A brief intermezzo on stacks

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The goal of this lecture will be to introduce the notion of a stack of groupoids on a site. Our main example of a stack will be vector bundles on topological spaces. First we motivate the notion of a stack by studying the behaviour of gluing vector bundles. We introduce the language of sites and sheaves, after which we are ready to define what a stack is. We end with a more geometric interpretation of what a stack is. The introduction is inspired by the notes by Groechenig [Gro14], and the rest has been adapted from the notes by Vistoli [Vis08] and Meier [Mei20]. Especially the notes by Vistoli serve as a good follow-up for further reading about stacks (from an algebro-geometric perspective).

1 Introduction: gluing and descending vector bundles

The running theme of this talk is the notion of gluing objects. Consider the example of vector bundles. If X is a topological space and E a vector bundle over X, then we can find an open cover $\{ U_i \subseteq X \}_i$ such that $E_i := E\big|_{U_i}$ is a trivial vector bundle on U_i. One might ask, can we go back? In other words, if one is given a trivial vector bundle on U_i for every i, can one glue these to form a vector bundle on E?

One cannot do this as stated; there is additional data that is needed. As discussed in any course introducing vector bundles, the bit of data that is needed are *transition functions* $\varphi_{ij} \colon U_i \cap U_j \to \mathrm{GL}(n, \mathbf{R})$ for every i and j. Equivalently, such a transition function is an automorphism of the trivial vector bundle on $U_i \cap U_j$. We will use this formulation of the φ_{ij} for the remainder of the talk. These transition functions should moreover satisfy the *cocycle condition*

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$$

when restricted to $U_i \cap U_j \cap U_k$. One can then show that given such transition functions, one can glue these trivial bundles to a bundle on E.

We will now describe a different formulation for this data. Write Y for the disjoint union $\bigsqcup_i U_i$, and $p \colon Y \to X$ for the map induced by the inclusions. Then $\widetilde{E} := p^*E$ is a trivial bundle on Y. Observe that

$$Y \times_X Y \cong \bigsqcup_{i,j} U_i \cap U_j.$$

The transition functions φ_{ij} now combine to yield a morphism

$$p_2^* \widetilde{E} \longrightarrow p_1^* \widetilde{E}$$

of vector bundles over $Y \times_X Y$, where p_1 and p_2 denote the two projections $Y \times_X Y \to Y$. Also observe that

$$Y \times_X Y \times_X Y \cong \bigsqcup_{i,j,k} U_i \cap U_j \cap U_k.$$

The cocycle conditions on the φ_{ij} now combine to yield the identity

$$p_{13}^* \varphi = p_{23}^* \varphi \circ p_{12}^* \varphi$$

as maps of vector bundles on $Y \times_X Y \times_X Y$. Here p_{12} denotes the projection away from the third component, and similarly for p_{13} and p_{23}.

Let us now consider a different, but similar situation to gluing vector bundles: that of *descending* vector bundles. One could do this for a general covering map $p\colon Y \to X$, but let us for simplicity focus on a specific example, namely that of the double cover of the circle, i.e., $Y = X = S^1$, and $p\colon Y \to X$ wraps the circle around itself twice. We may identify the fibres of p with the group C_2, where C_2 acts on Y by the antipodal action. In this way we can identify the map p with the quotient $Y \to Y/C^2 \cong X$. Now, if E is a vector bundle over X, we can pull it back to a vector bundle $\widetilde{E} := p^* E$ on Y. The question again is: what data do we need to go back?

Let $\widetilde{E}$ be a vector bundle over Y. In order to descend this to a bundle on X, it turns out we need a C_2-action on $\widetilde{E}$ that is compatible with the action of C_2 on Y, and that is fibre-wise linear. Indeed, if we have such an action, then we can define $E := \widetilde{E}/C_2$, and the conditions on the action ensure that this is indeed a vector bundle on X.

