Stacky Intermezzo II
Quasi-coherent sheaves on stacks

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Previously, we defined what a stack on a site is, and ended with a note on how to think of stacks as geometric objects. This time we pursue this much further, by asking the question what a sheaf on a stack ought to be. Specifically, we are interested in quasi-coherent sheaves on algebraic stacks. This is a generalisation of quasi-coherent sheaves on stacks. Once we have defined these notions, we then explain how to study such quasi-coherent sheaves by using comodules over Hopf algebroids. Lastly, we end with a sketch of how to apply these ideas to the moduli stack of formal groups.

Except for in §1, we take the functor of points perspective toward schemes. That is, we write $\mathrm{Spec}\colon \mathrm{CRing}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CRing}, \mathrm{Set})$ for the Yoneda embedding. A functor in the image of Spec we call an *affine scheme*; thus Spec is an equivalence onto the full subcategory Aff of affine schemes. A *scheme* is a functor $\mathrm{CRing} \to \mathrm{Set}$ that is a Zariski sheaf and that can be covered (Zariski locally) by affine schemes. We write Sch for the full subcategory on schemes.

We draw heavily from the lecture notes by Meier [Mei20] in most places, which contains useful further references. The notes by Vistoli [Vis08] are once again a good source for facts about stacks. Furthermore, for those who speak a little French, the book [LaMo] is a good exposition, containing remarks that other sources do not. Also, since we will be introducing Hopf algebroids in this lecture, the parts about stacks in the notes by Hopkins [Hop99] should be more accessible from now on. The last section uses parts from Lurie's lectures (lecture 11, to be specific).

1 Reminder: quasi-coherent sheaves on schemes

Let us for now not yet take the functor of points perspective for schemes. In other words, for now a scheme is a particular kind of locally ringed space $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ a sheaf of rings on X. (Later on we will give a definition of the structure sheaf $\mathcal{O}_X$ of a scheme from the functor of points perspective.)

Definition 1. Let X be a scheme.

(1) A **sheaf of $\mathcal{O}_X$-modules** on X (or an $\mathcal{O}_X$*-module* for short) is a sheaf of abelian groups F on X, together with, for every open subscheme $U \subseteq X$, the structure of an $\mathcal{O}_X(U)$-module on the group $F(U)$. A morphism of $\mathcal{O}_X$-modules $F \to G$ is a morphism of sheaves of abelian groups, such that for every open $U \subseteq X$, the map $F(U) \to G(U)$ is $\mathcal{O}_X(U)$-linear.

(2) A **quasi-coherent sheaf** on X is a sheaf of $\mathcal{O}_X$-modules F on X such that for all open affine subschemes $V \subseteq U \subseteq X$, the natural map

$$F(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \longrightarrow F(V)$$

is an isomorphism. A morphism of quasi-coherent sheaves is a morphism of $\mathcal{O}_X$-modules. We write $\mathrm{QCoh}(X)$ for the category of quasi-coherent sheaves on X.

Example 2. The structure sheaf $\mathcal{O}_X$ is quasi-coherent; if U is an affine scheme, we have more or less by definition that the map

$$\mathcal{O}_U(U) \otimes_{\mathcal{O}_U(U)} \mathcal{O}_U(V) \longrightarrow \mathcal{O}_U(V)$$

is an isomorphism. ▲

Thus if F is a quasi-coherent sheaf, and $U \subseteq X$ is an affine open, then the sheaf $F|_U$ can be reconstructed from the global sections $F(U)$ alone. Moreover, since a scheme can be covered by affine opens, this shows that if $\{ U_i \subseteq X \}_i$ is an affine open cover of X (i.e., a Zariski cover of X by affine schemes), then we can reconstruct F from the collection $\{ F(U_i) \}_i$ (where for every i, we view $F(U_i)$ as an $\mathcal{O}_X(U_i)$-module). In particular, if $X \cong \operatorname{Spec} A$ is affine, then a quasi-coherent sheaf on X is the same as an A-module.

Loosely speaking therefore, a quasi-coherent sheaf on X is completely captured by its value on affine schemes. However, one should be careful with the interpretation of the preceding sentence: the value on an affine scheme depends on how the affine is embedded in X. In the case of stacks, we will essentially turn this observation into a definition.

2 Stacks

Before we continue, we need some more facts about stacks. Recall that a stack on a site $\mathcal{C}$ is a pseudo-functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ satisfying the descent condition.

Suppose now that $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a functor. Every set X may be viewed as a discrete groupoid, meaning the category whose objects are the elements of X, with only the identity morphisms. If $\mathcal{C}$ is a site, then $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf if and only if $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ is a stack.

