Topological phases and K-theory

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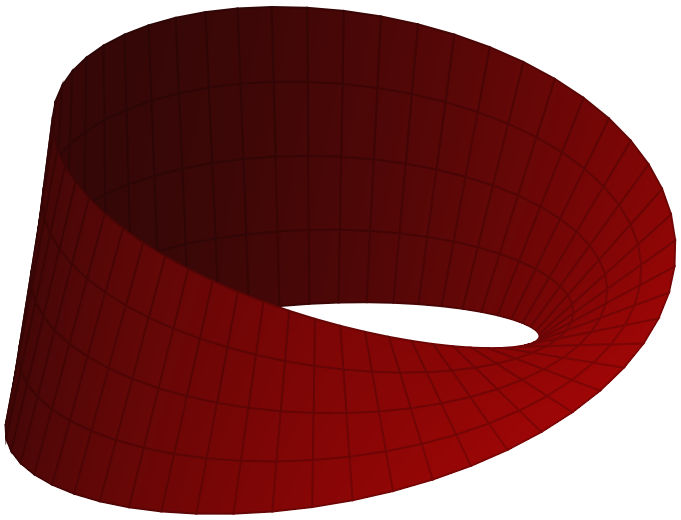

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ABSTRACT

Topological K-theory has shown up in the classification of symmetry-protected topological phases of free fermions. This started with the work of Kitaev, and was later continued by Freed and Moore, who modified K-theory to twisted K-theory in order to fit the physics. In more recent years, mathematically rigorous tools have been developed for the computation of this twisted K-theory. We present an introductory approach to both the mathematical and physical side of the subject, notably without assuming a prior knowledge of K-theory. The main results of Freed and Moore are summarised for the symmorphic case, and the basic cohomological tools for the computation of these K-groups are outlined. These tools are then applied to specific calculations. We classify all possible one-dimensional topological phases in all ten Altland–Zirnbauer classes, under a few assumptions. We do this mainly by using the Mayer–Vietoris exact sequence, and we find that for one-dimensional systems this method coincides with the Atiyah–Hirzebruch spectral sequence approach. We also treat the three-dimensional space group $F222$ in class A, i.e., topological phases protected by crystal symmetries only. This was already done by Shiozaki et al., but not in much detail; we provide a detailed version of this calculation. Using the Atiyah–Hirzebruch spectral sequence, we find the same result; in particular, we find a $\mathbb{Z}_2$-invariant.

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CONTENTS

CONVENTIONS

- The set $\mathbb{N}$ of natural numbers does not contain zero. We write $\mathbb{N}_0$ for $\mathbb{N} \cup \{ 0 \}$.
- The set of integers modulo n is denoted by $\mathbb{Z}_n$.
- The set of quaternions is denoted by $\mathbb{H}$.
- Unless otherwise indicated, the unit element of a group is denoted by 1.
- When R is a ring, we write $M_n(R)$ for the ring of $n \times n$ matrices over R.
- The exponential map, be it for numbers or operators, is denoted by exp. The elementary charge is denoted by e.
- The letter $\mathcal{H}$ is reserved for Hilbert spaces. The Hamiltonian is denoted by H.
- Boldface variables are used to denote 3-vectors, e.g., $\mathbf{E}$, $\mathbf{B}$, $\mathbf{F}$.
- We do not use the terms 'little co-group' or 'high symmetry point' common in condensed matter literature, but rather use the mathematical terms 'stabiliser' and 'fixed point', respectively.

INTRODUCTION

In 1980, a new state of matter was discovered. To create it, an insulating plane was cooled down to near absolute zero, and a strong magnetic field was put through it. The insulator remained an insulator, but it acquired the peculiar property of always being a one-way conductor on its boundary. Even when the system would have impurities in it, or be deformed slightly, this conductivity would remain. What is more, the conductivity that describes this is quantised: it takes on the values

$$\frac{e^2}{2\pi\hbar} \cdot \nu \quad \text{where } \nu \in \mathbb{Z}.$$

This is surprising because conductivity is not a microscopic quantity: it is a global property of the material, depending on all sorts of small intricacies in it. Yet in this setup, it is *exactly* an integer multiple of $e^2/2\pi\hbar$. Experimental physics found great use in this quantisation, for the fine-structure constant could be determined very precisely from it. For theoretical physicists it raised many questions: what is this integer, and why is this state so stable?

Multiple points of view were presented, but only one would later generalise to describe similar phenomena. In this theory, an insulator has associated with it the mathematical structure of a *vector bundle*. Very briefly, a vector bundle can be said to be a generalisation of a vector space. From this vector bundle a number called its *Chern number* could be computed. This Chern number is an *invariant* in the sense that two isomorphic vector bundles have the same Chern number. The integer appearing in the conductivity is this Chern number. The stability then gets a mathematical explanation: it is because the Chern number is invariant under continuous deformations of the vector bundle. Vector bundles were already a well-known concept in mathematics, and more specifically in topology. As such, these states with nonzero Chern numbers became known as *topological phases* of matter.

But this involvement of mathematics in describing special states of matter was only the beginning. As time went on, different kinds of states similar to the above one were discovered, one of which is the *topological insulator*. These states are now also called topological phases, but there was no overarching theory that described them. Finding this theory turned out to require turning topological phases into a subject in mathematical physics also. In 2009 it was discovered that a particular theory from algebraic topology is the natural way to describe these phenomena: *K-theory*.

Just as vector bundles had been an established part of mathematics, K-theory had been an established tool in algebraic topology ever since its development in the early 1960s. Many tools were created to compute *K-groups*, the central object of K-theory. The first use of K-theory in topological phases was by Kitaev [24], who used it to construct

a periodic table of topological phases. It became clear however that this periodic table could not be the whole story: there were missing entries. In 2013 Freed and Moore [11] described the K-theory objects that should be put in these missing entries. To do this they had to modify the existing K-theory in order to perfectly fit the physics. The only trouble is that these modified K-groups are notoriously difficult to compute, and that the number of entries is massive. Additionally, it was not clear if the toolbox that was previously available for K-theory still works for this modified version. This makes the classification an ongoing subject of research, both in physics and in mathematics, even at the University of Amsterdam.

On the side of theoretical physics, computational methods were described that are not fully rigorous, but which can be used to calculate cases were rigorous mathematical arguments (still) are unavailable. This is done by Kruthoff et al. [28, 29]. Mathematical physicists meanwhile were concerned with the mathematical nature of this new K-theory, trying to re-develop the tools from classical K-theory. Stehouwer [39] does this in this Master's thesis. Later Stehouwer et al. [40] used these tools in a number of cases, and also studied the physicists' method from a mathematical viewpoint.

How to read this thesis

In this work we give a first introduction to topological phases, and perform some new calculations. We take a more theoretical and mathematical viewpoint. However, the subject is too broad and the mathematics too detailed for this thesis to be fully self-contained. In developing the theory we therefore do not give all the proofs, but rather direct the reader to existing literature when the proof carries over without problem. Whenever subtleties arise that are not clearly explained in the literature, we do discuss these in the necessary detail. Accordingly, the calculations we do are also treated in detail.

This thesis is divided into three parts.

- Part I is concerned with the mathematics of K-theory, with a focus on giving the tools necessary to perform calculations. Chapter 1 lays out the basics of K-theory, and Chapter 2 expands upon this by outlining classical variants of K-theory.
- Part II has as its goal the discussion of symmetry-protected topological phases. It is a combination of both physics and mathematics, for ultimately we wish to mathematically calculate the different phases. One cannot describe symmetry-protected phases adequately without first discussing symmetries; this is done in Chapter 3. Chapter 4 treats the necessary theory of insulators, and Chapter 5 summarises the modifications to K-theory that are needed for the physics. Chapter 6 then applies this knowledge to discuss topological phases, starting with the first examples that were discovered, and ending with the relation to K-theory. It is possible to start this part in Chapter 6, reading previous chapters when necessary, because it does not immediately require a knowledge of the preceding chapters.
- Whereas Parts I and II contain only condensed versions of the existing literature, Part III contains original work. It turns out that the one-dimensional case is

simple enough that it can be fully classified, but (surprisingly) this has not yet been done. This is the subject of Chapter 7. Lastly, Chapter 8 studies a specific three-dimensional case, one for which different sources give different answers.

PRELIMINARIES

This thesis is written at the advanced undergraduate level for both physics and mathematics. On the side of physics, we assume a prior knowledge of condensed matter, quantum mechanics and classical electrodynamics. In particular, the reader is assumed to be familiar with the concepts of the Brillouin zones, valence and conduction bands, the reciprocal lattice, and the Berry phase.

On the side of mathematics the preliminaries are more varied, but they are not all equally important. A firm understanding of group theory and representation theory is assumed, as well as a knowledge of the theory around (short) exact sequences. Topology is used throughout. A previous familiarity with the concept of a vector bundle is helpful, although it is very briefly reviewed. Lastly, some basic concepts from category theory are used in various places.

ACKNOWLEDGEMENTS

I am tremendously grateful for being able to do this project, making me owe particular thanks to my supervisors Jan en Hessel. I greatly appreciate the time they took off their busy schedules to make the subject accessible to my level, and for the feedback they provided on my work.

Jan, thank you for suggesting the subject to me. I'm not sure if you realised it at first, but the one-dimensional classification was the perfect problem to get me introduced to the subject, and I think it can be for others as well. Thank you for making sure I kept close to physics, instead of running off into strictly mathematical territory constantly.

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Soli Deo Gloria.

Part I.

K-theory

1 TOPOLOGICAL K-THEORY

In algebraic topology one studies topological spaces by assigning algebraic structures to these spaces, usually a group or a ring. The structures are called *topological invariants*, in the sense that two homotopy equivalent (or, more strongly, homeomorphic) spaces have isomorphic structures assigned to them. A common first example of such a tool is the fundamental group. Another example is *topological K-theory*, which was developed by Atiyah and Hirzebruch in the late 1950's and early 1960's. It assigns groups to spaces, but very notably these groups are made up of objects that are themselves interesting to study: *vector bundles*. As such, its use becomes two-fold: not only does it provide a means to distinguish between topological spaces, but it also classifies vector bundles over a fixed topological space. Ultimately it is the latter use that we are interested in, though history has shown the former should not (and indeed cannot) be ignored.

For ease of discussion, we shall refer to topological K-theory simply as *K-theory*. We limit our presentation of K-theory to compact Hausdorff spaces. The goal of this chapter is to give a brief introduction to the construction of the K-group and its calculation.

The construction of the K-groups (Sections 1.1–1.3) is adapted from Atiyah [3]. In this pursuit we encounter the *graded K-groups*. At first sight these additional groups may seem unnatural or unmotivated, but they aid the calculation of the K-group. In later chapters their importance will be further emphasised by their appearance in physics.

After giving this construction, in Section 1.4 we give the basic axioms of K-theory that we use for the calculation of the K-groups. We then use these in Section 1.5 to calculate the K-theory of the spheres. Lastly, we briefly comment upon KO-theory in Section 1.6.

1.1. THE GROTHENDIECK GROUP

A **semigroup** is a (nonempty) set with a binary, associative operation on it. In other words, it is a group without a neutral element or inverses. A **homomorphism of semigroups** is a map that preserves the semigroup operation. This turns semigroups into a category. Notice that any group is also a semigroup.

If the operation on a semigroup is commutative, then we call the semigroup commutative. There exists a universal way to turn a commutative semigroup into an abelian group. This group is called its *Grothendieck group*. We provide a sketch of how it is constructed. Curious readers may consult Lang [30, pp. 39, 40] in case more details are desired.

Theorem 1.1. *Let S be a commutative semigroup. Then there exists an abelian group $K(S)$ and a semigroup homomorphism $i\colon S \to K(S)$, unique up to unique isomorphism, satisfying the following universal property. If A is an abelian group and $\varphi\colon S \to A$ a semigroup homomorphism, then there is a unique group homomorphism $\psi\colon K(S) \to A$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & A \\ & \searrow^{i} & \uparrow{\scriptstyle \psi} \\ & & K(S) \end{array}$$

*commutes. We call $K(S)$ the **Grothendieck group** of S.*

Proof. Uniqueness follows from the universal property. We give a construction of $K(S)$ as 'formal differences' of elements in S. Define an equivalence relation on $S \times S$ by

$$(a,b) \sim (c,d) \iff \text{there exists } s \in S \text{ such that } a+d+s = c+b+s.$$

This is readily verified to be an equivalence relation. We define $K(S)$ as $S \times S/\!\sim$, with the operation on S extended to $K(S)$ entry-wise. The associativity of S implies this operation is associative. An element $[(a,b)]$ is intended to represent $a-b$. With that in mind, fix some $s_0 \in S$. Then $[(s_0,s_0)] \in K(S)$ is a neutral element under this operation, independent of s_0. Reversing the order in a pair yields an inverse. Thus $K(S)$ forms an abelian group. Define

$$i\colon S \longrightarrow K(S)\colon \quad s \longmapsto [(s+s_0,\ s_0)].$$

Note that i is also independent of the choice of s_0. It is not difficult to verify that this satisfies the desired universal property. ∎

When S and T are two semigroups and $\varphi\colon S \to T$ a semigroup homomorphism, the above universal property gives us a unique map $K(\varphi)\colon K(S) \to K(T)$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ K(S) & \xrightarrow{K(\varphi)} & K(T) \end{array}$$

commutes. The induced map $K(\varphi)$ depends functorially on φ. In this manner, K becomes a functor from the category of commutative semigroups to the category of abelian groups.