We can rephrase this situation too. Observe that we have a homeomorphism

$$C_2 \times Y \xrightarrow{\cong} Y \times_X Y, \quad (\sigma, y) \longmapsto (\sigma \cdot y,\ y),$$

and under this identification, the two projection maps become

$$\begin{aligned} p_1\colon C_2 \times Y &\longrightarrow Y, \quad (\sigma, y) \longmapsto \sigma \cdot y, \\ p_2\colon C_2 \times Y &\longrightarrow Y, \quad (\sigma, y) \longmapsto y. \end{aligned}$$

Similarly, we have a homeomorphism

$$C_2 \times C_2 \times Y \xrightarrow{\cong} Y \times_X Y \times_X Y, \quad (\sigma, \tau, y) \longmapsto ((\sigma\tau) \cdot y,\ \sigma \cdot y,\ y),$$

under which we can identify the projections with the maps

$$\begin{aligned} p_{12}\colon C_2 \times C_2 \times Y &\longrightarrow C_2 \times Y, \quad (\sigma, \tau, y) \longmapsto (\sigma,\ \tau \cdot y), \\ p_{13}\colon C_2 \times C_2 \times Y &\longrightarrow C_2 \times Y, \quad (\sigma, \tau, y) \longmapsto (\sigma \circ \tau,\ y), \\ p_{23}\colon C_2 \times C_2 \times Y &\longrightarrow C_2 \times Y, \quad (\sigma, \tau, y) \longmapsto (\tau, y). \end{aligned}$$

Tracing through these maps, one can show that a C_2-action on $\widetilde{E}$ of the above form is equivalent to a vector bundle homomorphism

$$\varphi\colon p_2^* \widetilde{E} \longrightarrow p_1^* \widetilde{E}$$

such that the identity

$$p_{13}^* \varphi = p_{23}^* \varphi \circ p_{12}^* \varphi$$

is satisfied. (This last identity turns out to encode the mixed associativity $\sigma(\tau y) = (\sigma\tau)y$ of the action.)

Notice that the problems of gluing and descending vector bundles look more or less the same, once we rewrite them in terms of a map φ satisfying a cocycle condition. This is a useful abstraction: it allows one to treat both situations using the same formalism. This formalism is given by the language of stacks.

If $\mathcal{C}$ is a category, a stack on $\mathcal{C}$ is a functor sending an $X \in \mathcal{C}$ to some class of "objects over X" such that these objects can be glued together along *certain* maps of $\mathcal{C}$, provided that one has gluing data that satisfies a cocycle condition. The "certain" maps will determine the meaning of what a stack is. Roughly speaking, the two examples above consider either the class of open embeddings, and the class of covering maps. The above discussion then shows that vector bundles form a stack for both types of maps. Most of this lecture is devoted to formalising this statement.

2 Sites

The formal name for such a class of morphisms is called a *Grothendieck topology* on $\mathcal{C}$, and a category with a Grothendieck topology is called a *site*.

Definition 1. Let $\mathcal{C}$ be a category with pullbacks. A **Grothendieck topology** on $\mathcal{C}$ consists of the following data: an assignment to every $X \in \mathcal{C}$ a collection of collections of arrows $\{ U_i \to X \}_i$, called **coverings** of X, subject to the following conditions.

(a) If $f \colon Y \to X$ is an isomorphism, then $\{ f \}$ is a covering of X.

(b) If $X \in \mathcal{C}$ and $\{ U_i \to X \}_i$ is a covering of X, and for every i the collection $\{ U_{ij} \to U_i \}_j$ is a covering of U_i, then the collection $\{ U_{ij} \to X \}_{i,j}$ is a covering of X.

(c) If $\{ U_i \to X \}_i$ is a covering of X and $V \to X$ is a morphism, then $\{ U_i \times_X V \to V \}$ is a covering.

A **site** is a pair of a category with pullbacks and a Grothendieck topology on it.

The terminology is (unashamedly) stolen from the field of topology. One can think of the second condition as being similar to unions of open sets being open; the third condition is similar to the intersection of opens being open. This is illustrated by the following example.

Example 2 (Open topology). Consider $\mathcal{C} = \mathrm{Top}$. If $X \in \mathrm{Top}$, we define a collection $\{ U_i \to X \}_i$ to be a cover of X when it is

(a) jointly surjective, i.e., $\bigsqcup_i U_i \to X$ is surjective;

(b) every map $U_i \to X$ is an open embedding.