In this way, we can think of objects of $\mathcal{C}$ as pseudo-functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$. Namely, X represents the functor $h_X \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, and this we may view as a pseudo-functor to groupoids. We may abuse notation and write X instead of h_X going forward.

There is a notion of a morphism of stacks (or more generally of pseudo-functors), and a morphism between such morphisms.

Definition 3. Let $\mathcal{C}$ be a category, and $\mathscr{F}$ and $\mathscr{G}$ be two stacks on $\mathcal{C}$.

- A **pseudo-natural transformation** $F \colon \mathscr{F} \to \mathscr{G}$ consists of:
  (i) for every $X \in \mathcal{C}$, a map $F_X \colon \mathscr{F}(X) \to \mathscr{G}(X)$;

(ii) for every morphism $f\colon X \to Y$ in $\mathcal{C}$, a natural transformation

$$\begin{array}{ccc} \mathscr{F}(Y) & \xrightarrow{F_Y} & \mathscr{G}(Y) \\ {\scriptstyle f^*}\downarrow & \swArrow & \downarrow{\scriptstyle f^*} \\ \mathscr{F}(X) & \xrightarrow[F_X]{} & \mathscr{G}(X) \end{array}$$

satisfying a number of coherence conditions.

- Let $F, G\colon \mathscr{F} \to \mathscr{G}$ be two pseudo-natural transformations. A **2-morphism** η from F to G consists of, for every $X \in \mathcal{C}$, a natural transformation $\eta_X\colon F_X \to G_X$ of functors, such that the necessary squares commute.

The pseudo-natural transformations $\mathscr{F} \to \mathscr{G}$ form a category, denoted $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$.

Remark 4. The conditions become less laborious to write down when working with fibered categories instead: see, e.g., [Vis08, §3.5].

We say two pseudo-functors $\mathscr{F}$ and $\mathscr{G}$ are **equivalent** if there exist pseudo-natural transformations $F\colon \mathscr{F} \to \mathscr{G}$ and $g\colon \mathscr{G} \to \mathscr{F}$ such that $G \circ F$ is isomorphic to $\mathrm{id}_{\mathscr{F}}$ in $\mathrm{Hom}(\mathscr{F}, \mathscr{F})$, and similarly $F \circ G$ is isomorphic to $\mathrm{id}_{\mathscr{G}}$. We say such F and G are **equivalences**. It turns out that a pseudo-natural transformation F is an equivalence if and only if F_X is an equivalence for every $X \in \mathcal{C}$ (see, e.g., [Vis08, Prop. 3.36]).

Theorem 5 (2-categorical Yoneda lemma). *Let $\mathcal{C}$ be a category, $\mathscr{F}$ a pseudo-functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$, and $X \in \mathcal{C}$. Write $h_X\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ for the functor represented by X. Then we have a natural equivalence of categories*

$$\mathrm{Hom}(h_X, \mathscr{F}) \xrightarrow{\simeq} \mathscr{F}(X).$$

Proof. See, e.g., [Vis08, §3.6.2]. ■

This result recovers the usual Yoneda lemma when we take $\mathscr{F}$ to be an actual functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. It moreover justifies our immanent abuse of notation of writing X for h_X.

Lastly, we need the notion of a pullback of pseudo-functors.

Definition 6. Let $\mathcal{C}$ be a category, let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be pseudo-functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$, and let $f\colon \mathscr{F} \to \mathscr{H}$ and $g\colon \mathscr{G} \to \mathscr{H}$ be morphisms. We define the **2-pullback** of $\mathscr{F}$ and $\mathscr{G}$ to be the pseudo-functor $(\mathscr{F} \times_{\mathscr{H}} \mathscr{G})\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ as follows. Its value on $X \in \mathcal{C}$ is the groupoid with

- objects are triples (x, y, α) with $x \in \mathscr{F}(X)$, and $y \in \mathscr{G}(X)$, and $\alpha\colon f(x) \to g(y)$ a morphism in $\mathscr{H}(X)$;
- a morphism $(x, y, \alpha) \to (x', y', \alpha')$ is a pair (φ, ψ) with $\varphi\colon x \to x'$ a morphism in $\mathscr{F}(X)$ and $\psi\colon y \to y'$ a morphism in $\mathscr{G}(X)$, such that the square

$$\begin{array}{ccc} f(x) & \xrightarrow{\varphi} & f(x') \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ g(y) & \xrightarrow{\psi} & g(y') \end{array}$$

commutes.

