we introduce higher homotopy groups, so that we need stricter and stricter bounds on the Ext-degree before we can guarantee that the relevant Ext-groups vanish. This bound grows with r, which is a bad thing if we want to prove something for all $r \geqslant 0$.

What saves us is that we can restrict r to be below $\alpha + 3$, relying on Barkan's earlier arity approximation results.

Theorem 6.4 (Barkan [Bar23a]). *Let $\mathscr{E}$ be a complete symmetric monoidal m-category. Then there is a natural equivalence*

$$\mathrm{CMon}(\mathscr{E}) \simeq \mathrm{CMon}^{\leqslant m+2}(\mathscr{E}).$$

Remark 6.5. Barkan gives the following beautiful motivation for this result. In the definition of a commutative ring, every axiom involves at most three variables; note that abelian groups form a 1-category. In the definition of a symmetric monoidal category, every axiom involves at most four variables; note that categories form a 2-category. Barkan's result is that this pattern continues: in an m-category, to define a commutative monoid, you only need to specify the coherences up to artiy $m+2$.

We apply this to $\mathscr{E} = \mathrm{Cat}_\alpha$. As α-categories form an $(\alpha+1)$-category Cat_α, we find that for any ∞-operad $\mathscr{O}$, the approximation $h_\alpha\mathscr{O}$ is uniquely determined by its multi-mapping spaces of arities $\leqslant \alpha+3$. Applying this to our problem shows we only need to consider the cases where $r \leqslant \alpha+3$. We then use that if $p > n+1$, then the Tor dimension of Comod_{E_*E} is n. When taking a tensor product of r objects, we are really taking $r-1$ tensor products. In the end, this means that we can guarantee that $h_\alpha\mathscr{M}^\otimes_{k+1}(\mathscr{C}) \to h_\alpha\mathscr{M}^\otimes_k(\mathscr{C})$ is an equivalence whenever

$$k+2-\alpha \geqslant \underbrace{n^2+n}_{\text{Ext dimension}} + \underbrace{(\alpha+2)n}_{\alpha+2\text{ tensor products}} +1,$$

i.e., whenever

$$k \geqslant n^2+(\alpha+3)n-1.$$

Combining this with the second range $k \leqslant 2(p-1)-1$, we arrive at the bound

$$2(p-1) \geqslant n^2+(\alpha+3)n,$$

which is the bound in Theorem B.

References

[Bar23a] S. Barkan. *Arity Approximation of ∞-Operads*. 25th Apr. 2023. arXiv: `2207.07200 [math]`. preprint.

[Bar23b] S. Barkan. *Chromatic Homotopy Is Monoidally Algebraic at Large Primes*. 5th June 2023. arXiv: `2304.14457v2 [math]`. preprint.

[Bou85] A. K. Bousfield. "On the Homotopy Theory of K-Local Spectra at an Odd Prime". *American Journal of Mathematics* 107.4 (Aug. 1985), pp. 895–932. DOI: `10.2307/2374361`.

[HA] J. Lurie. "Higher Algebra". 18th Sept. 2017. URL: `https://www.math.ias.edu/~lurie/`.

[PP21] I. Patchkoria and P. Pstrągowski. *Adams Spectral Sequences and Franke's Algebraicity Conjecture*. 7th Oct. 2021. arXiv: `2110.03669v1 [math]`.

[Pst21] P. Pstrągowski. "Chromatic homotopy theory is algebraic when $p > n^2+n+1$". *Advances in Mathematics* 391 (Nov. 2021), p. 107958. DOI: `10.1016/j.aim.2021.107958`.

[Pst22] P. Pstrągowski. “Synthetic Spectra and the Cellular Motivic Category”. *Inventiones mathematicae* (20th Dec. 2022). DOI: 10.1007/s00222-022-01173-2.

[PV22] P. Pstrągowski and P. VanKoughnett. “Abstract Goerss-Hopkins Theory”. *Advances in Mathematics* 395 (Feb. 2022), p. 108098. DOI: 10.1016/j.aim.2021.108098.