Now suppose that a semigroup S is also a **semiring**, i.e., it has a binary, associative operation with unit that distributes over its semigroup operation. This induces a multiplication on $K(S)$ by requiring the map $i\colon S \to K(S)$ to be a semiring homomorphism (i.e., a semigroup homomorphism that is multiplicative and preserves the multiplicative unit). This extends to a multiplication on all of $K(S)$, and under this multiplication $K(S)$ becomes a ring. In the universal property, if A is a ring and φ a semiring homomorphism, the induced map ψ is a ring homomorphism. Thus, by a

similar argument, K is also a functor from the category of semirings to the category of rings.

To shed some light on these definitions, we present two examples: one elementary, the other a little more involved.

Example 1.2. The set of natural numbers $\mathbb{N}$ forms a semigroup under addition. The set of integers $\mathbb{Z}$ under addition together with the inclusion map $i\colon \mathbb{N} \hookrightarrow \mathbb{Z}$ satisfies the universal property for the Grothendieck group of $\mathbb{N}$. So we may identify $K(\mathbb{N})$ with $\mathbb{Z}$. Moreover, when we endow $\mathbb{N}$ with its usual multiplication, it becomes a semiring, and the induced multiplication on $K(\mathbb{N})$ coincides with the ordinary multiplication on $\mathbb{Z}$. An entirely analogous consideration applies to $\mathbb{N}_0$; it also has $\mathbb{Z}$ as its Grothendieck group.

Example 1.3 (Representation ring). Let G be a group. Denote by $\mathrm{Rep}(G)$ the set of isomorphism classes of finite-dimensional complex representations of G. When φ and ψ are two representations of G, then we may form their direct sum $\varphi \oplus \psi$, which is again a representation. This is associative in the sense that

$$(\varphi \oplus \psi) \oplus \chi \cong \varphi \oplus (\psi \oplus \chi)$$

as representations. Since $\mathrm{Rep}(G)$ is made up of *isomorphism classes* of representations, the direct sum descends to an associative operation on $\mathrm{Rep}(G)$. In addition, we give the set $\mathrm{Rep}(G)$ a multiplication via the tensor product of representations. The corresponding Grothendieck group (with its induced ring structure) is called the **representation ring** of G and is denoted by $R(G)$.

1.2. Vector bundles

For the convenience of the reader, we briefly recall the definition of a (topological) vector bundle. Here we restrict our attention to complex vector bundles, though the concept is easily modified to yield real vector bundles.

Definition 1.4. Let X be a topological space. A **complex vector bundle** over X is a pair (E, π) of a topological space E and a projection $\pi\colon E \to X$, satisfying the following conditions.

(i) For every $x \in X$, the **fibre** $E_x := \pi^{-1}(\{x\})$ is endowed with the structure of a finite-dimensional complex vector space.

(ii) Around every $x \in X$ there exists a neighbourhood U and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^n$ (called a **local trivialisation**) for some $n \in \mathbb{N}_0$ such that:

- $\pi_U \circ \Phi = \pi$, with $\pi_U\colon U \times \mathbb{C}^n \to U$ the projection onto the first component;
- for every $y \in X$, the restriction $\Phi\colon E_y \to \{y\} \times \mathbb{C}^n$ is a linear isomorphism.

Remark 1.5. We do not add the usual restriction that the dimensions of the fibres be constant throughout X; our definition implies it is only *locally constant*. This definition is more useful for K-theory and is as such employed by standard texts like Atiyah [3]. If however the fibres of a bundle all happen to have the same dimension k, the vector bundle is said to have **rank k**. Bundles over a connected base space thus always have a well-defined rank.

Let X be a topological space and let E and F be vector bundles over X. A **homomorphism** from E to F is a continuous map $f\colon E \to F$ satisfying the following conditions.

(i) We have $\pi_F \circ f = \pi_E$.

(ii) For every $x \in X$, the restriction $f\colon E_x \to F_x$ is linear.

With these definitions, vector bundles over a fixed topological space form a category.

We also recall some standard constructions with vector bundles. Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous map.

- **Direct sum.** If E and F are vector bundles over X, their direct sum $E \oplus F$ is a bundle over X with fibres $(E \oplus F)_x = E_x \oplus F_x$.
- **Tensor product.** If E and F are vector bundles over X, their tensor product $E \otimes F$ is a bundle over X with fibres $(E \otimes F)_x = E_x \otimes F_x$.
- **Pullback.** If E is a vector bundle over Y, then the pullback bundle f^*E is a bundle over X with fibres $(f^*E)_x = E_{f(x)}$.

Isomorphisms like $V \otimes (W \oplus U) \cong (V \otimes W) \oplus (V \otimes U)$ generalise directly to vector bundles in the place of vector spaces. Moreover, the pullback under a continuous map preserves these isomorphisms.

Lastly, recall that a vector bundle over X is called **trivial** if it is isomorphic to a bundle of the form $X \times \mathbb{C}^n$.

1.3. The K-groups

The direct sum of vector bundles is associative up to isomorphism. It is tempting to think of this as an addition of isomorphism classes of vector bundles, but the analogy quickly breaks down as there are no inverses. But previously we have seen how inverses can be added in a universal way, namely by taking the Grothendieck group. Applying this to isomorphism classes of vector bundles is the *K-group*.

Definition 1.6. Let X be a compact Hausdorff space. Denote by $\mathrm{Vect}(X)$ the set of isomorphism classes of complex vector bundles over X. The direct sum of vector bundles gives an associative, commutative operation on $\mathrm{Vect}(X)$, turning it into a semigroup. The tensor product gives it the additional structure of a semiring. The **(complex) K-group** of X, denoted by $K(X)$, is the Grothendieck group of the semiring $\mathrm{Vect}(X)$.

Remark 1.7. Although the K-group has the structure of a ring, most of the time we shall only be concerned with its structure as an abelian group.

We make K a *contravariant* functor from compact Hausdorff spaces to rings as follows. When $f\colon X \to Y$ is continuous, it induces a map $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ via

$$[E] \longmapsto [f^*E],$$

the pullback of vector bundles. This is a well-defined semiring homomorphism. By functoriality of the Grothendieck group, it induces a ring homomorphism $K(Y) \to K(X)$, which we also denote by f^*.

Example 1.8 (K-group of a point). Take X to be a point, $X = \mathrm{pt}$. Then a vector bundle over X is nothing but a finite-dimensional vector space. (In this sense a vector bundle can be thought of as a generalisation of a vector space.) A finite-dimensional vector space is classified up to isomorphism by its dimension. Thus $\mathrm{Vect}(\mathrm{pt})$ is isomorphic to $\mathbb{N}_0$ as a semigroup, with isomorphism the dimension map. From Example 1.2 we know that $K(\mathbb{N}_0) \cong \mathbb{Z}$ as rings, so $K(\mathrm{pt}) \cong \mathbb{Z}$ as rings.

1.3.1. The reduced K-group

Recall that a **pointed topological space** is a pair (X, x_0) of a topological space X and a point $x_0 \in X$ called its **basepoint**. Equivalently, it is a space together with the choice of a map $\mathrm{pt} \hookrightarrow X$. Pointed spaces form a category, where morphisms are the continuous maps that preserve basepoints.

What makes a space interesting in terms of its K-group is how many nontrivial vector bundles it can have. If all bundles over the space are trivial, its K-group is simply (isomorphic to) $\mathbb{Z}$. This motivates the definition of the *reduced K-group* of a pointed space. It is obtained by 'removing all data' at the fibre over the basepoint; in this case, this means we 'forget' the dimension of its fibre. What is left is a group that keeps track of the possible ways in which vector bundles over the space can be nontrivial.

Definition 1.9. Let X be a pointed compact Hausdorff space. Then the inclusion map $\mathrm{pt} \hookrightarrow X$ induces a map $K(X) \to K(\mathrm{pt})$. The kernel of this map is called the **reduced K-group** of X and is denoted by $\widetilde{K}(X)$.

Remark 1.10. In contrast to the K-group of a space, the reduced K-group is not a ring: it is an ideal in $K(X)$. In particular it is still an abelian group, and this structure is the one of interest to us.

Note that the map $\mathrm{pt} \hookrightarrow X$ has a continuous left-inverse $X \twoheadrightarrow \mathrm{pt}$ given by projection onto one point. This gives us a section of the short exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \overset{\longleftarrow}{\longrightarrow} K(\mathrm{pt}) \longrightarrow 0,$$

which induces an isomorphism

$$K(X) \cong \widetilde{K}(X) \oplus K(\mathrm{pt}). \tag{1.3.1}$$

This direct sum formalises the idea that the reduced K-group 'forgets' the information at the basepoint. A diagram chasing argument now shows that a map $f\colon X \to Y$ also induces a map between reduced K-groups, $f^*\colon \widetilde{K}(Y) \to \widetilde{K}(X)$. Thus it is a contravariant functor from pointed compact Hausdorff spaces to abelian groups.

Remark 1.11. Some authors define $\widetilde{K}(X)$ differently. Namely, the map $K(\mathrm{pt}) \to K(X)$ induced by $X \twoheadrightarrow \mathrm{pt}$ has as its image the trivial bundles. Then $\widetilde{K}(X)$ is defined as the cokernel of this map. Because this map $K(\mathrm{pt}) \to K(X)$ is the section that splits the short exact sequence above, these views are entirely equivalent.

Example 1.12. Again taking $X = \mathrm{pt}$, this automatically becomes a pointed space. The inclusion of the basepoint is a homeomorphism, hence the induced map of K-groups is an isomorphism. As such it has trivial kernel, which means that $\widetilde{K}(\mathrm{pt}) = 0$.

1.3.2. Graded K-groups

In addition to just one K-group associated with a space, K-theory associates many groups to a space. Before we can define these, we need to define some preliminary operations on pointed spaces: the *wedge sum* and *smash product*. They are the pointed analogues of the disjoint union and Cartesian product of ordinary spaces, respectively.

Definition 1.13. Let X and Y be pointed topological spaces, with basepoints x_0 and y_0 respectively. Their **wedge sum** $X \vee Y$ is the pointed space

$$X \vee Y := (X \times \{ y_0 \}) \cup (\{ x_0 \} \times Y)$$

(in the subspace topology of $X \times Y$) with $\{ x_0, y_0 \}$ as its basepoint. Their **smash product** $X \wedge Y$ is the pointed space

$$X \wedge Y := {}^{X \times Y}\,/_{X \vee Y}$$

with $X \vee Y$ as its basepoint.

Remark 1.14. A different definition of the wedge sum is the quotient $X \sqcup Y / x_0 \sim y_0$. One quickly verifies that this definition is homeomorphic to ours.

Example 1.15 (Smash product of spheres). In this example we take $\mathbb{S}^n$ to be the (pointed) space

$$\mathbb{S}^n := {}^{I^n}\,/_{\partial I^n},$$

where $I = [0, 1]$ is the unit interval. With this definition, it is easy to see that when $n, m \in \mathbb{N}$, we have a homeomorphism of pointed spaces

$$\mathbb{S}^n \wedge \mathbb{S}^m \cong {}^{I^{n+m}}\,/_{\partial I^{n+m}} \cong \mathbb{S}^{n+m}.$$

We can extend this result to account for $n = 0$ if we take the standard definition $\mathbb{S}^0 = \{ \pm 1 \}$, seeing the 0-sphere as the elements of length 1 in $\mathbb{R}$. In fact, for any

pointed space X we then have a homeomorphism of pointed spaces

$$\mathbb{S}^0 \wedge X \cong X \wedge \mathbb{S}^0 \cong X.$$

Definition 1.16. Let X be a pointed topological space and $n \in \mathbb{N}$ a natural number. The ***n*-fold reduced suspension** of X, denoted $\Sigma^n X$, is the pointed space

$$\Sigma^n X := \mathbb{S}^n \wedge X.$$

We abbreviate $\Sigma^1 X$ as ΣX.

Each n-fold suspension can also be turned into a functor. For this, when $f\colon X \to Y$ is a continuous map of pointed spaces, we define a map $\mathbb{S}^n \times X \to \mathbb{S}^n \times Y$ via $(\alpha, x) \mapsto (\alpha, f(x))$. Since f preserves basepoints by assumption, this is compatible with passing to the quotient to get a continuous map $\Sigma^n f\colon \Sigma^n X \to \Sigma^n Y$ of pointed spaces. The reader may verify that this assignment of maps is functorial.

One reason for the interest in suspensions is a classic theorem of algebraic topology called the *Freudenthal suspension theorem*. A consequence of this theorem is that (for n large enough) we have a group isomorphism $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$, where π_n denotes the n-th homotopy group. In the case of K-theory, we use the suspensions to *define* more K-groups.

Definition 1.17. Let X be a compact Hausdorff space and $n \in \mathbb{N}$ a natural number. We define the **K-group in degree $-n$** of X, denoted by $K^{-n}(X)$, and the **reduced K-group in degree $-n$**, denoted by $\widetilde{K}^{-n}(X)$, as

$$\begin{aligned} \widetilde{K}^{-n}(X) &:= \widetilde{K}(\Sigma^n X) \qquad \text{when } X \text{ is pointed;} \\ K^{-n}(X) &:= \widetilde{K}(\Sigma^n(X^+)), \end{aligned}$$

where X^+ is the pointed space $X \sqcup \mathrm{pt}$ with basepoint pt. By the **K-theory** of X we shall mean all of the K-groups of X together.

Remark 1.18. Because the groups K^{-n} and $\widetilde{K}^{-n}$ are defined in terms of a reduced K-group, neither of them has the structure of a ring; see Remark 1.10.

Remark 1.19. The reason why we give the n-fold suspension a negative index as K-group is rather dull and will be explained later.