This defines a topology on Top, which we will call the **open topology** on Top. Indeed, the first and second conditions are obviously satisfied. For the third, note that if $f \colon V \to X$ is a map and $\{ U_i \subseteq X \}_i$ is a covering by open subsets, then $U_i \times_X V$ is (isomorphic to) $f^{-1}(U_i)$, and under this identification the projection $U_i \times_X V \to U_i$ is given by the inclusion $f^{-1}(U_i) \subseteq V$. More generally, if $f_i \colon U_i \to X$ is an open embedding, then $U_i \times_X V$ is isomorphic to $f^{-1}(f_i(U_i))$. ▲

Example 3. Let X be a topological space, and let $\mathrm{Open}(X)$ denote the category whose objects are open subsets of X, and the morphisms are given by inclusions of sets. Then the topology from Example 2 induces a topology on $\mathrm{Open}(X)$. ▲

An important difference between point-set topology and Grothendieck topologies lies in the behaviour of the pullbacks $U_i \times_X U_j$. While in the previous examples this behaviour was more or less the same, in general it can be quite different. For instance, in the case where $U_i = U_j = U$, the two projections $U \times_X U \to U$ need not be the same, as the following example illustrates.

Example 4 (Covering topology). Again consider $\mathcal{C} = \mathrm{Top}$, but now with the following topology: a collection $\{ U_i \to X \}_i$ is defined to be a cover of X when it is

(a) jointly surjective;

(b) every map $U_i \to X$ is a local homeomorphism.

For lack of a better name, we call this the **covering topology** on Top. In this topology, a surjective covering map $p\colon Y \to X$ defines a covering $\{ p \}$ of X. If $p\colon Y \to X$ is such a surjective covering map with fibre F, then we have

$$Y \times_X Y \cong F \times Y.$$

Note that the two projections $Y \times_X Y \to Y$ need not be the same. For example, consider again the two-fold covering map of the circle, i.e., $p\colon Y \to X$ where $X = Y = S^1$. In the introduction we saw that the two projections $Y \times_X Y \to Y$ are not the same. (The case for a general fibre F with size $\#F > 1$ is analogous, by identifying F with the group of deck transformations of the cover.) ▲

Examples 2 and 4 show that the language of sites captures both examples from the introduction. In a general site, one can think of the coverings as maps one would like to 'glue' or 'descend' objects along. (Whether one should think of this as 'gluing' or 'descending' depends on the topology.) Before we discuss what we mean by gluing objects along maps in a Grothendieck topology, we will take a step back and consider an easier examples: *sheaves* on sites.

3 Sheaves

A **presheaf of sets** F on a topological space X is an assignment of a set $F(U)$ to every open subset U of X, together with restriction maps $F(U) \to F(V)$ whenever $V \subseteq U$, and these restrictions should be compatible with chains of inclusions. Important examples include the sheaf of smooth functions $C^\infty(M)$ on a smooth manifold M, or the structure sheaf $\mathcal{O}_X$ on a scheme X. These examples satisfy a special property: functions on open subsets can be compared locally, and can be constructed locally. This is called the *sheaf condition*. Formally, a **sheaf of sets** on X is a presheaf of sets F on X satisfying: for every open subset U of X and every open covering $\{ U_i \subseteq U \}_i$ of U, the natural diagram

$$F(U) \longrightarrow \textstyle\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

should be an equaliser diagram of sets. Note that the injectivity of $F(U) \to \prod_i F(U_i)$ says functions can be compared locally; the additional statement that the diagram is an equaliser diagram says that functions can be glued together when constructed on the opens in an open cover.

We can generalise these definitions to general sites.

Definition 5. Let $\mathcal{C}$ be a category. A **presheaf of sets** on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

If F is a presheaf of sets on $\mathcal{C}$ and $U \to X$ and $V \to X$ are two morphisms in X, then we get maps $F(U) \to F(U \times_X V)$ and $F(V) \to F(U \times_X V)$ induced by the two projections $U \times_X V \to U$ and $U \times_X V \to V$, respectively.

Definition 6. Let $\mathcal{C}$ be a site. A **sheaf of sets** on $\mathcal{C}$ is a presheaf of sets $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ that satisfies the **sheaf condition**: for every $X \in \mathcal{C}$ and every covering $\{ U_i \to X \}_i$ of X, the diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

is an equaliser diagram.