Be warned that the diagram (where $X \in \mathcal{C}$)

$$\begin{array}{ccc} (\mathscr{F} \times_{\mathscr{H}} \mathscr{G})(X) & \longrightarrow & \mathscr{G}(X) \\ \big\downarrow & & \big\downarrow {\scriptstyle g_X} \\ \mathscr{F}(X) & \xrightarrow[f_X]{} & \mathscr{H}(X) \end{array}$$

does not commute; instead, one can write down a canonical natural transformation from one composite to the other. The diagram should rather be thought of as a commutative diagram in the 2-category of groupoids, i.e., a diagram with a chosen natural transformation from one composite to the other.

For ease of language, we shall refer to the 2-pullback simply as the *pullback*.

3 Sheaves on stacks

Definition 7. Let $\mathcal{C}$ be a site and let $\mathscr{F}\colon \mathcal{C}^{\text{op}} \to \text{Grpd}$ be a pseudo-functor. The **site associated to $\mathscr{F}$**, denoted $\mathcal{C}/\mathscr{F}$, is the category whose

- objects are maps $X \to \mathscr{F}$, where $X \in \mathcal{C}$;
- morphisms are commutative triangles

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \swarrow\!\!\!\!\Leftarrow & \\ \searrow & & \swarrow \\ & \mathscr{F}; & \end{array}$$

- covers of an object $X \to \mathscr{F}$ are covers of X in the sense of $\mathcal{C}$ (in other words, by forgetting the map to $\mathscr{F}$).

We shall also refer to objects of $\mathcal{C}/\mathscr{F}$ as **morphisms in $\mathcal{C}$ over $\mathscr{F}$**.

By the Yoneda lemma, a morphism $X \to \mathscr{F}$ is equivalent to an element of $\mathscr{F}(X)$. As such, $\mathcal{C}/\mathscr{F}$ is equivalent to the following category. The objects are pairs (X, x) with $X \in \mathcal{C}$ and $x \in \mathscr{F}(X)$. A morphism $(X, x) \to (Y, y)$ is a pair (f, φ) where $f\colon X \to Y$ is a morphism in $\mathcal{C}$ and φ is an isomorphism $f^*(y) \to x$ in $\mathscr{F}(X)$.

Remark 8. The category $\mathcal{C}/\mathscr{F}$ goes by many names: it is also called the *category of elements of $\mathscr{F}$*, and is also denoted by $\int_{\mathcal{C}} \mathscr{F}$ or $\text{Elt}(\mathscr{F})$. It is the category that features in the equivalence between pseudo-functors $\mathcal{C}^{\text{op}} \to \text{Grpd}$ and fibred categories $\mathcal{E} \to \mathcal{C}$. The projection $\mathcal{C}/\mathscr{F} \to \mathcal{C}$ sends $X \to \mathscr{F}$ to X (or from the other viewpoint, sends (X, x) to X).

Example 9. Let $X\colon \text{Aff}^{\text{op}} \to \text{Set}$ be (the functor of points of) a scheme. Then the category Aff/X consists of all maps $\text{Spec}\, R \to X$ from an affine scheme to X. In particular, this category contains the inclusions of all affine open subschemes $U \subseteq X$ into X. ▲

Definition 10. Let $\mathcal{C}$ be a site and let $\mathscr{F}$ be a stack of groupoids on $\mathcal{C}$. A **sheaf of sets** on $\mathscr{F}$ is a sheaf of sets on the site $\mathcal{C}/\mathscr{F}$, i.e., a functor $(\mathcal{C}/\mathscr{F})^{\text{op}} \to \text{Set}$ satisfying the sheaf condition. A morphism of sheaves of sets is a natural transformation of functors. We write $\text{Shv}(\mathscr{F})$ for the category of sheaves on $\mathscr{F}$.

Remark 11. Technically speaking, the category $\mathrm{Shv}(\mathscr{F})$ may not be locally small. There are standard ways to deal with any issues that might arise, so we will not worry about it.

The category $\mathrm{Shv}(\mathscr{F})$ naturally has a symmetric monoidal structure: to compute the product, first one takes the value-wise product, but this will generally not be a sheaf, so one then has to sheafify.

If $f\colon \mathscr{F} \to \mathscr{G}$ is a morphism of stacks, then we get a functor $\mathcal{C}/\mathscr{F} \to \mathcal{C}/\mathscr{G}$ given by postcomposition. Thus on sheaves, the direction is reversed, and we get a functor $f^*\colon \mathrm{Shv}(\mathscr{G}) \to \mathrm{Shv}(\mathscr{F})$. Moreover, this functor is symmetric monoidal.

Similarly, one can look at sheaves of abelian groups instead of sheaves of sets. The category of Ab-valued sheaves also has a natural symmetric monoidal structure, given by the sheafification of the value-wise tensor product.