We turn these other K-groups into contravariant functors in the following way. The reduced group $\widetilde{K}^{-n}$ is the composition of the functor Σ^n and the contravariant functor $\widetilde{K}$. The assignment $X \mapsto X^+$ becomes a functor when we let it extend continuous maps to preserve the newly added basepoints. Then K^{-n} is a composition of two functors and one contravariant functor, and is thus a contravariant functor. In conclusion, for every $n \in \mathbb{N}$ we have contravariant functors K^{-n} and $\widetilde{K}^{-n}$. For convenience we shall define K^0 as K, and $\widetilde{K}^0$ as $\widetilde{K}$.

1.4. Axioms of K-theory

It turns out that the above construction of K-theory satisfies certain axioms that help the calculation of these groups. We shall not prove the majority of the axioms we list; most omitted proofs may be found in Atiyah [3]. In these, the reader will find the (so far hidden) reasons why we restrict our definitions to compact Hausdorff spaces. Many of the maps in the diagrams below are natural, but we shall comment below how they are constructed. Note that some of these illustrate that the graded K-groups impact the calculation of the K-group.

(1) **Contravariance.** For every n, the assignments $X \mapsto K^{-n}(X)$ and $f \mapsto f^*$ make K^{-n} into a contravariant functor, and similarly for $\widetilde{K}^{-n}$.

(2) **Homotopy invariance.** If $f, g\colon X \to Y$ are homotopic, then $f^* = g^*$ for every n.

(3) **Exactness.** When $Y \subseteq X$ is a closed subspace, we have a long exact sequence (infinite to the left) of natural maps

$$\cdots \xrightarrow{\delta} \widetilde{K}^{-n}(X/Y) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(Y) \xrightarrow{\delta} \widetilde{K}^{-n+1}(X/Y) \longrightarrow \cdots$$

When X is a pointed space and $Y \subseteq X$ a closed subspace that shares the basepoint, we have an analogous long exact sequence (infinite to the left)

$$\cdots \xrightarrow{\delta} \widetilde{K}^{-n}(X/Y) \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow \widetilde{K}^{-n}(Y) \xrightarrow{\delta} \widetilde{K}^{-n+1}(X/Y) \longrightarrow \cdots$$

(4) **Excision.** If $X = A \cup B$ for closed subspaces $A, B \subseteq X$, then for every n the composition $A/(A \cap B) \hookrightarrow X/(A \cap B) \twoheadrightarrow X/B$ induces an isomorphism

$$\widetilde{K}^{-n}(X/B) \cong \widetilde{K}^{-n}(A/(A \cap B)).$$

(5) **Additivity.** If $X = A \sqcup B$ for closed subspaces A and B, then for every n the inclusion maps induce isomorphisms

$$K^{-n}(A \sqcup B) \cong K^{-n}(A) \oplus K^{-n}(B).$$

For $\widetilde{K}^{-n}$, the same holds for a wedge sum $X = A \vee B$ of pointed spaces in the place of a disjoint union.

(6) **Mayer–Vietoris sequence.** If $X = A \cup B$ for some closed subspaces $A, B \subseteq X$, then we have a long exact sequence (infinite to the left) of natural maps

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\delta} & K^{-n}(X) & \longrightarrow & K^{-n}(A) \oplus K^{-n}(B) & \longrightarrow & K^{-n}(A \cap B) \\ & & & & & & \downarrow \delta \\ \cdots & \xleftarrow{\delta} & K^{-n+1}(A \cap B) & \longleftarrow & K^{-n+1}(A) \oplus K^{-n+1}(B) & \longleftarrow & K^{-n+1}(X) \end{array}$$

(7) **Bott periodicity.** For every n, we have an isomorphism

$$K^{-n-2}(X) \cong K^{-n}(X).$$

We already proved Axiom (1). Axiom (2) implies that the K-theory of a space depends only on its homotopy type; in particular, any contractible space has the K-theory of a point.

In Axiom (3), the maps labelled δ are called **coboundary maps**. The minus signs in our definition of K-groups (Definition 1.17) make these maps increase the degree by one instead of decreasing it, which is a standard custom (cf. Remark 1.19). The existence of these coboundary maps is often more important than a precise construction. As for the other morphisms, the map $X^+ \to X/Y$ given by collapsing $Y \cup \mathrm{pt}$ to a single point induces $\widetilde{K}^{-n}(X/Y) \to K^{-n}(X)$. The map $K^{-n}(X) \to K^{-n}(Y)$ is just the one induced by inclusion.

Axiom (5) follows from our definition that a vector bundle only has locally constant rank (see Remark 1.5). The analogous property for reduced K-theory follows from Axiom (3).

Roughly speaking, any 'theory' that satisfies Axioms (1)–(5) but with the exact sequences infinite in both directions is called a *cohomology theory*. For a precise definition of a cohomology theory, see, e.g., Hilton [18, Ch. 1]. From these, Axiom (6) follows in general. Like in Axiom (5), often the existence of the maps labelled δ is more important than a construction of them. The inclusion map $A \hookrightarrow X$ induces a map $K^{-n}(X) \to K^{-n}(A)$, and similarly for $B \hookrightarrow X$. The map $K^{-n}(X) \to K^{-n}(A) \oplus K^{-n}(B)$ is the direct sum of these two. Similarly, the inclusion $A \cap B \hookrightarrow A$ induces a map $K^{-n}(A) \to K^{-n}(A \cap B)$, and analogously for $A \cap B \hookrightarrow B$. The difference of these two is the map indicated $K^{-n}(A) \oplus K^{-n}(B) \to K^{-n}(A \cap B)$ in the diagram.

Lastly, Axiom (7) is perhaps the most surprising of all. This property allows us to extend the definition of K^n for *any* integer n, by making K-theory fully periodic with period 2. Reading the other axioms in light of this one, we extend all the exact sequences to be infinite to the right as well, thereby making K-theory a full-fledged cohomology theory. In fact, the exact sequences also become periodic.

1.5. The K-theory of spheres

We have already computed the K-group of a point in Example 1.8. Computing the K-group $K^{-1}(\mathrm{pt})$ suffices to compute the entire K-theory of a point by Bott periodicity (Axiom (7)). From Definition 1.17 we see that this amounts to $\widetilde{K}(\mathbb{S}^1)$. Indeed, adding a disjoint point to pt yields $\mathbb{S}^0$, and by Example 1.15 we have $\Sigma^1(\mathbb{S}^0) \cong \mathbb{S}^1$. To determine $\widetilde{K}(\mathbb{S}^1)$ it suffices to first compute $K(\mathbb{S}^1)$, for then we have the splitting

$$K(\mathbb{S}^1) \cong \widetilde{K}(\mathbb{S}^1) \oplus K(\mathrm{pt})$$

given in Equation (1.3.1). It turns out that there are no nontrivial *complex* vector bundles over the circle; see for instance Hatcher [17, pp. 22–24]. This is equivalent to saying that the rank map $\mathrm{Vect}(\mathbb{S}^1) \to \mathbb{N}_0$ is an isomorphism of semigroups. By taking Grothendieck groups we then find

$$K(\mathbb{S}^1) \cong \mathbb{Z}.$$

In particular we see that $\widetilde{K}(\mathbb{S}^1) = 0$. As noted, this completes the computation of the K-theory of a point:

$$K^{-n}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \tag{1.5.1}$$

This result has deeper implications than it might first appear to have. Because the spheres are suspensions of a point, and lower-degree K-groups are defined using suspensions, we can derive the K-theory of all spheres from this. Let us illustrate this in more detail.

Similarly to the above, we have by definition that $K^{-n}(\mathrm{pt}) = \widetilde{K}(\mathbb{S}^n)$. Since we know this group for all possible n, the splitting $K(\mathbb{S}^n) \cong \widetilde{K}(\mathbb{S}^n) \oplus K(\mathrm{pt})$ from Equation (1.3.1) tells us how to compute the $K(\mathbb{S}^n)$ for every n. This argument even generalises to all degrees, as follows.

Proposition 1.20. *Let X be a pointed compact Hausdorff space. Then for all $n \in \mathbb{N}_0$ we have a split short exact sequence*

$$0 \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(\mathrm{pt}) \longrightarrow 0.$$

In particular, for all n we have an isomorphism $K^{-n}(X) \cong \widetilde{K}^{-n}(X) \oplus K^{-n}(\mathrm{pt})$.

Proof. Take Y equal to the basepoint of X; then we have a homeomorphism $X/Y \cong X$. Part of the exact sequence in Axiom (3) then reads

$$K^{-n-1}(X) \longrightarrow K^{-n-1}(\mathrm{pt}) \xrightarrow{\delta} \widetilde{K}^{-n}(X) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(\mathrm{pt}).$$

The last map in this sequence induced by the inclusion $\mathrm{pt} \hookrightarrow X$. This has a left-inverse $X \twoheadrightarrow \mathrm{pt}$ given by projection onto one point. So by contravariance of K^{-n}, the map $K^{-n}(X) \to K^{-n}(\mathrm{pt})$ has a right-inverse $K^{-n}(\mathrm{pt}) \to K^{-n}(X)$. It is therefore surjective. Notice that the same argument applies to the first map in this sequence, so it is surjective as well. By exactness this means that the map δ is the zero map, and therefore the map $\widetilde{K}^{-n}(X) \to K^{-n}(X)$ following it is injective. Consequently we have a short exact sequence

$$0 \longrightarrow \widetilde{K}^{-n}(X) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(\mathrm{pt}) \longrightarrow 0,$$

for which we already constructed a section $K^{-n}(\mathrm{pt}) \to K^{-n}(X)$, so it splits. ∎

Remark 1.21. The proof of the above immediately generalises to the case where Y is a retraction of X (i.e., when the inclusion $Y \hookrightarrow X$ has a continuous left-inverse). In that case we get a splitting $K^{-n}(X) \cong \widetilde{K}^{-n}(X/Y) \oplus K^{-n}(Y)$.

Notice that for all n and d,

$$\widetilde{K}^{-d}(\mathbb{S}^n) = \widetilde{K}(\Sigma^d \mathbb{S}^n) \cong \widetilde{K}(\mathbb{S}^{n+d}) = K^{-n-d}(\mathrm{pt}).$$

By the above proposition we know that $K^{-d}(\mathbb{S}^n) \cong \widetilde{K}^{-d}(\mathbb{S}^n) \oplus K^{-d}(\mathrm{pt})$. In conclusion, the K-theory of a sphere can be deduced from the K-theory of a point.

1.6. KO-THEORY

The above discussion only treated complex vector bundles, i.e., vector bundles where the fibres have the structure of a complex vector space. The discussion applies verbatim with the real numbers in the place of the complex numbers. If we repeat the construction in Section 1.3 for vector bundles over the real numbers, the groups we end up with are called the **KO-groups** and are denoted by $KO^{-n}(X)$, and $\widetilde{KO}^{-n}(X)$ for the reduced versions. The KO-groups are also referred to as real K-groups, in contrast to the complex K-groups outlined previously. Section 1.4 applies as well,[1] except for Bott periodicity (Axiom (7)): though there still is a periodicity, it is now *eight*-fold:

$$KO^{-n-8}(X) \cong KO^{-n}(X).$$

This periodicity is fittingly called *real Bott periodicity*. In a similar fashion to K-theory we extend KO-theory to have indices in $\mathbb{Z}$, making it a cohomology theory too.

The most basic yet crucial calculation is the KO-theory of a point. This turns out to be a much more involved task than the complex variant. Atiyah, Bott, and Shapiro [5] first computed this and found that

$$KO^{-n}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \bmod 8, \\ \mathbb{Z}_2 & n \equiv 1 \bmod 8, \\ \mathbb{Z}_2 & n \equiv 2 \bmod 8, \\ 0 & n \equiv 3 \bmod 8, \\ \mathbb{Z} & n \equiv 4 \bmod 8, \\ 0 & n \equiv 5 \bmod 8, \\ 0 & n \equiv 6 \bmod 8, \\ 0 & n \equiv 7 \bmod 8. \end{cases} \tag{1.6.1}$$

In particular, we see that

$$KO(\mathbb{S}^1) \cong KO(\mathrm{pt}) \oplus \widetilde{KO}(\mathbb{S}^1) \cong KO(\mathrm{pt}) \oplus KO^{-1}(\mathrm{pt}) \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

so the circle does have a nontrivial *real* vector bundle. This is the Möbius bundle. Indeed, a direct sum of two Möbius bundles is a trivial bundle, hence it manifests as $\mathbb{Z}_2$ in the KO-group. Because its real dimension is odd, it does not appear in $K(\mathbb{S}^1)$.

[1] In particular, Proposition 1.20 also holds for KO-theory, because we derived it from these axioms alone.

2 VARIANTS OF K-THEORY

The groups $K(X)$ and $KO(X)$ are made up of complex and real vector bundles, respectively. Atiyah introduced a K-group that generalises both of these: the group $KR(X)$. Its definition is slightly more difficult because it is made up of vector bundles with additional structure. Each of these three K-theories also comes with its own *equivariant* version. Throughout, the word 'equivariant' signifies that a group action is involved. In this case it means that, given a group G, we consider vector bundles over X that have a group action of G. The K-groups we obtain from these bundles are denoted by $K_G(X)$, $KO_G(X)$ and $KR_G(X)$.

Upon first reading however, the equivariant K-group $K_G(X)$ is an easier introduction to the idea of additional structure than the KR-group is. Hence we first discuss equivariant K-theory in Section 2.1, and define KR-theory and equivariant KR-theory afterwards in Section 2.2. For all of these theories, the difficulty lies in defining the appropriate type of vector bundle. Once established, the definitions of the K-groups are entirely analogous to the ones from the previous chapter. The axioms for their calculation also carry over naturally. In some cases these new groups reduce to ones we have seen before; such cases will be of use in later calculations.