Example 7. Let X be a topological space. Then a (pre)sheaf of sets on $\mathrm{Open}(X)$ is the same as a (pre)sheaf of sets on X, where $\mathrm{Open}(X)$ has the topology from Example 3. Thus, one example of a sheaf on $\mathrm{Open}(X)$ is the sheaf of continuous real-valued functions $C(-, \mathbf{R})$, sending an open $U \subseteq X$ to the set $C(U, \mathbf{R})$ of continuous functions $U \to \mathbf{R}$. ▲

Example 8. One can also consider a global variant of $C(-, \mathbf{R})$, by defining it as a functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ sending X to $C(X, \mathbf{R})$. This is a sheaf in the open topology on Top, and also for the covering topology on Top. In fact, the sheaf $C(-, \mathbf{R})$ on $\mathrm{Open}(X)$ from the previous example is obtained by restricting it to the subcategory $\mathrm{Open}(X)$. ▲

4 Stacks

The key reason that things like $C^\infty(M)$ or $\mathcal{O}_X$ form sheaves is that functions can be glued together whenever they agree on intersections. In other words, you do not need data to glue them: checking their equality is enough. This breaks down when we consider more difficult objects, such as vector bundles: equality of vector bundles is a useless notion. We have already seen that to glue vector bundles, you need data (namely, an isomorphism on intersections, subject to a cocycle condition). The reason that checking equality no longer works for vector bundles is that vector bundles can have nontrivial automorphisms. Objects that have nontrivial automorphisms do not live in a set, but instead live in a *groupoid*.

Definition 9. A **groupoid** is a category such that all morphisms are isomorphisms. A **map of groupoids** is a functor.

Example 10. Let X be a topological space, and n a natural number. Then vector bundles of rank n over X form a groupoid $\mathrm{Vect}_n(X)$, where the objects are vector bundles of rank n, and the morphisms are the isomorphisms of vector bundles. ▲

We would like to say that Vect_n forms a contravariant functor from Top to groupoids. However, life is not this simple. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of topological spaces, then while we do have pullback maps

$$f^*\colon \mathrm{Vect}_n(Y) \longrightarrow \mathrm{Vect}_n(X) \qquad \text{and} \qquad g^*\colon \mathrm{Vect}_n(Z) \longrightarrow \mathrm{Vect}_n(Y),$$

the composite f^*g^* is *not* equal to $(g \circ f)^*$. Instead, we only have a natural isomorphism $f^*g^* \cong (g \circ f)^*$.

To remedy this, we have to consider groupoids as living in a 2-*category*. A 2-category is a category which does not merely have morphisms between objects, but can also have morphisms between morphisms (called 2-*morphisms*). In the above example, the natural isomorphism $f^* g^* \cong (g \circ f)^*$ is an example of a 2-morphism in the 2-category Grpd of groupoids. While one could formalise this by developing the theory of 2-categories, we will only need the following definition.

Definition 11. Let $\mathcal{C}$ be a category. A **pseudo-presheaf of groupoids** $\mathscr{F}$ on $\mathcal{C}$ consists of the data

(i) for every $X \in \mathcal{C}$, a groupoid $\mathscr{F}(X)$;

(ii) for every morphisms $f\colon X \to Y$ in $\mathcal{C}$, a map of groupoids $f^*\colon \mathscr{F}(Y) \to \mathscr{F}(X)$;

(iii) for every two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, a natural isomorphism $\varphi_{f,g}\colon f^* g^* \simeq (g \circ f)^*$,

satisfying the following conditions:

(a) for all $X \in \mathcal{C}$, we have $\mathrm{id}_X^* = \mathrm{id}_{\mathscr{F}(X)}$;

(b) for all morphisms $g\colon Y \to Z$, we have $\varphi_{\mathrm{id}_Y, g} = \mathrm{id}$;

(c) for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ and $h\colon Z \to W$, we have

$$\varphi_{f, h \circ g}(f^* \varphi_{g,h}) = \varphi_{g \circ f, h}(\varphi_{f,g} h^*).$$

Remark. A functor between 2-categories is also called a *pseudo-functor*. Every 1-category naturally defines a 2-category, where the 2-morphisms are only the identies $f \to f$. A pseudo-presheaf of groupoids is then the same as a pseudo-functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ to the 2-category of groupoids, justifying the term 'presheaf'.