4 Quasi-coherent sheaves

Henceforth, unless otherwise noted, we consider Aff to have the so-called *fpqc topology* (see [Vis08, §2.3.2]). We can define the notion of a quasi-coherent sheaf on an fpqc stack $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{Grpd}$.

Definition 12. Let $\mathcal{X}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Grpd}$ be an fpqc stack of groupoids.

(1) The **structure sheaf** of $\mathcal{X}$ is the fpqc sheaf of rings $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$ (i.e., a sheaf of rings on $\mathrm{Aff}/\mathcal{X}$) given by sending $\operatorname{Spec} R \to \mathcal{X}$ to R, and by sending a morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ over $\mathcal{X}$ to the morphism $S \to R$. This is a ring object in the symmetric monoidal category of sheaves of abelian groups on $\mathcal{X}$.

(2) A **sheaf of $\mathcal{O}_{\mathcal{X}}$-modules** on $\mathcal{X}$ (or an $\mathcal{O}_{\mathcal{X}}$*-module* for short) is a module object over $\mathcal{O}_{\mathcal{X}}$ in the category of Ab-valued fpqc sheaves on $\mathcal{X}$.

(3) A **quasi-coherent sheaf** F on $\mathcal{X}$ is an fpqc sheaf of $\mathcal{O}_{\mathcal{X}}$-modules such that for every morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ over $\mathcal{X}$, the natural map

$$F(\operatorname{Spec} S \to \mathcal{X}) \otimes_S R \longrightarrow F(\operatorname{Spec} R \to \mathcal{X})$$

is an isomorphism. We denote the category of quasi-coherent sheaves on $\mathcal{X}$ by $\mathrm{QCoh}(\mathcal{X})$.

More concretely, if M is an $\mathcal{O}_{\mathcal{X}}$-module, then the abelian group $M(\operatorname{Spec} A \to \mathcal{X})$ will be equipped with the structure of an A-module.

Remark 13. Compared to the definition of quasi-coherent sheaves on schemes given at the beginning, there seems to be missing data in this new definition. For we only record its value on affines, whereas in the original case we also recorded, e.g., the global sections $F(X)$. The reason we do not lose data is that the global sections can be recovered from this definition. Indeed, we can cover a scheme X by open affine subschemes $\operatorname{Spec} A_i \to X$, which is a Zariski cover of X. Given a quasi-coherent sheaf F on X in the new sense, we *define* the global sections $F(X)$ by

$$F(X) := \lim_i F(\operatorname{Spec} A_i \to X).$$

This limit is independent of the cover chosen on X, precisely because F is a sheaf in the fpqc topology on Aff/X, so in particular a Zariski sheaf.

Remark 14. A priori, this definition seems to be asking too much compared to the classical formulation from §1. Namely, originally we only asked for a Zariski sheaf, whereas right now we are asking for the much stronger condition of an fpqc sheaf. Grothendieck showed that quasi-coherent sheaves over schemes also satisfy fpqc descent, i.e., that they also define fpqc sheaves. This is a special case of Theorem 26 below, although that theorem is formulated in a slightly different way.

Remark 15. We can motivate the definition for the structure sheaf $\mathcal{O}_{\mathcal{X}}$ further in the case where $\mathcal{X} = X$ is a scheme. In the non-functor of points definition of schemes, then if $\operatorname{Spec} A \cong U \subseteq X$ is an open affine scheme, we have $\mathcal{O}_X(U) \cong A$. Note that if we embed $\operatorname{Spec} A$ as an open affine subscheme $V \subseteq X$ in a different way, we still have $\mathcal{O}_X(V) \cong A$. Now if $\operatorname{Spec} R \to X$ is any map of schemes, then étale locally this map factors through the inclusion of an open affine subscheme of X. Even in the non-functor of points perspective therefore, we could extend $\mathcal{O}_X$ by defining $\mathcal{O}_X(\operatorname{Spec} R)$ to be the limit

$$\mathcal{O}_X(\operatorname{Spec} R) := \lim_i \mathcal{O}_X(U_i),$$

with $U_i \subseteq X$ the affine open subschemes such that $\operatorname{Spec} R \to X$ factors through the U_i on an étale cover of X. In the end, this would result in an isomorphism $\mathcal{O}_X(\operatorname{Spec} R) \cong R$ for any choice of factoring.

If $f\colon \mathcal{X} \to \mathcal{X}'$ is a morphism of stacks, then the functor $f^*\colon \operatorname{Shv}(\mathcal{X}') \to \operatorname{Shv}(\mathcal{X})$ is symmetric monoidal, and hence induces a functor on module objects. It preserves the quasi-coherence condition, and thus we get a functor $f^*\colon \operatorname{QCoh}(\mathcal{X}') \to \operatorname{QCoh}(\mathcal{X})$.