2.1. EQUIVARIANT K-THEORY

Equivariant K-theory is a modification of K-theory that considers spaces with a group action on it. The vector bundles that make up these modified K-groups are required to carry a group action that is compatible with the one on the base space. To define this rigorously, we first need to specify the category of spaces that we will define this K-theory on. Our presentation of equivariant K-theory is based on the paper by Segal [37], with some influences from Atiyah [3]. To save ourselves unnecessary effort, we shall discuss only finite groups.

2.1.1. Equivariant spaces

A further discussion of equivariant spaces is given by Atiyah [3, Ch. 1]. Fix a finite group G. A ***G*-equivariant space** X (or *G-space*) is a topological space X together with a continuous group action of G on it. That is, it has a group action where multiplication by a group element is a continuous map. (Note that this implies it is a homeomorphism.) When the group G is clear from the context, we may leave it implicit and simply call the space an *equivariant space*.

A ***G*-equivariant map** (or *G-map*) between two equivariant spaces X and Y is a continuous map $f\colon X \to Y$ that commutes with the group action. With these morphisms, G-equivariant spaces form a category. A ***G*-equivariant subspace** (or *G-subspace*) is a subspace of X that is closed under the group action. In particular, the inclusion map of an equivariant subspace is an equivariant map.

A **pointed *G*-equivariant space** is a pointed space and a G-equivariant space where the basepoint is fixed under the G-action. A morphism between pointed equivariant spaces is a morphism of pointed spaces that is also an equivariant map. So pointed equivariant spaces form a category as well.

Definition 2.1. Let G be a finite group and let X be a G-equivariant space. A **complex *G*-equivariant vector bundle** over X is a complex vector bundle E over X that is simultaneously a G-equivariant space, such that the following conditions are satisfied.

(i) The projection $\pi\colon E \to X$ is an equivariant map.

(ii) The action of G on E is fibre-wise linear, i.e., for every $g \in G$ the restriction $E_x \to E_{g\cdot x}$ of multiplication by g is linear for every $x \in X$.

Suppose X is a G-equivariant space and E and F are two G-equivariant vector bundles over X. A ***G*-equivariant homomorphism** from E to F is a map $f\colon E \to F$ that is both a homomorphism of vector bundles and a G-equivariant map.

Operations like the direct sum, tensor product and pullback are all possible for equivariant bundles too. This is done by constructing the non-equivariant bundle first from the underlying vector bundles, and then giving it the natural G-action. Note though that the pullback of an equivariant bundle is only defined over an equivariant map.

Notice the following special cases of equivariant bundles. When G is the trivial group, a G-equivariant bundle is just a vector bundle. When the base space has trivial group action, the G-action restricts to give *representations* of G on each fibre. More generally, the fibre over $x \in X$ has a representation of the stabiliser G_x of x on it. In particular, a G-equivariant bundle over a point is the same as a finite-dimensional representation of G. Thus equivariant vector bundles generalise both vector bundles and representations.

2.1.2. Equivariant K-groups

Fix a finite group G. Let X be a compact Hausdorff G-equivariant space. Denote by $\mathrm{Vect}_G(X)$ the set of equivariant isomorphism classes of complex G-equivariant vector bundles over X. This set forms a commutative semigroup under the direct sum. The **(complex) *G*-equivariant K-group** of X, denoted by $K_G(X)$, is the Grothendieck group of this semigroup. The tensor product of equivariant vector bundles induces a ring structure on it. When $f\colon X \to Y$ is an equivariant map, it induces a homomorphism $K_G(Y) \to K_G(X)$ via the pullback of equivariant vector bundles. This turns K_G into a contravariant functor from G-equivariant compact Hausdorff spaces to rings.

It is important to note that $K_G(X)$ is constructed out of *G-equivariant* isomorphism classes of vector bundles. Being equivariantly isomorphic is stronger than being isomorphic as vector bundles. Indeed, two equivariant vector bundles may be isomorphic as vector bundles, while not being equivariantly isomorphic. This is illustrated in a simple case by the following example.

Example 2.2. Previously we remarked that (complex) G-equivariant bundles over a point are simply finite-dimensional (complex) representations of G. Back in Example 1.3, we denoted the semigroup of equivalence classes of such representations by $\mathrm{Rep}(G)$. So we see that this semigroup is isomorphic to $\mathrm{Vect}_G(\mathrm{pt})$. Therefore $K_G(\mathrm{pt}) = R(G)$, the representation ring of G defined in Example 1.3.

Several special cases of G-actions give a relation between the equivariant K-group and the ordinary K-group. If G is the trivial group, then $K_G(X)$ is simply $K(X)$. If the G-action on X is trivial, then $K_G(X)$ may be viewed as modifying $K(X)$ by giving each fibre a G-representation. For this, first notice that $R(G) = K_G(\mathrm{pt})$ has a natural map to $K_G(X)$ induced by $X \twoheadrightarrow \mathrm{pt}$. This yields a natural ring homomorphism $R(G) \otimes_{\mathbb{Z}} K(X) \to K_G(X)$.

Proposition 2.3. *Suppose X is a G-equivariant space with trivial G-action. Then the natural map*

$$R(G) \otimes_{\mathbb{Z}} K(X) \longrightarrow K_G(X)$$

is a ring isomorphism.

Proof. See Segal [37, Prop. 2.2]. ∎

When the G-action on X is *free* (i.e., only the identity element of G has fixed points), we can quotient out by the G-action and obtain the K-group.

Proposition 2.4. *Suppose X is a free G-equivariant space. Then we have a ring isomorphism*

$$K_G(X) \xrightarrow{\sim} K(X/G): \quad [E] \longmapsto [E/G].$$

More generally, if N is a normal subgroup of G that acts freely on X, then we have an isomorphism $K_G(X) \cong K_{G/N}(X/N)$.

Proof. See Segal [37, Prop. 2.1] and Atiyah [3, Prop. 1.6.1]. ∎

When X is a pointed equivariant space, this means the map $\mathrm{pt} \hookrightarrow X$ that defines the basepoint is an equivariant map. Thus it induces a map $K_G(X) \to K_G(\mathrm{pt})$. The **reduced G-equivariant K-group** of X, denoted by $\widetilde{K}_G(X)$, is defined as the kernel of this map. Similar to Equation (1.3.1), we have an isomorphism

$$K_G(X) \cong \widetilde{K}_G(X) \oplus R(G),$$

where we used $K_G(\mathrm{pt}) \cong R(G)$ as explained in Example 2.2. Notice that the fibre over the basepoint has the structure of a G-representation: the basepoint is fixed under the group action. So now the reduced group 'forgets' not just the dimension of the

fibre over the basepoint, but also the representation on it. Like in ordinary K-theory, $\widetilde{K}_G$ becomes a contravariant functor from pointed G-equivariant compact Hausdorff spaces to abelian groups.

To define graded equivariant K-groups, we need to turn the suspensions $\Sigma^n X$ into G-equivariant spaces. We first give the spheres $\mathbb{S}^n$ trivial G-action; we shall denote this space by $\mathbb{S}^n_{\text{triv}}$. The product $\mathbb{S}^n_{\text{triv}} \times X$ is then also an equivariant space. Taking the quotient, we get a G-action on $\Sigma^n X$ too. We can then use the same definition as before for the lower degree K-groups (cf. Definition 1.17):

$$\begin{aligned} \widetilde{K}_G^{-n}(X) &:= \widetilde{K}_G(\Sigma^n X) \qquad \text{when } X \text{ is pointed;} \\ K_G^{-n}(X) &:= \widetilde{K}_G(\Sigma^n(X^+)). \end{aligned}$$

Example 2.5 (Equivariant K-theory of a point). Let G be a finite group. In Example 2.2 we argued why $K_G(\text{pt}) = R(G)$. We will now compute the entire G-equivariant K-theory of a point. First notice that since pt has trivial G-action, so does pt^+, and hence so do all suspensions $\Sigma^n(\text{pt}^+)$. Thus the suspension $\Sigma^n(\text{pt}^+)$ is *equivariantly* homeomorphic to the sphere $\mathbb{S}^n_{\text{triv}}$ (i.e., the n-sphere $\mathbb{S}^n$ with trivial group action). Proposition 2.3 together with the K-theory of spheres from Section 1.5 yields for $n > 0$

$$K_G(\mathbb{S}^n_{\text{triv}}) \cong R(G) \otimes_{\mathbb{Z}} K(\mathbb{S}^n) \cong \begin{cases} R(G) \oplus R(G) & n \text{ even}, \\ R(G) & n \text{ odd}. \end{cases}$$

Taking reduced groups, we conclude for all n

$$K_G^{-n}(\text{pt}) = \widetilde{K}_G(\mathbb{S}^n_{\text{triv}}) \cong \begin{cases} R(G) & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Notice the striking similarity with the K-theory of a point from Equation (1.5.1).

Equivariant K-theory also satisfies the axioms of Section 1.4, but with some modifications made. Most notable are the replacement of maps with equivariant maps. This includes replacing the homotopy with an *equivariant homotopy*: a homotopy that is also an equivariant map (with the unit interval carrying a trivial action). These make equivariant K-theory an *equivariant cohomology theory*; see Bredon [6, §1.2] for a precise definition. Two-fold Bott periodicity also still holds, allowing us to extend equivariant K-theory to have degrees in $\mathbb{Z}$ as well.

2.1.3. Equivariant KO-theory

Parallel to Section 1.6, KO-theory can also be adapted to an equivariant version. All that is needed is to replace the complex fibres with real fibres. This yields a ring $KO_G(X)$ and an abelian group $\widetilde{KO}_G(X)$. The graded versions $KO_G^{-n}(X)$ and $\widetilde{KO}_G^{-n}(X)$ are

defined in the exact same manner as the complex variant. All the axioms for KO-theory carry over to equivariant KO-theory, *mutatis mutandis*, making it an equivariant cohomology theory as well. It too has an eight-fold Bott periodicity.

2.2. KR-THEORY

Central to KR-theory is the idea of an antilinear involution on a complex vector space. An **involution** is a map that is its own inverse. A map $T\colon V \to V$ on a complex vector space V is **antilinear** when

$$T(x + \lambda \cdot y) = Tx + \bar{\lambda} \cdot Ty \quad \text{for all } x, y \in V \text{ and } \lambda \in \mathbb{C}.$$

This can be thought of as a conjugation on the vector space: $x \mapsto \bar{x}$. Antilinearity then looks like $\overline{\lambda \cdot x} = \bar{\lambda} \cdot \bar{x}$. The space $\mathbb{C}^n$ automatically has such a conjugation, namely component-wise complex conjugation: $(z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n)$. For a general complex vector space however, there is no canonical conjugation: it is an additional piece of information. This additional information gives it the structure of the *complexification* of a real vector space.

Proposition 2.6. *Let V be a finite-dimensional complex vector space, and let $T\colon V \to V$ be an antilinear involution on V. Then there is a real vector subspace W of V such that we have an isomorphism $V \cong \mathbb{C} \otimes_{\mathbb{R}} W$ under which T takes the form $z \otimes x \mapsto \bar{z} \otimes x$.*

Proof. The complex vector space V also has the structure of a real vector space. The map T is linear over the real numbers, so we can consider its eigenvalues. Since T squares to one (it is an involution), it can only have eigenvalues $+1$ and -1. It is diagonalisable since any $x \in V$ can be written as

$$x = \tfrac{1}{2}(x + Tx) + \tfrac{1}{2}(x - Tx).$$

The eigenspaces of T are of equal real dimension because multiplication by $i \in \mathbb{C}$ restricts to an isomorphism between them. Denote the eigenspace of eigenvalue $+1$ (i.e., the set of fixed points of T) by W. The complex dimension of V is half its real dimension, so

$$\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} V.$$

The map $\mathbb{C} \times W \to V$ given by scalar multiplication is bilinear over the real numbers, so it induces a map $\mathbb{C} \otimes_{\mathbb{R}} W \to V$. This map is surjective, and because $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} V$, it is also an isomorphism. The elements of W are fixed by T, so because T is antilinear, conjugating it with this isomorphism yields the map $z \otimes x \mapsto \bar{z} \otimes x$ on $\mathbb{C} \otimes_{\mathbb{R}} W$. ∎

We shall use this proposition to view a complex vector space with conjugation as a real vector space. One can think of KR-theory as consisting of bundles where some fibres are real, and others are complex. An involution that is fibre-wise antilinear will accomplish this. This involution will depend on the base-space; the category of such spaces we shall have to define first. In the remainder of our presentation of KR-theory, we mostly follow the original paper by Atiyah [2], which can be consulted when we omit proofs or details.

Remark 2.7. Sadly, Atiyah's original terminology in KR-theory has the tendency to be confusing. A particular variation of his original terms seems to have become standard in much of the literature. In particular, it was employed by Atiyah and Segal [4] in 1969 (Atiyah's original paper came out in 1966). It is this variant we shall adopt throughout.

2.2.1. Real spaces

A **Real space** X (note the capital 'R') is a topological space X together with a continuous involution $T\colon X \to X$. A **Real map** between two Real spaces X and Y is a continuous map that commutes with the involutions on X and Y. A **pointed Real space** is a pointed space and a Real space where the basepoint is fixed under the involution.

Strictly speaking, the notion of a Real space is equivalent to that of a $\mathbb{Z}_2$-equivariant space. The reason why we do not use this terminology is because we would like to view it as a conjugation, which becomes apparent in the appropriate kind of vector bundle.

Definition 2.8. Let X be a Real space. A **Real vector bundle** over X is a complex vector bundle E over X that is simultaneously a Real space, such that the following conditions are satisfied.

(i) The projection $\pi\colon E \to X$ is a Real map.