Remark. An alternative way to define pseudo-functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ (where $\mathcal{C}$ is an ordinary category) is to work with so-called *Grothendieck fibrations* on $\mathcal{C}$, also called *fibered categories*. A Grothendieck fibration on $\mathcal{C}$ is a functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is an ordinary category, satisfying a number of conditions. One can then show that the category of pseudo-functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ is equivalent to the category of categories fibered in groupoids. If $\mathscr{F}'\colon \mathcal{D} \to \mathcal{C}$ is such a fibration, then the value $\mathscr{F}(X)$ of the corresponding pseudo-functor $\mathscr{F}$ is given by the subcategory of $\mathcal{D}$ on objects

$$\{\, d \in \mathcal{D} \mid \mathscr{F}'(d) = X \,\}$$

and morphisms

$$\mathrm{Hom}_{\mathscr{F}(X)}(d, e) = \{\, f\colon d \to e \text{ in } \mathcal{D} \mid \mathscr{F}'(f) = \mathrm{id}_X \,\}.$$

This has the benefit of avoiding any 2-categorical language in the definition, but it would require additional work to make precise. We will not pursue this, but refer the interested reader to [Vis08, Ch. 3].

Example 12. The groupoid $\mathrm{Vect}_n(X)$ from Example 10 forms a pseudo-presheaf of groupoids Vect_n on Top. Indeed, as we already remarked, we have natural isomorphisms $f^* g^* \cong (g \circ f)^*$, and one can check that these satisfy the conditions above. ▲

Taking Vect_n as our leading example, we note that this pseudo-presheaf has some special features.

(1) If X is a space and E and F are rank-n vector bundles over X, then we can construct maps $E \to F$ locally. More precisely, let $\{ U_i \subseteq X \}_i$ be an open cover of X, and write $E_i := E\big|_{U_i}$ and $F_i = F\big|_{U_i}$. If we have maps $f_i \colon E_i \to F_i$ such that $f_j\big|_{U_i \cap U_j} = f_i\big|_{U_i \cap U_j}$, then these maps glue to a map $f \colon E \to F$. In addition to being able to construct maps locally, we can also compare maps locally: if for two maps f, g their restrictions f_i and g_i to U_i are equal for every i, then f and g are equal. (In other words, the gluing of maps is unique.)

(2) We can construct vector bundles over X locally; this we explained in the introduction.

A stack is a generalisation of these two phenomena. First, we generalise the data we need to glue objects over X.

Definition 13. Let $\mathcal{C}$ be a site, and let $\mathscr{F}$ be a pseudo-presheaf of groupoids on $\mathcal{C}$. Let $X \in \mathcal{C}$, and let $\mathcal{U} = \{ U_i \to X \}_i$ be a cover of X. The category $\mathscr{F}(\mathcal{U})$ of **descent data** is the category whose

- objects are pairs $(\{ E_i \}_i, \{ \varphi_{ij} \}_{i,j})$, where $E_i \in \mathscr{F}(U_i)$ and $\varphi_{ij} \colon p_2^* E_j \to p_1^* E_i$ is a morphism in $\mathscr{F}(U_i \times_X U_j)$, such that the cocycle condition
$$p_{13}^* \varphi_{ik} = p_{23}^* \varphi_{jk} \circ p_{13}^* \varphi_{ij}$$
is satisfied (as maps in $\mathscr{F}(U_i \times_X U_j \times_X U_k)$);
- morphisms are sets $\{ \alpha_i \}_i$, where $\alpha_i \colon E_i \to F_i$ is a morphism in $\mathscr{F}(U_i)$ such that the diagram
$$\begin{array}{ccc} p_2^* E_j & \xrightarrow{p_2^* \alpha_j} & p_2^* F_j \\ {\scriptstyle \varphi_{ij}^E}\downarrow & & \downarrow{\scriptstyle \varphi_{ij}^F} \\ p_1^* E_i & \xrightarrow[p_1^* \alpha_i]{} & p_1^* F_i \end{array}$$
commutes.

An object in $\mathscr{F}(\mathcal{U})$ is called a **descent datum**.