Proposition 16. *Let A be a commutative ring. Then we have an equivalence of categories*

$$\operatorname{QCoh}(\operatorname{Spec} A) \simeq \operatorname{Mod}_A.$$

Proof. We construct an equivalence $\operatorname{Mod}_A \to \operatorname{QCoh}(\operatorname{Spec} A)$ as follows. If M is an A-module, then let F_M be the sheaf on $\operatorname{Aff}/\operatorname{Spec} A$ defined by

$$F_M(\operatorname{Spec} R \to \operatorname{Spec} A) := M \otimes_A R.$$

If $\operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism over $\operatorname{Spec} A$ (i.e., a map $S \to R$ of A-algebras), then F_M sends it to the map

$$M \otimes_A S \longrightarrow M \otimes_A R$$

given by the tensor product with id_M. We equip F_M with the obvious $\mathcal{O}_X$-module structure. The sheaf F_M is quasi-coherent because the natural map

$$(M \otimes_A S) \otimes_S R \longrightarrow M \otimes_A R$$

is an isomorphism.

In the other direction, let $\operatorname{QCoh}(\operatorname{Spec} A) \to \operatorname{Mod}_A$ be given by sending F to the A-module $F(\mathrm{id}_{\operatorname{Spec} A})$. It is immediate that the composite $\operatorname{Mod}_A \to \operatorname{QCoh}(\operatorname{Spec} A) \to \operatorname{Mod}_A$ is equivalent to the identity, so we only have to check the other composite. Let $F \in \operatorname{QCoh}(\operatorname{Spec} A)$, and write $M = F(\mathrm{id}_{\operatorname{Spec} A})$. By definition of a quasi-coherent sheaf, the natural map

$$M \otimes_A R \longrightarrow F(\operatorname{Spec} R \to \operatorname{Spec} A)$$

is an isomorphism for every map $\operatorname{Spec} R \to \operatorname{Spec} A$, proving the claim. ■

Going forward, we shall use the previously constructed equivalences when identifying Mod_A with QCoh(Spec A).

Remark 17. There is a different, more categorical manner of defining QCoh, for which the technology of ∞-categories is very helpful. We wish to define QCoh as a functor $\mathrm{Fun}_{\mathrm{sh}}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}$, which we can do by defining it as a functor to Cat_∞ that lands in 1-categories. On representable functors, we define QCoh to send Spec A to Mod_A (turning Proposition 16 into a definition rather than a theorem). We can then take a right Kan extension to extend it to the entire category of sheaves on Aff.

$$\begin{array}{ccc} \mathrm{Aff}^{\mathrm{op}} & \longrightarrow & \mathrm{Cat}_\infty \\ \downarrow & \overset{\mathrm{QCoh}}{\nearrow} & \\ \mathrm{Fun}_{\mathrm{sh}}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set})^{\mathrm{op}} & & \end{array}$$

Note that this would *not* result in the right functor QCoh if we had worked with 1-categories: taking limits in Cat_∞ is different from taking limits in Cat. One can then show that his has nice properties, and coincides with the above definition. See [Pst21, §4] for more information.

5 Hopf algebroids

Definition 18. Let $\mathcal{C}$ be a category with pullbacks. A **groupoid object** in $\mathcal{C}$ is a pair (X, Y) of $X, Y \in \mathcal{C}$, together with

(i) two maps $s, t \colon Y \to X$ called the *source* and *target map*, respectively;

(ii) a map $e \colon X \to Y$ called the *unit map*;

(iii) a map $Y \times_X Y \to X$ (the pullback being formed using the maps s and t) called *composition*;

(iv) a map $Y \to Y$ called the *inverse map*,

satisfying a number of compatibility axioms. We call X the *objects*, and Y the *morphisms* of the groupoid object.

If $\mathcal{C}$ is a category with pushouts, then a **cogroupoid object** in $\mathcal{C}$ is a groupoid object in $\mathcal{C}^{\mathrm{op}}$.

If (X, Y) is a groupoid object, and if $A \in \mathcal{C}$, then we get a groupoid whose set of objects is $\mathrm{Hom}_{\mathcal{C}}(A, X)$ and whose set of morphisms is $\mathrm{Hom}_{\mathcal{C}}(A, Y)$. This is a functorial assignment, yielding a (pseudo-)functor

$$[X/Y]' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Grpd}, \quad A \longmapsto (\mathrm{Hom}_{\mathcal{C}}(A, X),\ \mathrm{Hom}_{\mathcal{C}}(A, Y)).$$

We say that this is the functor *corepresented* by the cogroupoid object (X, Y).