(ii) The involution on E is fibre-wise antilinear, i.e., the restriction $E_x \to E_{Tx}$ of the involution is antilinear for every $x \in X$.

If the involution T on the base-space fixes a point, the fibre over that point is the complexification of a real space, as in Proposition 2.6. Points that are not fixed by the involution do not have this property. It is in this sense that a Real vector bundle can have both real and complex fibres, even though strictly speaking all fibres are complex vector spaces.

2.2.2. The KR-groups

The definition of the KR-group is very much similar to that of the equivariant K-group. In short, when X is a Real compact Hausdorff space, the **KR-group** $KR(X)$ is the Grothendieck group of isomorphism classes of Real vector bundles over X under the direct sum. The tensor product turns it into a ring. In this manner KR is a contravariant functor from Real compact Hausdorff spaces to rings.

Special types of Real spaces also yield a relation between KR-theory and ordinary K-theory. When X is a Real space, denote by X_R the set of **Real points** of X, i.e., all points that are fixed under the involution. Previously we discussed why the fibre of a Real point can be viewed as a real vector space. This idea works on the level of K-groups as well.

Proposition 2.9. *Let X be a Real compact Hausdorff space. Then we have a ring isomorphism*

$$KO(X_R) \xrightarrow{\sim} KR(X_R)$$

given by fibre-wise complexification (and the involution induced thereby). In particular, if X has trivial Real structure, then $KR(X) \cong KO(X)$.

Proof. See Atiyah [2]. ∎

On the other hand, KR-theory also generalises K-theory in the following manner.

Proposition 2.10. *Let X be a compact Hausdorff space. Give the space $X \sqcup X$ the involution that swaps the two copies of X. Then we have a ring isomorphism*

$$K(X) \cong KR(X \sqcup X).$$

Proof. If E is a vector bundle over X, denote by $\bar{E}$ the *conjugate-bundle* over X. As a set it is equal to E, but we give the fibres a different scalar multiplication: if $v \in \bar{E}_x$ and $\lambda \in \mathbb{C}$, then we take $\lambda * v := \bar{\lambda} \cdot v$ as scalar multiplication (where the latter is the scalar multiplication on E_x). Give $E \sqcup \bar{E}$ the Real involution sending $x \in E$ to $x \in \bar{E}$ and vice-versa. This is antilinear by definition of $\bar{E}$. Then the map

$$K(X) \longrightarrow KR(X \sqcup X): \quad [E] \longmapsto [E \sqcup \bar{E}],$$

with $\bar{E}$ over the second X in $X \sqcup X$, is an isomorphism. ∎

When X is also pointed, the **reduced KR-group** $\widetilde{KR}(X)$ is the kernel of the map $KR(X) \to KR(\mathrm{pt})$ induced by inclusion of the basepoint. We have an isomorphism $KR(X) \cong \widetilde{KR}(X) \oplus KR(\mathrm{pt})$. In the same way as before, $\widetilde{KR}$ then becomes a contravariant functor from pointed Real compact Hausdorff spaces to abelian groups.

It is possible to define graded KR-groups by giving the spheres trivial involution, analogous to the trivial action in equivariant K-theory. Later however we will need a slightly more complicated definition. The method using a trivial involution on the spheres will be a special case of this. With $p, q \geq 0$ not both zero, define $\mathbb{S}^{p,q}$ as follows: as a topological space it equals S^{p+q-1}, and its involution is given by

$$(\alpha_1, \ldots, \alpha_{p+q}) \longmapsto (-\alpha_1, \ldots, -\alpha_p, \alpha_{p+1}, \ldots, \alpha_{p+q}).$$

In other words, the first p components get a minus sign, and the remaining q ones remain fixed. Notice that $\mathbb{S}^n_{\mathrm{triv}}$, the n-sphere with trivial involution, is equal to $\mathbb{S}^{0,n+1}$ in this notation. Now when X is a pointed Real space, define the **(p,q)-suspension** of X as

$$\Sigma^{p,q} X := \mathbb{S}^{p,q+1} \wedge X,$$

with the induced Real structure on the quotient. It is defined in this way to ensure that

$$\Sigma^{p,q}(\mathbb{S}^{r,s}) \cong \mathbb{S}^{p+r,q+s}$$

as Real spaces. In particular, $\Sigma^{0,0}$ is the identity. Now we define

$$\begin{aligned} \widetilde{KR}^{p,q}(X) &:= \widetilde{KR}(\Sigma^{p,q} X) \qquad \text{when } X \text{ is pointed;} \\ KR^{p,q}(X) &:= \widetilde{KR}(\Sigma^{p,q}(X^+)). \end{aligned}$$

It turns out that, up to isomorphism, the group $KR^{p,q}(X)$ only depends on the value of $p-q$ mod 8. In this spirit we abbreviate

$$KR^{p-q}(X) := KR^{p,q}(X) \quad \text{and} \quad \widetilde{KR}^{p-q}(X) := \widetilde{KR}^{p,q}(X).$$

As promised, the definition using a trivial involution on the spheres is a special case of this: take $p = 0$. In fact, this periodicity even shows it is sufficient to use only the cases with $p = 0$. Nevertheless, being able to take p nonzero is useful, as it gives an easy way to compute the KR-groups of spaces that are a suspension of that form. This will come up in Section 7.2.

Example 2.11 (KR-theory of a point). Since the point has a trivial involution, Proposition 2.9 implies that $KR(\mathrm{pt}) \cong KO(\mathrm{pt})$. We can generalise this, noting that the suspensions $\Sigma^{0,n}(\mathrm{pt}^+)$ all have trivial involution as well. Therefore we can use Proposition 2.9 again, finding for all n that

$$KR^{-n}(\mathrm{pt}) = KR^{0,n}(\mathrm{pt}) \cong KO^{-n}(\mathrm{pt}).$$

The KO-theory of a point is given in Equation (1.6.1).

The axioms from Section 1.4 carry over to the groups KR^{-q} as well, *mutatis mutandis*. Like with KO-theory (see Section 1.6), its periodicity is *eight*-fold:

$$KR^{-q-8}(X) \cong KR^{-q}(X).$$

Because these axioms carry over to KR-theory, Proposition 1.20 carries over directly to KR-theory: if X is pointed,

$$KR^{-q}(X) \cong \widetilde{KR}^{-q}(X) \oplus KR^{-q}(\mathrm{pt}).$$

2.2.3. Equivariant KR-theory

The most involved modification of K-theory we will encounter here is *equivariant KR-theory*. As its name suggests, it involves a Real space that also carries a group action. Though more generality is possible (like in the work of Atiyah and Segal [4, §6]), all we need is the following.[1] Fix a finite group G. A ***G*-equivariant Real space** is a Real space and a G-equivariant space, such that the group action commutes with the involution. A ***G*-equivariant Real vector bundle** E over X is a Real vector bundle E that is also a G-equivariant vector bundle, such that the group action again commutes with the involution. A homomorphism of Real G-equivariant vector bundles is a homomorphism of G-equivariant vector bundles that is also a Real map.

These definitions yield, like we have seen a number of times, a ring $KR_G(X)$ and an abelian group $\widetilde{KR}_G(X)$. The suspensions from KR-theory become G-equivariant spaces by giving the spheres trivial G-action. This allows us to define $KR_G^{p,q}(X)$ and $\widetilde{KR}_G^{p,q}(X)$, and also $KR_G^{-q}(X)$ and $\widetilde{KR}_G^{-q}(X)$, in the obvious way. As is the case with equivariant K-theory, the axioms of KR-theory hold for its equivariant version as well (mutatis mutandis). Eight-fold Bott periodicity in particular also applies.

[1] Our definition is a special case of Atiyah and Segal's, namely by taking G to have trivial involution.

Part II.

Symmetry-protected topological phases

3 SYMMETRIES IN QUANTUM MECHANICS

This chapter gives a mathematical description of the type of group actions that arise in quantum mechanics. A first guess at the relevant action would be a representation on the Hilbert space, i.e., a group acting on the Hilbert space by linear maps. It turns out this is not sufficiently general for quantum mechanics. First, the requirement of linear maps has to be relaxed: some symmetries cannot be implemented by linear maps. Second, the group action cannot even be a homomorphism in general: it is only multiplicative up to phases. In Sections 3.1 and 3.2 we demonstrate these phenomena with physical examples, and then abstract these ideas into the notion of a *quantum symmetry group*. Later, when asking the question of what symmetries commute with the Hamiltonian, we are led to the definition of an *extended quantum symmetry group*. We conclude the chapter with Section 3.3, where we discuss certain symmetries that occur often in condensed matter; these will play a central role in the rest of the work.

The mathematics presented in this chapter is mostly an adaptation of the work by Freed and Moore [11], with influences from Stehouwer [39].

3.1. QUANTUM AUTOMORPHISMS

In quantum mechanics, the state of a particle or system is described by a (nonzero, normalised) element of the Hilbert space of the system. The inner product on the Hilbert space is physically relevant: inner products give probability amplitudes. However, it is not immediately clear to what extent this inner product is physical. This is a relevant concern: if the inner product is not fully physical, then there are maps from the Hilbert space to itself that do not preserve the inner product, yet do preserve all relevant physical information. In other words, answering this question is directly relevant to deciding which maps should be called *quantum automorphisms*: maps from the Hilbert space to itself that preserve all physically relevant structure on it.

In any case, unitary maps should definitely be considered automorphisms. Wondrously enough, there are relatively simple examples of automorphisms that cannot be implemented as a unitary map. One such example is that of **time-reversal**: the exchange $t \mapsto -t$. Though a particular system may or may not be invariant under this exchange, it should still be a valid automorphism on the quantum level. Suppose that we were to implement time-reversal by a unitary operator T acting on the Hilbert space. Now consider the Schrödinger equation for the system:

$$i\hbar \cdot \partial_t \psi = H\psi,$$

with H the Hamiltonian of the system. Clearly T anticommutes with the time-derivative operator ∂_t. Since it is complex-linear, we find by applying T to both sides that

$$-i\hbar \cdot \partial_t T\psi = TH\psi.$$

Now we have run into a problem as far as our physics is concerned. Suppose ψ is an eigenstate of the Hamiltonian, i.e., $H\psi = E\psi$ for some energy E. Then apparently $T\psi$ will have energy $-E$. Assuming that the system has arbitrarily large energies (i.e., H has arbitrarily large eigenvalues), it must therefore also have arbitrarily negative energies. In other words, there would be no ground state for this system!

The way out of this problem is to let go of the complex-linearity condition. For if we let T be an *antilinear* operator, meaning that $T(\lambda\psi) = \bar{\lambda} \cdot T\psi$ for all ψ and λ, the problem goes away in full. However, the length of the inner product should still be conserved (i.e., $|\langle\psi_1, \psi_2\rangle|$ must be invariant). The appropriate notion of such a map is called an *antiunitary* operator.

Definition 3.1. Let V be a complex vector space with inner product $\langle\cdot,\cdot\rangle$. An **antiunitary map** from V to V is a bijective map $T\colon V \to V$ satisfying the following conditions.

(i) It is **antilinear**: $T(v + \lambda \cdot w) = Tv + \bar{\lambda} \cdot Tw$ for all $v, w \in V$ and $\lambda \in \mathbb{C}$.

(ii) It satisfies $\langle Tv,\, Tw\rangle = \overline{\langle v, w\rangle}$ for all $v, w \in V$.

The inverse of an antiunitary map is quickly verified to be antiunitary as well. Also notice that the composition of two antiunitary maps is unitary, and the composition of a unitary with an antiunitary map is antiunitary (whatever the order). Hence the set of all unitary and antiunitary operators on a (Hilbert) space forms a group under composition. It is these maps that we shall take for quantum automorphisms.

Definition 3.2. Let $\mathcal{H}$ be a Hilbert space. A **(linear) quantum automorphism** is an operator on $\mathcal{H}$ that is either unitary or antiunitary. The group of all linear quantum automorphisms on $\mathcal{H}$ is denoted by $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$.

Remark 3.3. The reason for the term "linear" in the above definition is to contrast it with so-called *projective quantum automorphisms*, which we do not discuss in this work. (For more information, see Freed and Moore [11, Ch. 1].) As such, we shall often forget the term "linear" and just refer to "quantum automorphisms" instead.

We have seen why we need to consider antiunitary operators as quantum automorphisms as well. However, it is not at all clear if we have not overlooked any other types of maps. The fact that we have not is known as *Wigner's theorem*. A full discussion would take too long for our purposes, but let us briefly comment on this theorem. Let $\mathcal{H}$ be a Hilbert space; then the **transition probability** of two nonzero elements $\psi_1, \psi_2 \in \mathcal{H}$ is defined as

$$P(\psi_1, \psi_2) = \frac{|\langle\psi_1, \psi_2\rangle|^2}{\|\psi_1\|^2 \cdot \|\psi_2\|^2}.$$

An automorphism of a quantum system should preserve this quantity since it can be measured (see Stehouwer [39, §1.1]). Roughly speaking, Wigner's theorem states that

any bijective map that preserves this transition probability may be 'implemented' as either a unitary or antiunitary operator. A precise formulation and its proof are not necessary for our purposes, but may be found in the exposition by Freed [10].