Note that for every $X \in \mathcal{C}$ and every covering $\mathcal{U} = \{ f_i \colon U_i \to X \}_i$ of X, we have a natural map
$$\begin{aligned} \mathscr{F}(X) &\longrightarrow \mathscr{F}(\mathcal{U}), \\ E &\longmapsto \Big(\{ f_i^* E \}_i, \{ \varphi_{ij} \colon p_2^* f_j^* E \cong p_1^* f_i^* E \}_i \Big), \\ (\alpha \colon E \to F) &\longmapsto \{ f_i^* \alpha \}_i, \end{aligned}$$
where the isomorphism $p_2^* f_j^* \cong p_1^* f_i^*$ comes from the equality $f_j \circ p_2 = f_i \circ p_1$ of the commutative diagram
$$\begin{array}{ccc} U_i \times_X U_j & \xrightarrow{p_2} & U_j \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_j} \\ U_i & \xrightarrow[f_i]{} & X. \end{array}$$

Definition 14. Let $\mathcal{C}$ be a site, and let $\mathscr{F}$ be a pseudo-presheaf of groupoids on $\mathcal{C}$.

- We call $\mathscr{F}$ a **prestack of groupoids** if for every $X \in \mathcal{C}$ and every covering $\mathcal{U}$ of X, the natural map $\mathscr{F}(X) \to \mathscr{F}(\mathcal{U})$ is fully faithful.

- We call $\mathscr{F}$ a **stack of groupoids** if for every $X \in \mathcal{C}$ and every covering $\mathcal{U}$ of X, the natural map $\mathscr{F}(X) \to \mathscr{F}(\mathcal{U})$ is an equivalence.

Informally, a prestack is a pseudo-presheaf of groupoids where maps between two objects $E, F \in \mathscr{F}(X)$ can be compared locally and can be constructed locally. A stack is a prestack where in addition we can also construct the objects locally.

Remark. Some authors use the term 'prestack' to mean a pseudo-presheaf, which would make the terminology prestack/stack similar to the terminology presheaf/sheaf. The above definition of a prestack is due to Grothendieck, and is also used by [Vis08].

Remark. One can phrase the condition that maps can be constructed and compared locally in a more formal way: to any pseudo-presheaf of groupoids $\mathscr{F}$, there is an associated presheaf of maps of $\mathscr{F}$. This presheaf is a sheaf if and only if $\mathscr{F}$ is a prestack. See, e.g., [Vis08, Def. 2.58, §3.7, Prop. 4.7] for a precise discussion.

Unlike the notion of a pseudo-presheaf, the notion of a (pre)stack is a statement that is only defined on sites; in other words, changing the topology changes the meaning of what a stack is.

Example 15. The pseudo-presheaf Vect_n is a stack on Top, for both the open topology (from Example 2) and the covering topology (from Example 4) on Top. Saying that Vect_n is a stack in the open topology says that vector bundles can be constructed by restricting to an open cover of X. Saying that Vect_n is a stack in the covering topology says that vector bundles can be descended along covering maps. ▲

5 A geometric interpretation of stacks

If $\mathcal{C}$ is a category (thinking of an example like $\mathcal{C} = \mathrm{Top}$ or $\mathcal{C} = \mathrm{Aff}$, or some other category of 'spaces' will be helpful) and $X \in \mathcal{C}$ an object, we get a functor

$$h_X \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

In other words, the functor h_X remembers how other spaces can be mapped into X. The Yoneda embedding

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \quad X \longmapsto h_X$$

is fully faithful, so we can think of the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ as an enlargement of $\mathcal{C}$. An object $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ not in the image of the Yoneda embedding can be thought of as a 'generalised object of $\mathcal{C}$'. In this picture, we should think of $F(X)$ as the set of maps from X into this generalised object. A map between two 'generalised objects' $F, G \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is a natural transformation $F \to G$. The Yoneda lemma gives a natural isomorphism

$$\mathrm{Nat}(h_X, F) \cong F(X).$$

Thus, morphisms out of an 'honest' object of $\mathcal{C}$ to a generalised object of $\mathcal{C}$ agrees with $F(X)$, justifying our heuristic for $F(X)$.