Now suppose $\mathcal{C}$ is a site, and that this topology is subcanonical (i.e., representable functors are sheaves). In general, $[X/Y]'$ need not be a stack, but it will always be a prestack (i.e., one can glue morphisms locally): this is essentially because $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ is a sheaf (by assumption of subcanonicity of $\mathcal{C}$). We can turn the prestack $[X/Y]'$ into a stack by *stackification*, and we denote the resulting stack by $[X/Y]$.

Definition 19. A **Hopf algebroid** is a cogroupoid object in CRing.

For a Hopf algebroid, the following notation and terminology is often used for the structure maps:

(i) the *left* and *right unit maps* $\eta_L, \eta_R\colon A \to \Gamma$;

(ii) the *counit* or *augmentation* $\varepsilon\colon \Gamma \to A$;

(iii) the *comultiplication* $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$;

(iv) the *conjugation* $c\colon \Gamma \to \Gamma$.

Remark 20. If the left and right unit coincide, then the Hopf algebroid is called a *Hopf algebra*.

A Hopf algebroid (A, Γ) gives rise to a stack

$$[A/\Gamma]\colon \mathrm{CRing} \simeq \mathrm{Aff}^{\mathrm{op}} \longrightarrow \mathrm{Grpd}.$$

Conversely, if we begin with a stack on Aff, we can ask if there is a Hopf algebroid corepresenting it in this way. This is not true in general, but in cases we are interested in (i.e., the moduli stack of formal groups) it does hold. While we could focus on this stack alone, the procedure will be the same for any so-called *algebraic stack*.

6 Algebraic stacks

An algebraic stack is one that is 'close' to being a scheme. Roughly speaking, an algebraic stack is to schemes what an orbifold is to manifolds. We will make this precise by defining an algebraic stack to be a stack $\mathcal{X}$ which admits a 'cover' $\operatorname{Spec} A \to \mathcal{X}$ from an affine scheme. To make this precise, we need to put requirements on this cover; this we do as follows.

Definition 21. Let $\mathcal{X}, \mathcal{X}'$ be stacks on Aff (or more generally, pseudo-presheaves of groupoids).

- A map $f\colon \mathcal{X} \to \mathcal{X}'$ is called **representable** if for every map $S \to \mathcal{X}'$ with X a scheme, the pullback $S \times_{\mathcal{X}'} \mathcal{X}$ is equivalent to a scheme.
- Let P be a property of morphisms of schemes that is closed under pullback. Then we say that a representable morphism $\mathcal{X} \to \mathcal{X}'$ satisfies P if $S \times_{\mathcal{X}'} \mathcal{X} \to X$ satisfies P for every map $S \to \mathcal{X}'$ with S a scheme.

We can thus make sense of when a map between stacks is affine or fpqc.

Definition 22. An **algebraic stack** is an fpqc stack of groupoids $\mathcal{X}$ on Aff, such that there exists an affine fpqc map $\operatorname{Spec} A \to \mathcal{X}$ for some ring A. Such a map $\operatorname{Spec} A \to \mathcal{X}$ is called a **presentation** of $\mathcal{X}$.

Remark 23. There are other niceness conditions on stacks around. Most common are *Artin stacks* and *Deligne–Mumford stacks*; the latter of the two is the strongest assumption. For Artin stacks, one ought to work with the *smooth topology* instead of the fpqc topology, and for Deligne–Mumford stacks one should work with the *étale topology*. Algebraic geometers often use the term 'algebraic stack' to mean an Artin stack. We cannot afford this luxury however, because the moduli stack of formal groups is not an Artin stack, but only an algebraic stack in the sense above.

If $\operatorname{Spec} A \to \mathcal{X}$ is an affine fpqc map to an algebraic stack, then we can pull it back against itself.

By assumption, the pullback will itself be an affine scheme again:

$$\begin{array}{ccc} \operatorname{Spec}\Gamma & \longrightarrow & \operatorname{Spec} A \\ \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{Spec} A & \longrightarrow & \mathcal{X}. \end{array}$$

Then the pair $(\operatorname{Spec} A,\ \operatorname{Spec}\Gamma)$ forms a groupoid object in the category Aff, with the following maps.

- The source and target maps $\operatorname{Spec}\Gamma \to \operatorname{Spec} A$ come from the two projections in the pullback.
- The unit map $\operatorname{Spec} A \to \operatorname{Spec}\Gamma$ is the diagonal.
- The composition is the map

  $$\operatorname{Spec}\Gamma \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma \simeq \operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A \longrightarrow \operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A \simeq \operatorname{Spec}\Gamma$$

  given by the projection onto the first and third coordinate.
- Lastly, the inverse is the switch map $\operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A \to \operatorname{Spec}_A \times_{\mathcal{X}} \operatorname{Spec} A$.