3.2. QUANTUM SYMMETRIES

Knowing what automorphisms of Hilbert space we consider as relevant, we can raise the question what the relevant type of group action is. This question is of also physical importance: often we shall consider a crystal living in Euclidean space $\mathbb{R}^3$, or in other dimensions, $\mathbb{R}^d$. This space comes with its own automorphism group: the group of all transformations of $\mathbb{R}^d$, consisting of translations, rotations, reflections, and glide-reflections. The Hilbert space $\mathcal{H}$ of complex-valued wave functions on $\mathbb{R}^d$ (i.e., $L^2(\mathbb{R}^d, \mathbb{C})$) then inherits an action of this group: if R is a transformation of $\mathbb{R}^d$ and $\psi\colon \mathbb{R}^d \to \mathbb{C}$ a wave function,

$$(R \cdot \psi)(x) := \psi(R^{-1} \cdot x) \quad \text{for all } x \in \mathbb{R}^d.$$

The inverse appears to make this a left-action instead of a right-action. Thus Euclidean transformations of $\mathbb{R}^d$ act naturally on wave functions defined on $\mathbb{R}^d$. One might think this yields a group homomorphism into $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$, but this is not so. In general the action is not multiplicative, but only up to a phase.

Example 3.4 (Rotation of fermions). Consider a three-dimensional system with half-integral angular momentum j, with $\mathbf{J}^2$ and J_z as the corresponding angular momentum operators. A rotation around the quantisation axis (i.e., the z-axis) is generated by $-i/\hbar \cdot J_z$, so that a rotation of an angle θ around the z-axis is represented by the operator

$$R_z(\theta) := \exp(-\theta i/\hbar \cdot J_z).$$

Using the familiar bra-ket notation for this example, we have a basis $\{\, |j,m\rangle \,\}_m$ for our Hilbert space of states. Any state $|\psi\rangle$ can be expressed in this basis via

$$|\psi\rangle = \sum_{m=-j}^{j} |j,m\rangle\langle j,m|\psi\rangle.$$

Because j is half-integral, so too are all possible values of m. Therefore, acting on this state with $R_z(2\pi)$ yields

$$\begin{aligned} R_z(2\pi)|\psi\rangle &= \sum_{m=-j}^{j} \exp(-2\pi i/\hbar \cdot J_z) \cdot |j,m\rangle\langle j,m|\psi\rangle \\ &= \sum_{m=-j}^{j} \exp(-2\pi i \cdot m) \cdot |j,m\rangle\langle j,m|\psi\rangle \end{aligned}$$

$$= \exp(-\pi i) \sum_{m=-j}^{j} |j,m\rangle\langle j,m|\psi\rangle = -|\psi\rangle.$$

Thus $R_z(\pi) \cdot R_z(\pi) = -R_z(0)$. Even though rotating π radians twice is the identity on Euclidean space, on a quantum system of half-integral spin this is not the case.

This example shows we need a modified version of a group action to describe how groups act on quantum systems. The approach we take was originally thought of by Mackey [31]. If G is a group, we will impose on a map $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ that

$$\rho(g) \cdot \rho(h) = \tau(g,h) \cdot \rho(gh) \tag{3.2.1}$$

for some map $\tau\colon G \times G \to U(1)$, where $U(1)$ is the complex circle (i.e., all $z \in \mathbb{C}$ of modulus 1). In Example 3.12 we will calculate the τ belonging to Example 3.4 above. The map τ we will view as an intrinsic part of the group and will be called the **(quantum) anomaly** associated to the group. Nevertheless, its effects are only seen in the representations of the group, not in the multiplication on the group itself. Before we give a general definition, we derive some properties of this map τ.

3.2.1. Cocycles

Let G be a group, and let $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ be a map satisfying Equation (3.2.1). We must first comment upon what happens when G has elements that act antiunitarily under ρ (e.g., when G contains time-reversal). When $\rho(g)$ is antiunitary, we have

$$\rho(g) \cdot z = \bar{z} \cdot \rho(g) = z^{-1} \cdot \rho(g) \quad \text{for all } z \in U(1),$$

while $\rho(g)$ commutes with all of $U(1)$ when it is unitary. Define a map $\varphi\colon G \to \{\pm 1\}$ by setting $\varphi(g) = +1$ when $\rho(g)$ is unitary, and $\varphi(g) = -1$ when $\rho(g)$ is antiunitary. This is a homomorphism by previous comments. We can write the above more compactly as

$$\rho(g) \cdot z = z^{\varphi(g)} \cdot \rho(g) \quad \text{for all } z \in U(1).$$

With this notation established, let us study the anomaly τ. Let $g,h,k \in G$. Multiplication in $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ is associative, so $\rho(g) \cdot \rho(h) \cdot \rho(k)$ is equal to

$$\begin{aligned} \rho(g) \cdot (\rho(h) \cdot \rho(k)) &= \rho(g) \cdot \tau(h,k) \cdot \rho(hk) \\ &= \tau(h,k)^{\varphi(g)} \cdot \rho(g) \cdot \rho(hk) \\ &= \tau(h,k)^{\varphi(g)} \cdot \tau(g,hk) \cdot \rho(ghk), \end{aligned}$$

but also equals

$$\begin{aligned} (\rho(g) \cdot \rho(h)) \cdot \rho(k) &= \tau(g,h) \cdot \rho(gh) \cdot \rho(k) \\ &= \tau(g,h) \cdot \tau(gh,k) \cdot \rho(ghk). \end{aligned}$$

The map $\rho(ghk)$ is invertible, so we must conclude that

$$\tau(h,k)^{\varphi(g)} \cdot \tau(g,hk) = \tau(g,h) \cdot \tau(gh,k). \tag{3.2.2}$$

In summary, this identity encodes the fact that multiplication from Equation (3.2.1) must be associative. This identity is not a new one: it has shown up before in the mathematical subject of *group cohomology*. Because the terminology from this field is commonplace in the literature, we give the necessary definitions to make further reading easier. More material on group cohomology may be found in Dummit and Foote [7, Ch. 17].

Definition 3.5. Let G be a group. A **G-module** is an abelian group A together with a group homomorphism $G \to \mathrm{Aut}(A)$.

More concretely, a G-module A is an abelian group along with a multiplication by elements of G, satisfying (where we write A additively)

$$1 \cdot a = a, \qquad (gh) \cdot a = g \cdot (h \cdot a) \qquad \text{and} \qquad g \cdot (a + b) = g \cdot a + g \cdot b$$

for all $g, h \in G$ and $a, b \in A$.

Example 3.6. Let G be a group and $\varphi \colon G \to \{\pm 1\}$ a homomorphism. Then the abelian group $U(1)$ of complex numbers of modulus one is a G-module under the multiplication

$$g \cdot z := z^{\varphi(g)} \quad \text{for } g \in G \text{ and } z \in U(1).$$

We denote $U(1)_\varphi$ for the group $U(1)$ with this G-module structure.

Definition 3.7. Let G be a group, and let A a be G-module (written additively). A **2-cocycle** from G to A is a map $\tau \colon G \times G \to A$ satisfying

$$g \cdot \tau(h, k) + \tau(g, hk) = \tau(g, h) + \tau(gh, k)$$

for all $g, h, k \in G$.

Unwrapping definitions, we see that Equation (3.2.2) says that τ is a 2-cocycle from G to the G-module $U(1)_\varphi$. Note that we write $U(1)$ multiplicatively rather than additively.

Another requirement on τ arises if we want ρ to map the identity element of G to the identity of $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$. In this case, we have for all $g \in G$,

$$\rho(g) = \rho(1) \cdot \rho(g) = \tau(1, g) \cdot \rho(g), \quad \text{meaning} \quad \tau(1, g) = 1,$$

and $\tau(g, 1) = 1$ follows similarly. A 2-cocycle satisfying $\tau(1, g) = \tau(g, 1) = 1$ for all $g \in G$ is called **unital**. In our application, this notion is equivalent to $\rho(1) = \mathrm{Id}$. There are no physical obstructions to having this be the case; in fact, it seems rather desirable. Mathematically speaking it is harmless to choose τ to be unital.

Remark 3.8. In group cohomology, one not only has 2-cocycles, but n-cocycles for a natural number n. If G is a group and A is a G-module, one writes $C^n(G, A)$ for the group of n-cocycles with values in A (with pointwise addition as operation). This forms a cochain complex, and the cohomology at position n is called the *n-th cohomology group of G* with values in A, denoted by $H^n(G, A)$. A more mathematically pleasing version

of the above is to use the class of τ in the group $H^2(G, A)$ instead of τ itself. We may do this because equivalent cocycles describe twistings of groups that are physically indistinguishable. However, doing this is not necessary for the calculations we do in this work, so we shall not pursue this further. Our approach can be thought of as choosing a representative for a class in $H^2(G, A)$. The assumption that τ is unital is justified because every class in $H^2(G, A)$ has a unital representative; see, e.g., Stehouwer [39, Lem. A.16].

3.2.2. Symmetry groups

With the proper terminology fixed, we define a quantum version of a symmetry group and a representation thereof. This is simply a generalisation of our earlier discussion.

Definition 3.9. A **quantum symmetry group** is a triple (G, φ, τ) of a group G, a group homomorphism $\varphi\colon G \to \{\pm 1\}$, and a unital 2-cocycle τ from G to the G-module $U(1)_\varphi$. The cocycle τ is called the **(quantum) anomaly** of the quantum symmetry group.

The meaning of the maps φ and τ lies in the conditions they impose on representations of G, as follows.

Definition 3.10. Let $\mathcal{H}$ be a Hilbert space and (G, φ, τ) a quantum symmetry group. A **(φ, τ)-twisted representation** of G on $\mathcal{H}$ is a map $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ satisfying the following conditions.

(i) The map $\rho(g)$ is unitary when $\varphi(g) = +1$, and antiunitary when $\varphi(g) = -1$.

(ii) For all $g, h \in G$, we have $\rho(g) \cdot \rho(h) = \tau(g, h) \cdot \rho(gh)$.

Quantum symmetry groups and twisted representations also have a notion of a homomorphism. A homomorphism between two quantum symmetry groups (G, φ, τ) and (G', φ', τ') is a group homomorphism $f\colon G \to G'$ such that

$$\varphi'(f(g)) = \varphi(g) \quad \text{and} \quad \tau'(f(g), f(h)) = \tau(g, h)$$

for all $g, h \in G$. A homomorphism between twisted representations is a map between the Hilbert spaces that 'intertwines' the multiplication of G.

Remark 3.11. In all our definitions, G is a group without any further structure. In the work of Freed and Moore, G is taken to be a *topological group*, and both φ and τ are required to be continuous, as are twisted representations. This would require $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ to be given a topology. For our more modest purposes, this detail is not necessary to discuss.

Example 3.12 (Rotation of fermions, continued). We continue Example 3.4 and phrase the situation in our new terminology. For our group G we take the circle. Since all rotations defined previously are unitary maps (because J_z is hermitian), the morphism φ is trivial in this case. However, as shown before, the cocycle τ

cannot be trivial. To find an expression for it, we first describe our desired (twisted) representation ρ, because this determines τ uniquely. Recall the map (where $\mathcal{H}$ is our Hilbert space)

$$R_z\colon \mathbb{R} \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})\colon \quad \theta \longmapsto \exp(-i\theta/\hbar \cdot J_z).$$

For convenience we will represent G as $[0, 2\pi)$ with addition modulo 2π. Define the map $s\colon [0, 2\pi) \to \mathbb{R}$ to be the inclusion. Then we define $\rho := R_z \circ s$. Notice that

$$R_z(\theta \pm 2\pi) = -R_z(\theta)$$

since $\exp(\mp 2\pi i/\hbar \cdot J_z)$ acts as $-\mathrm{Id}$ on $\mathcal{H}$ due to the half-integral angular momentum of the system.

When $\theta, \eta \in [0, 2\pi)$ are such that $0 \leq \theta + \eta < 2\pi$, then $\rho(\theta + \eta) = \rho(\theta) \cdot \rho(\eta)$ since s is additive under these conditions. But when $2\pi \leq \theta + \eta < 4\pi$, then

$$s(\theta + \eta) = s(\theta) + s(\eta) - 2\pi = \theta + \eta - 2\pi,$$

so that $\rho(\theta + \eta) = R_z(\theta + \eta - 2\pi) = -R_z(\theta + \eta)$. So in this case we find

$$\begin{aligned} \tau(\theta, \eta) &= \rho(\theta) \cdot \rho(\eta) \cdot \rho(\theta + \eta)^{-1} \\ &= -R_z(\theta) \cdot R_z(\eta) \cdot R_z(\theta + \eta)^{-1} \\ &= -R_z(\theta + \eta) \cdot R_z(\theta + \eta)^{-1} = -1. \end{aligned}$$

We thereby conclude that, for $\theta, \eta \in [0, 2\pi)$,

$$\tau(\theta, \eta) = \begin{cases} +1 & \text{when } \theta + \eta < 2\pi, \\ -1 & \text{when } \theta + \eta \geq 2\pi. \end{cases}$$

3.2.3. Extended symmetry groups

In our applications of quantum symmetry groups, we will encounter groups that have elements that reverse the time direction of the system. This can be encoded abstractly by a group homomorphism $\theta\colon G \to \{\pm 1\}$, where $\theta(g)$ is $+1$ or -1 when it preserves or flips the time direction, respectively. From this information we can derive whether (the representation of) an element $g \in G$ will commute with the Hamiltonian.