Of course, not all functors are very nice objects. If we want to think of these functors as geometric objects (e.g., if $\mathcal{C} = \mathrm{Top}$ or $\mathcal{C} = \mathrm{Aff}$), then we still want to be able to construct maps of generalised spaces locally. Thus, if $\mathcal{C}$ is also a site, it is more natural to only consider the sheaves $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. For a general site $\mathcal{C}$, representable functors need not be sheaves, but most (if not all) sites one deals with in practise have this property. (A site is called *subcanonical* if all representable functors are sheaves.)

Remark. The category of schemes Sch arises in this fashion. Namely, a scheme S gives rise to a functor of points $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$ given by $\operatorname{Spec} R \mapsto \operatorname{Hom}(\operatorname{Spec} R, S)$. This functor is a sheaf in the Zariski topology on Aff (briefly, we say it is a *Zariski sheaf*). This defines a fully faithful functor

$$\mathrm{Sch} \longrightarrow \mathrm{Fun}_{\mathrm{sh}}(\mathrm{Aff}^{\mathrm{op}}, \mathrm{Set})$$

with essential image those sheaves that are (Zariski) locally representable. One can formulate the entire theory of schemes using this perspective; this is done in, e.g., [Ras18].

In this sense, sheaves of sets on a site $\mathcal{C}$ can be thought of as generalised objects of $\mathcal{C}$. If we now have a stack of groupoids $\mathscr{F}$ on a site $\mathcal{C}$, one can try to use the same intuition for it, thinking of $\mathscr{F}$ as an even further 'generalised object' of $\mathcal{C}$. This intuition is helpful in dealing with stacks, but can only go so far. Namely, if $X \in \mathcal{C}$, then $\mathscr{F}(X)$ is no longer a set, but a groupoid; therefore maps into $\mathscr{F}$ do not form merely a set, but a groupoid. This is somewhat odd behaviour: for example, if $\mathcal{C} = \mathrm{Top}$, then we tend to think of $\mathscr{F}(*)$ as the 'points' of $\mathscr{F}$, but since $\mathscr{F}(*)$ is a groupoid, this means that 'points of $\mathscr{F}$' now have automorphisms!

The reason this intuition is still used comes from the original motivation for stacks, namely the search for moduli spaces. The moduli space for a geometric object (such as vector bundles, elliptic curves, formal groups, etc.) is a space X such that for any other space Y, geometric objects over Y are in one-to-one correspondence with maps $Y \to X$. However, these spaces do not always exist. For instance, there is no topological space X such that we have an isomorphism

$$\operatorname{Hom}_{\mathrm{Top}}(Y, X) \cong \{\text{ isomorphism classes of rank } n \text{ vector bundles on } Y\,\}.$$

The reason such a space X does not exist is because vector bundles over Y can have nontrivial automorphisms. However, we have seen that there is a stack Vect_n on Top classifying vector bundles. In the above intuition for stacks as geometric objects, Vect_n is exactly the moduli stack of vector bundles: maps $Y \to \mathrm{Vect}_n$ are classified by vector bundles over Y.

Remark. Using homotopy theory, there is a moduli space for vector bundles: we have a natural isomorphism

$$[Y, BO(n)] \cong \{\text{ isomorphism classes of rank } n \text{ vector bundles on } Y\,\},$$

where $[Y, BO(n)]$ indicates *homotopy classes* of maps $Y \to BO(n)$. The usage of homotopy classes rather than honest maps is crucial here.

This picture so far is merely an intuition for stacks. One can also put precise conditions on a stack to turn it into an object of algebraic geometry. Such stacks are called **algebraic stacks**, although the precise definition may depend on the author. (In particular, homotopy theorists have some different conventions than algebraic geometers.) Further examples of such conditions are *Artin stacks* or *Deligne–Mumford stacks*. While we shall not go into much detail about this, we note that the main idea is that an algebraic stack is a stack on $\mathrm{Aff} = \mathrm{CRing}^{\mathrm{op}}$ which 'locally' looks like an affine scheme. The crux here is that the word 'locally' does *not* mean in the Zariski topology on Aff. (In fact, a stack on Sch/S that Zariski-locally looks like an affine scheme is in fact a scheme: see [Mei20, Prop. 4.26] for a precise discussion.) For some more details, see, e.g., [Mei20, §4.5].

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