Pulling this through the equivalence $\mathrm{Aff} \simeq \mathrm{CRing}^{\mathrm{op}}$, we find that the pair (A, Γ) is a Hopf algebroid. Because $\mathcal{X}$ is algebraic, it turns out that the stack $[A/\Gamma]$ is equivalent to $\mathcal{X}$: see [Nau07, §3.3].

Definition 24. Let (A, Γ) be a Hopf algebroid. A (left) **comodule** over (A, Γ) is an A-module M together with a map $M \to \Gamma \otimes_A M$ satisfying counitality and coassociativity. A map of comodules is a map intertwining the coaction maps. We write $\mathrm{coMod}_{(A,\Gamma)}$ for the category of (left) comodules over (A, Γ).

Example 25. The A-module A is naturally a left (A, Γ)-comodule: take $A \to \Gamma \otimes_A A \cong \Gamma$ to be the counit map η_L. ▲

If we buy the intuition that an algebraic stack $\mathcal{X}$ is a scheme Spec A quotiented by some type of group action, then it seems reasonable to expect that a quasi-coherent sheaf over $\mathcal{X}$ should be the same as a quasi-coherent sheaf over Spec A (i.e., an A-module) together with a type of group action on the sheaf, compatible with the action on $\mathcal{X}$. A comodule over (A, Γ) looks exactly like the datum of such a quasi-coherent sheaf with a group action.

Let $\mathcal{X}$ be an algebraic stack and $\operatorname{Spec} A \to \mathcal{X}$ an affine fpqc map, and write (A, Γ) for the resulting Hopf algebroid. If F is a quasi-coherent sheaf on $\mathcal{X}$, then $M := F(\operatorname{Spec} A)$ is an A-module. We can give it the structure of a (left) (A, Γ)-comodule as follows. In the diagram

$$\begin{array}{ccccc} M \otimes_A \Gamma & \xrightarrow{\cong} & F(\operatorname{Spec}\Gamma) & \xleftarrow{\cong} & \Gamma \otimes_A M \\ {\scriptstyle M\otimes\eta_L}\big\uparrow & & & \nearrow & \\ M & & & & \end{array}$$

the solid vertical arrow is given by tensoring M with the left unit η_L. The horizontal map on the left is induced by η_L, and the horizontal map on the right is induced by η_R. These two horizontal maps are isomorphisms because F is a quasi-coherent sheaf. Thus there is a unique dashed arrow making the diagram commute. This we take to be the coaction that turns M into an (A, Γ)-comodule.

Theorem 26 (Faithfully flat descent, Grothendieck). *Let $\mathcal{X}$ be an algebraic stack, and let $p\colon \operatorname{Spec} A \to \mathcal{X}$ be an affine fpqc map. Write (A, Γ) for the corresponding Hopf algebroid. The lift of the functor $p^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ described above is an equivalence.*

$$\begin{array}{ccc} & & \mathrm{coMod}_{(A,\Gamma)} \\ & \overset{\simeq}{\nearrow} & \downarrow \\ \mathrm{QCoh}(\mathcal{X}) & \xrightarrow[p^*]{} & \mathrm{Mod}_A \end{array}$$

If (A, Γ) is a Hopf algebroid, then the category $\mathrm{coMod}(A, \Gamma)$ is an abelian category with enough injectives, and thus we can define Ext-groups in that category. Thus, we can do the same in $\mathrm{QCoh}(\mathcal{X})$.

Definition 27. Let $\mathcal{X}$ be an algebraic stack, and let F be a quasi-coherent sheaf on $\mathcal{X}$. We define the ***n*-th sheaf cohomology group** of F by

$$H^n(\mathcal{X}; F) := \mathrm{Ext}^n_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, F).$$

If (A, Γ) is a Hopf algebroid representing $\mathcal{X}$, and M the comodule corresponding to F, then we have an isomorphism

$$H^n(\mathcal{X}; F) \cong \mathrm{Ext}^n_{(A,\Gamma)}(A, M).$$

The group $H^0(\mathcal{X}; F) = \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, F)$ is also called the **global sections** of F. It is a module over the global sections of $\mathcal{O}_{\mathcal{X}}$, i.e., over the ring $\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$.