Proposition 3.13. *Let (G, φ, τ) be a quantum symmetry group, let $\mathcal{H}$ be a Hilbert space with a Hamiltonian H, and let ρ be a (φ, τ)-twisted representation of G on $\mathcal{H}$. Let $\theta\colon G \to \{\pm 1\}$ be the homomorphism defined by $\theta(g) = +1$ when $\rho(g)$ preserves the time direction, and $\theta(g) = -1$ when it reverses it. Then for all $g \in G$,*

$$H \cdot \rho(g) = \varphi(g) \cdot \theta(g) \cdot \rho(g) \cdot H.$$

Proof sketch. Time translations in quantum mechanics are generated by $-iH/\hbar$, i.e.,

translating the system t units of time into the future is implemented by the operator

$$\exp(-itH/\hbar).$$

When $g \in G$, the map $\rho(g)$ reverses the time direction if and only if $\theta(g) = -1$, which means that

$$\exp(-itH/\hbar) \cdot \rho(g) = \rho(g) \cdot \exp(-\theta(g) \cdot itH/\hbar).$$

From this it follows that

$$(-itH/\hbar) \cdot \rho(g) = \rho(g) \cdot (-\theta(g) \cdot itH/\hbar).$$

Remember that $\rho(g)$ commutes with i when $\varphi(g) = +1$, and anticommutes with i when $\varphi(g) = -1$. So this proves the result. ■

Writing $c := \varphi \cdot \theta$, this homomorphism keeps track whether elements of g should commute or anticommute with the Hamiltonian. A quantum symmetry group equipped with such a homomorphism c we shall call an *extended quantum symmetry group*.

Definition 3.14. An **extended quantum symmetry group** is a quadruple (G, φ, τ, c) where (G, φ, τ) is a quantum symmetry group, and where $c\colon G \to \{\pm 1\}$ is a homomorphism.

Remark 3.15. Since any two of φ, θ and c determine the third, we have lost no information by not making θ part of this definition. Choosing c in this definition is a useful convention, which will become more and more clear as we progress.

Like with quantum symmetry groups before, the meaning of the homomorphism c is in the requirements it puts on representations. However, a representation of an extended quantum symmetry group only makes sense when we have also chosen a Hamiltonian.

Definition 3.16. Let (G, φ, τ, c) be an extended quantum symmetry group. An **extended quantum system** with extended quantum symmetry group (G, φ, τ, c) is a triple $(\mathcal{H}, H, \rho)$ of a Hilbert space $\mathcal{H}$, a self-adjoint operator H on $\mathcal{H}$, and a (φ, τ)-twisted representation ρ of G on $\mathcal{H}$, such that

$$H \cdot \rho(g) = c(g) \cdot \rho(g) \cdot H$$

for all $g \in G$. We call H the **Hamiltonian** of the quantum system.

Remark 3.17. Note that we do not call the map ρ in the above definition a "(φ, τ, c)-twisted representation" — that terminology is reserved for a later concept (see Definition 4.19).

3.3. Altland–Zirnbauer classes

We have seen why time-reversal has to be implemented by an antiunitary operator. One would expect that reversing time twice is the same as not reversing it at all, but in quantum mechanics this need not be the case. It will always be a scalar, i.e., $T^2 = z$ for some $z \in \mathbb{C}^\times$, but there are two options for z.

Proposition 3.18. *Let $\mathcal{H}$ be a Hilbert space and let T be an antiunitary operator on $\mathcal{H}$ such that $T^2 = z$, where $z \in \mathbb{C}^\times$. Then z is either $+1$ or -1.*

Proof. Because T conjugates the inner product on $\mathcal{H}$, the operator T^2 must preserve it. Therefore z must be of modulus one: for all $v, w \in \mathcal{H}$,

$$\langle v, w\rangle = \langle T^2 v,\, T^2 w\rangle = \langle zv,\, zw\rangle = \bar{z}z \cdot \langle v, w\rangle = |z|^2 \cdot \langle v, w\rangle,$$

which implies $|z|^2 = 1$. The operator T commutes with itself, so

$$T \cdot z = T \cdot T^2 = T^2 \cdot T = z \cdot T,$$

while because T is antilinear,

$$T \cdot z = \bar{z} \cdot T.$$

The fact that T is invertible now implies $z = \bar{z}$. We conclude $z = \pm 1$. ∎

The possibility $T^2 = -1$ is very similar to the rotation of a fermionic system from Example 3.4. Indeed, on a fermionic system time-reversal will square to -1. Nonetheless, the argument above is limited only to antiunitary maps. For the square of a unitary operator can be altered at will: if U is unitary and $|w| = 1$, then wU is unitary and $(wU)^2 = w^2 \cdot U^2$. For an antiunitary operator T this reasoning does not work, since then $(wT)^2 = wTwT = w\bar{w} \cdot T^2 = T^2$.

But if this is the case, then it is not clear why we should care that a 180-degree rotation squares to -1 in Example 3.4. If there are no further symmetries present, then the above argument indeed shows the square is irrelevant. However, if we have a system with both a unitary symmetry U *and* an antiunitary symmetry T, then multiplying U with a phase changes the commutation relation with T. For example, if U and T commute, then iU and T anticommute: $iUT = iTU = -TiU$. In other words, we cannot change the square of U while leaving the physical system unchanged.

This distinction made in Proposition 3.18 is physically relevant. In 1962, Dyson [9] grouped single-particle quantum systems into three classes, according to what type of time-reversal symmetry it has. A system could either have no time-reversal symmetry, it could have time-reversal that squares to $+1$, or time-reversal that squares to -1. This classification into three classes became known as the *three-fold way*. For convenience, we shall abbreviate these three cases by $T^2 = 0$, $T^2 = +1$ and $T^2 = -1$, respectively. Note that this does not mean that time-reversal squares to the zero map in the first case: it is but an abbreviation.

Later in 1997, Altland and Zirnbauer [1] gave a similar classification that considered more symmetries. These classes are now called the **Altland–Zirnbauer classes**; it is also referred to as the *ten-fold way*, referring to how many classes there are. The second

symmetry they incorporated is **particle-hole symmetry** (also called *charge conjugation*). We denote this symmetry by C. As its name suggests, it is the reversal of a particle and its corresponding hole (i.e., the absence of a particle). It turns out to be implemented by an antiunitary map that anticommutes with the Hamiltonian. One can see this because a hole has the opposite energy of a particle, so C must anticommute with the Hamiltonian. If one then requires C to preserve the Schrödinger equation, one has no choice but to make it antiunitary. Proposition 3.18 then implies it squares to either $+1$ or -1 when it is present. We shall use the abbreviations $C^2 = 0$ and $C^2 = \pm 1$ in the same way as we do with time-reversal.

The final symmetry that Altland and Zirnbauer considered is **chiral symmetry** (also called *sublattice symmetry*): the product of time-reversal and particle-hole reversal, which we denote by S. Being the product of two antiunitary maps, it is a unitary map. When a system has time-reversal and particle-hole symmetry, then the square of chiral symmetry is fixed. It is also possible that a system has chiral symmetry while having neither time-reversal nor particle-hole symmetry. In that case we can take $S^2 = +1$ by the previous discussion. The square of S is thus always uniquely determined. Due of this lack of freedom it is customary to use the abbreviations $S = 0$ and $S = 1$ to indicate the absence and presence of chiral symmetry, respectively.

In total this gives ten classes: there are three options for T, three for C, and one additional case where only S is present. Each of the classes has a label called its **Cartan label**. This has to do with a different classification done by the mathematician Élie Cartan, though the connection is not obvious. (See the comments made by Ryu et al. [36] for more information.) Table 3.1 gives all the Altland–Zirnbauer classes, using our abbreviations to denote the symmetries in each class. As odd it may seem, the order of the classes in that table is chosen deliberately. It turns out this ordering is particularly useful for the calculations we intend to do in this work (more specifically, for Corollary 6.16). Take particular note that classes A and AIII are positioned separately: they are the only ones that do not have an antiunitary symmetry.

	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
T^2	0	0	$+1$	$+1$	0	-1	-1	-1	0	$+1$
C^2	0	0	0	$+1$	$+1$	$+1$	0	-1	-1	-1
S	0	1	0	1	0	1	0	1	0	1

Table 3.1: The ten Altland–Zirnbauer classes. The top row gives the Cartan label of the classes. The three rows below indicate whether time-reversal symmetry T, particle-hole symmetry C, and chiral symmetry S is present in a given class, respectively. A zero indicates the absence of a symmetry. If present, time-reversal and particle-hole reversal have two options: their square is either $+1$ or -1. For chiral symmetry there is no such distinction.

4 INSULATORS

Topological insulators are special states of insulators, so we cannot continue unless we have a proper understanding of insulators. The most important aspect of an insulator is its symmetries. Insulators are in particular crystals, giving them crystal symmetry, but they can also have time-reversing or particle-hole reversing symmetries. The most elegant way to unify these concepts is to realise that an insulator has a vector bundle that summarises all of its physics. We call this bundle its *Bloch bundle*. The symmetries of the insulator then act on this vector bundle in a similar way to the equivariant vector bundles from Chapter 2. As we will see in Chapter 6, recognising this vector bundle in an insulator is the crucial first step towards describing topological phases.

To ensure that this vector bundle does indeed describe all of the physics of an insulator, we are forced to work non-relativistically, and to use a non-interacting model of electrons. This is assumed throughout this chapter, and even this whole work. Most material in this chapter is drawn from Stehouwer [39, Ch. 2]. Before we define crystals, in Section 4.1 we fix some terminology to allow for easier discussion. We can then in Section 4.2 define crystals mathematically and re-introduce basic terminology from condensed matter in this context. The Bloch bundle is the subject of Section 4.3, which then allows us to discuss time-reversal and particle-hole reversal in Sections 4.4 and 4.5. The formalisation of these ideas given in Section 4.6 concludes the chapter.

4.1. AFFINE SPACES

Even though it is sometimes modelled that way, Euclidean space does not form a vector space. The trouble lies in that Euclidean space has no preferred origin. But once an origin is chosen, the vector space structure is recovered by making this origin the neutral element of the vector space. The reason why this recovers a vector space structure is because Euclidean space still has a notion of *translations*. More mathematically speaking, it has an action of a (real) vector space. The formalisation of this idea is an *affine space*.

Definition 4.1. An **affine space** is a set E together with a real vector space V that acts transitively and freely on E. In other words, for every $x, y \in E$, there is a unique $v \in V$ such that $x + v = y$, with $x + v$ denoting the result of v acting on x. The vector space V is called the **affine group** of E.

If E is an affine space with affine group V, then any choice $e_0 \in E$ of origin makes it

a vector space isomorphic to V. For then we can define a map

$$V \longrightarrow E: \quad v \longmapsto e_0 + v,$$

and since V acts transitively and faithfully on E, this is a bijection. Therefore it gives E a unique vector space structure, which in turn makes the map a linear isomorphism.

Our only interest in affine spaces is to describe Euclidean space. Thus any examples of affine spaces in our work may safely be replaced with Euclidean space (whence the label E for an affine space). But Euclidean space is more than just an affine space: it also has a distance function (i.e., metric) on it. The functions that preserve this metric (i.e., isometries) form the *Euclidean group*.

Definition 4.2. The ***d*-dimensional Euclidean space** $\mathbb{E}^d$ is, as a set, equal to $\mathbb{R}^d$. It is an affine space with affine group $\mathbb{R}^d$. Its group of isometries (where we give $\mathbb{E}^d$ has the usual Euclidean distance function) is called the ***d*-dimensional Euclidean group** and is denoted by $\mathbb{E}(d)$.

The group of translations $\mathbb{R}^d$ is a normal subgroup of $\mathbb{E}(d)$. Its quotient is (isomorphic to) $O(d)$, the group of d-dimensional orthogonal transformations. More visually, dividing out by translations means fixing an origin in space, and the group of all isometries that fix an origin is $O(d)$. In fact, $\mathbb{E}(d)$ is a semidirect product $O(d) \ltimes \mathbb{R}^d$, but not canonically: an isomorphism $\mathbb{E}(d) \cong O(d) \ltimes \mathbb{R}^d$ is the same as choosing an origin for Euclidean space $\mathbb{E}^d$.

4.2. CRYSTALS

Informally, a crystal is a collection of points that is positioned regularly. This property can be described more formally in terms of its symmetry group. More specifically, it has to do with its translational symmetry: it means that its group of translations is a *lattice*.

Definition 4.3. Let V be a d-dimensional vector space. A **lattice** Π in V is a subgroup of V that is isomorphic to $\mathbb{Z}^m$ for some $m \geq 1$, such that its linear span is an m-dimensional subspace of V. The lattice Π is called **full** when $m = d$.

Definition 4.4. A ***d*-dimensional crystal** is a subset C of $\mathbb{E}^d$, such that the group of translations of C,

$$\Pi(C) := \{\, v \in \mathbb{R}^d \mid C + v = C \,\},$$

is a full lattice. If $C \subseteq \mathbb{E}^d$ is a crystal, its **space group** is

$$S(C) := \{\, R \in \mathbb{E}(d) \mid R(C) = C \,\},$$

and its **point group** $P(C)$ is the quotient $S(C)/\Pi(C)$.

Two facts about the point group are of paramount importance for us. The first is that it may be naturally embedded in $O(d)$, as follows. The map $S(C) \hookrightarrow \mathbb{E}(d) \twoheadrightarrow O(d)$ has

$\Pi(C)$ as its kernel. It therefore induces an injection $P(C) \hookrightarrow O(d)$; we shall interpret it as an inclusion. The other fact about the point group is that it is always finite.