7 Complex bordism

Let E be a flat homotopy commutative ring spectrum, i.e., one for which $E_*E = \pi_*(E \otimes E)$ is a flat π_*E-module. Then the maps

(i) $\pi_*E \to E_*E$ induced by smashing with $\mathbf{S} \to E$ on the left/right;

(ii) $E_*E \to \pi_*E$ induced by multiplication;

(iii) $E_*E \to \pi_*(E \otimes E \otimes E) \cong E_*E \otimes_{\pi_*E} E_*E$ induced by smashing with $\mathbf{S} \to E$ in the middle;

(iv) $E_*E \to E_*E$ induced by the switching,

turn $(\pi_*E,\ E_*E)$ into a graded Hopf algebroid (i.e., a cogroupoid object in graded rings). Moreover, for every spectrum X, the homology E_*X will get the structure of a left $(\pi_*E,\ E_*E)$-comodule.

Henceforth we will study the case $E = MU$. We actually already computed the Hopf algebroid associated to MU: this is the graded Hopf algebroid $(L,\ L[b_1, b_2, \ldots])$, where L is the Lazard ring. From an algebro-geometric point of view, this is the Hopf algebroid associated to the moduli stack $\mathcal{M}^s_{\mathrm{FG}}$ of strict formal groups. We can compare this to the non-strict version $\mathcal{M}_{\mathrm{FG}}$ as follows.

Recall that a grading is the same as a $\mathbf{G}_m$-action (see, e.g., [Mei20, Prop. 3.26, Ex. 4.45]). As such, we can package the evenly graded Hopf algebroid $(L,\ L[b_1, b_2, \ldots])$ as the Hopf algebroid $(L,\ L[b_0^{\pm}, b_1, b_2, \ldots])$. By "package", we mean that we get an equivalence of categories

$$\mathrm{coMod}(L,\ L[b_0^{\pm}, b_1, b_2, \ldots]) \simeq \mathrm{coMod}^{\mathrm{gr,\ ev}}(L,\ L[b_1, b_2, \ldots])$$

between comodules on the left, and evenly graded comodules on the right.

Recall that we have equivalences of stacks

$$\mathcal{M}_{\mathrm{FG}}^{s} \simeq [\operatorname{Spec} L / \operatorname{Spec} L[b_1, b_2, \dots]] \qquad \text{and} \qquad \mathcal{M}_{\mathrm{FG}} \simeq [\operatorname{Spec} L / \operatorname{Spec} L[b_0^{\pm}, b_1, b_2, \dots]].$$

In conclusion, we find that

$$\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^{s}) \simeq \mathrm{coMod}(L,\ L[b_1, b_2, \dots]) \qquad \text{and} \qquad \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}) \simeq \mathrm{coMod}(L,\ L[b_0^{\pm}, b_1, b_2, \dots]).$$

Now for our applications, recall that in the Adams–Novikov spectral sequence for a spectrum X, the E_2-page consists of Ext-groups of the form

$$\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*,\ MU_*(X)).$$

Here the second index t was the internal degree, coming from the fact that we are dealing with graded groups. We can rewrite this into stack-theoretic terms as follows. Let $F_i(X)$ denote the graded object $MU_{2*+i}(X)$. (Thus F_0 is the even part of $MU_*(X)$, and F_2 is a shift of F_0, etc.) This is an $(MU_*,\ MU_*MU)$-comodule with a compatible grading, and so we can view it as a module over $\mathcal{M}_{\mathrm{FG}}$. We write ω for the quasi-coherent sheaf F_2. In the end, the E_2-page of the Adams–Novikov spectral sequence becomes

$$E_2^{s,t} \cong H^s(\mathcal{M}_{\mathrm{FG}};\ F_t(X)).$$

People often express this differently, as follows. Write ω for the quasi-coherent sheaf F_2. This turns out to be an invertible sheaf (a.k.a. a line bundle), which means that negative powers of ω also make sense. Distiguishing between when t is even or odd, we get

$$E_2^{s,2a} \cong H^s(\mathcal{M}_{\mathrm{FG}};\ F_0(X) \otimes \omega^{\otimes a}),$$
$$E_2^{s,2a+1} \cong H^s(\mathcal{M}_{\mathrm{FG}};\ F_1(X) \otimes \omega^{\otimes a}).$$

See [Mei20, §4.6] for more information. The upshot is that understanding $\mathcal{M}_{\mathrm{FG}}$ and quasi-coherent sheaves thereon will give us a lot of information about the stable homotopy category.

Remark 28. The line bundle ω is an inverse to the Lie algebra sheaf on $\mathcal{M}_{\mathrm{FG}}$. Specifically, the quasi-coherent sheaf ω^{-1} is equivalent to the quasi-coherent sheaf which sends $G\colon \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ (i.e., a formal group G over R) to the Lie algebra of G. For this reason, ω is called the *sheaf of invariant differentials* on $\mathcal{M}_{\mathrm{FG}}$.

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