Proposition 4.5. *If $C \subseteq \mathbb{E}^d$ is a crystal, then its point group P is a finite group.*

Proof. First we shall prove that the point group naturally acts on the crystal lattice Π by group automorphisms. For P can be embedded in $O(d)$ as we argued above. It therefore acts on $\mathbb{R}^d$. If $R \in P$ and $v \in \Pi$, then Rv lies in Π also:

$$C + Rv = R(C) + Rv = R(C + v) = R(C) = C,$$

because both R and v leave the crystal invariant. Hence the action of P on $\mathbb{R}^d$ restricts to an action on Π. One immediately sees that $v \mapsto Rv$ is a group automorphism of Π for any $R \in P$. Lastly, the action is also faithful: only the identity element acts trivially, because Π spans all of $\mathbb{R}^d$. Taken together, these facts imply we have an injective homomorphism $P \to \mathrm{Aut}(\Pi)$, i.e., an embedding of P into $\mathrm{Aut}(\Pi)$. Because Π is isomorphic to $\mathbb{Z}^d$, its automorphism group is isomorphic to $\mathrm{Aut}(\mathbb{Z}^d)$. But $\mathrm{Aut}(\mathbb{Z}^d)$ is a finite group; thus P is finite also. ∎

Instead of using crystals directly, we will only use the space group of the crystal. However, this group can be hard to work with because it is not finite. Sometimes we can quotient out the lattice, in which case we can use the point group instead, but this does not always happen. Two types of crystals will distinguish themselves: those for which P is a subgroup of S, and those for which it is not. More precisely, the distinction is whether S is isomorphic to a semidirect product $P \ltimes \Pi$ or not. If it is, we shall call the crystal **symmorphic**, and **nonsymmorphic** otherwise. In this context it is customary to denote elements in $P \ltimes \Pi$ as $\{ R \mid v \}$ instead of (R, v) for $R \in P$ and $v \in \Pi$. This is called **Seitz notation**.[1] An isomorphism $S \cong P \ltimes \Pi$ then gives rise to an embedding of P into S, via

$$R \longmapsto \{ R \mid 0 \}.$$

This should be thought of as choosing an origin on the crystal and interpreting elements of P as rotations and reflections around this origin. It allows one to *restrict* to the point group instead of having to take a quotient by the lattice. In particular, any S-action restricts to a P-action through the above embedding, even when the lattice acts nontrivially. It should be noted however that there is no preferred isomorphism $S \cong P \ltimes \Pi$: choosing one is the same as choosing an origin.

In this thesis we shall only treat symmorphic crystals. Stehouwer [39] gives a thorough treatment that is close to our presentation, but which includes nonsymmorphic crystals. Occasionally we will point out the places where nonsymmorphic crystals cause subtleties.

4.2.1. The Brillouin zone

In a crystal it is very natural to take the Hamiltonian to be invariant under the lattice of the crystal. Examples like

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V,$$

[1] Seitz notation also exists for nonsymmorphic crystals, but it is slightly more intricate in that case.

where V is a Π-periodic potential, often occur. Due to this periodicity, the wave functions of particles moving through the lattice depend periodically on the momentum of the particle. This allows us to think of a particle as not having a momentum, but a *quasi-momentum*: a momentum up to a *reciprocal lattice vector*. The set of all these momenta is called the (*first*) *Brillouin zone*. Mathematically the Brillouin zone is easier to work with because it is compact, which momentum-space (isomorphic to $\mathbb{R}^d$) is not. All these notions can be defined from the lattice alone.

Definition 4.6. Let V be a real vector space and let $\Pi \subseteq V$ be a full lattice in V. The **reciprocal space** of V is the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The **reciprocal lattice** of Π is the group

$$\Pi^* := \mathrm{Hom}(\Pi, 2\pi\mathbb{Z})$$

of group homomorphisms from Π to $2\pi\mathbb{Z}$.

The reciprocal space V^* is to be thought of as the space of all momenta, often called *k-space*. The dual coupling of V and V^* we shall denote as $\langle k, v\rangle := k(v)$ for $k \in V^*$ and $v \in V$, in order to think of this as an inner product between a vector v and a reciprocal vector k. This is also why we require dual lattice vectors to map the lattice to $2\pi\mathbb{Z}$: in condensed matter, if k is the dual lattice vector to a lattice vector v, one requires the inner product $\langle k, v\rangle$ to be 2π.

Notice that a functional $k\colon V \to \mathbb{R}$ is uniquely determined by its valued on a full lattice, since the span of a full lattice is the entire vector space. Conversely, if $\Pi \subseteq V$ is a full lattice, any group homomorphism $\Pi \to 2\pi\mathbb{Z}$ can be extended to a linear map $V \to \mathbb{R}$. This gives an embedding of the reciprocal lattice in reciprocal space. We may therefore also regard it as a subset of reciprocal space.

Definition 4.7. Let V be a real vector space and let $\Pi \subseteq V$ be a full lattice in V. The **Brillouin zone** of Π is the quotient group

$$X_\Pi := V^*/\Pi^*.$$

It becomes a topological space under the quotient topology of the projection $V^* \to V^*/\Pi^*$ (where V^* has the topology coming from its finite-dimensional vector space structure).

Proposition 4.8. *Let V be a d-dimensional real vector space, and let Π be a full lattice in V. Then the Brillouin zone X_Π is, as a topological space, homeomorphic to the d-torus $\mathbb{T}^d = (\mathbb{S}^1)^d$.*

Proof. We have a (non-canonical) isomorphism $V^* \cong \mathbb{R}^d$ under which Π gets mapped to $\mathbb{Z}^d$. Moreover, this map is a homeomorphism. The quotient group $\mathbb{R}^d/\mathbb{Z}^d \cong (\mathbb{R}/\mathbb{Z})^d$ is homeomorphic to the d-torus because the quotient group $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle. ∎

The above definitions do not require the lattice to come from a crystal. When it does, it inherits an action of P, as follows. In this case we have $V = \mathbb{R}^d$, making k-space equal

to $V^* = (\mathbb{R}^d)^*$. The point group P then acts naturally on $(\mathbb{R}^d)^*$ via, for $R \in P \subseteq O(d)$ and $k \in (\mathbb{R}^d)^*$,

$$(R, k) \longmapsto k \circ R^{-1}.$$

The inverse appears to make this a left-action instead of a right-action. This action preserves the dual lattice because $R(\Pi) = \Pi$ for all $R \in P$, as we showed in the proof of Proposition 4.5. Hence it induces an action on the quotient V^*/Π^*. In the language of Section 2.1.1, this makes the Brillouin zone a P-equivariant space.

Although this action depends on the crystal, one action on the Brillouin zone will recur often. Consider the involution on $V^* = (\mathbb{R}^d)^*$ given by $k \mapsto -k$, i.e., the reversal of the momentum direction. It is clear that this preserves the dual lattice Π^*, so it induces an involution on the Brillouin zone too. This involution we shall denote by σ. Its importance lies in the multitude of examples of symmetries that reverse the direction of momenta.

4.3. The Bloch bundle

The basic solutions to the Schrödinger equation with zero potential (i.e., free particles) are the plane waves, $\exp(i\langle k, x\rangle)$. Although these functions themselves are not L^2-functions, the L^2-solutions to the Schrödinger equation have a Fourier decomposition into these waves:

$$\psi(x) = \int \hat{\psi}(k) \cdot \exp(i\langle k, x\rangle)\, \mathrm{d}k.$$

Wave functions on crystals behave analogously, but with a decomposition into *Bloch waves*. This is known as *Bloch's theorem*. It is of fundamental importance for us because it reveals that a certain vector bundle is underlying the Hilbert space.

Definition 4.9. Let E be an affine space and let Π be a full lattice of translations of E. Let $k \in X_\Pi$. A **Bloch wave with momentum k** on E is a function $\psi\colon E \to \mathbb{C}$ such that

$$\psi(x + v) = \exp(i\langle k, v\rangle) \cdot \psi(x) \quad \text{for all } x \in E \text{ and } v \in \Pi.$$

The phase appearing in this definition is well-defined because dual lattice vectors map Π to $2\pi\mathbb{Z}$.

Remark 4.10. A more common definition of a Bloch wave is a function ψ that can be written as $\psi(x) = \exp(i\langle k, x\rangle) \cdot u(x)$, with u a Π-periodic function. Although it is equivalent to ours, inherent in this alternate definition is the choice of an origin for the affine space E.

Through a process of symmetrisation, any function can be turned into a Bloch wave of a certain momentum.

Definition 4.11. Let Π be a full lattice of translations of $\mathbb{E}^d$. Let $\psi\colon \mathbb{E}^d \to \mathbb{C}$ be an L^2-function on X. The **Bloch sum** of ψ is the function $\hat{\psi}\colon X_\Pi \times \mathbb{E}^d \to \mathbb{C}$ defined by

$$\hat{\psi}(k, x) := \sum_{v \in \Pi} \exp(-i\langle k, v\rangle) \cdot \psi(x + v).$$

We write $\hat{\psi}_k$ with $k \in X_\Pi$ for the function $\mathbb{E}^d \to \mathbb{C}\colon x \mapsto \hat{\psi}(k, x)$.

Remark 4.12. The above sum converges because ψ is an L^2-function.

Notice that $\hat{\psi}_k$ is a Bloch wave of momentum k. Define $\mathcal{E}_k$ (with $k \in X_\Pi$) to be the Hilbert space of all Bloch waves over $\mathbb{E}^d$ of momentum k. Then for every $k \in X_\Pi$, the Bloch sum yields a map

$$L^2(\mathbb{E}^d, \mathbb{C}) \longrightarrow \mathcal{E}_k \colon \quad \psi \longmapsto \hat{\psi}_k.$$

We define $\mathcal{E} := \bigsqcup_{k \in X_\Pi} \mathcal{E}_k$, the set of all Bloch waves. This can be thought of as a sort of bundle $\mathcal{E}$ over X_Π: it has a projection $\mathcal{E} \to X_\Pi$ assigning to each wave its momentum. If each $\mathcal{E}_k$ were finite dimensional, this would yield a vector bundle over X_Π, but this is not the case. The sets $\mathcal{E}_k$ do form a Hilbert space, so instead it forms a *Hilbert bundle*. A **Hilbert bundle** is a modification of a vector bundle, where the fibres are endowed with the structure of a Hilbert space instead of just a vector space. In particular, these Hilbert spaces are not assumed to be finite-dimensional. Local triviality is still assumed, in an analogous fashion to vector bundles (cf. Definition 1.4).

Definition 4.13. Let Π be a full lattice of translations of $\mathbb{E}^d$. With $k \in X_\Pi$, write $\mathcal{E}_k$ for the Hilbert space of Bloch waves over $\mathbb{E}^d$ of momentum k. The **Bloch bundle** $\mathcal{E}$ is the complex Hilbert bundle over X_Π with fibres $\mathcal{E}_k$.

Freed and Moore [11, App. D.2] prove that the Bloch bundle has a natural topology that makes it a Hilbert bundle, i.e., under which it is locally trivial.

The Bloch sum now gives a relationship between the Hilbert space $L^2(\mathbb{E}^d, \mathbb{C})$ and the Bloch bundle $\mathcal{E}$. For if $\psi \in L^2(\mathbb{E}^d, \mathbb{C})$, then $\hat{\psi}_k$ is a Bloch wave of momentum k. In other words, the assignment $\psi \mapsto \hat{\psi}$ in the Bloch sum maps wave functions to *sections* of the Bloch bundle $\mathcal{E}$: at every point $k \in X_\Pi$, we get a Bloch wave $\hat{\psi}_k \in \mathcal{E}_k$. Such a section will itself be an L^2-map from X_Π to $\mathcal{E}$; the set of such sections of the bundle $\mathcal{E}$ we denote by $\Gamma_{L^2}(\mathcal{E})$.

Theorem 4.14 (Bloch). *Let Π be a full lattice of translations in $\mathbb{E}^d$. The Bloch sum is an isomorphism of Hilbert spaces*

$$L^2(\mathbb{E}^d, \mathbb{C}) \xrightarrow{\sim} \Gamma_{L^2}(\mathcal{E}) \colon \quad \psi \longmapsto \hat{\psi},$$

with inverse given by

$$\varphi \longmapsto \left(x \longmapsto \int_{X_\Pi} \varphi_k(x) \, \mathrm{d}k \right).$$

Proof. See Freed and Moore [11, Prop. D.17]. ∎

In general, the Hilbert space of a system may not be all of $L^2(\mathbb{E}^d, \mathbb{C})$, but a (closed) subspace of it. A physically relevant subspace corresponds to a sub-Hilbert bundle of $\mathcal{E}$ under the isomorphism of Bloch sums. This subbundle we shall call the Bloch bundle of the system; by abuse of notation, we shall also denote it by $\mathcal{E}$. However, because we shall rarely use the 'total' Bloch bundle and almost always use the Bloch bundle of the system, this is unlikely to cause confusion.

4.3.1. The equivariant Bloch bundle

Previously, we saw how the point group of a crystal naturally acts on the Brillouin zone, making it a P-equivariant space. This idea also holds true for the Bloch bundle, but the details are more intricate. Difficulties arise in nonsymmorphic crystals, i.e., crystals for which S is not a semidirect product $P \ltimes \Pi$. As we stated before, we only treat symmorphic crystals, so we may fix an isomorphism $S \cong P \ltimes \Pi$ for this section. Recall that we write elements of $P \ltimes \Pi$ as $\{ R \mid v \}$ with $v \in \Pi$ and $R \in P$. The product is given by

$$\{ R \mid v \} \cdot \{ R' \mid v' \} = \{ RR' \mid v + Rv' \}.$$

Recall that this means we have an embedding of P into S via $R \mapsto \{ R \mid 0 \}$.

The Hilbert space $L^2(\mathbb{E}^d, \mathbb{C})$ naturally has an S-action associated with it, namely precomposition: if $\psi \in L^2(\mathbb{E}^d, \mathbb{C})$ and $s \in S$, define $s \cdot \psi$ by

$$(s \cdot \psi)(x) := \psi(s^{-1} \cdot x),$$

where the inverse appears to make this a left-action instead of a right-action. Written out under the isomorphism $S \cong P \ltimes \Pi$, we have

$$(\{ R \mid v \} \cdot \psi)(x) = \psi(\{ R^{-1} \mid -R^{-1}v \} \cdot x) = \psi(R^{-1}x - R^{-1}v).$$

By imposing the isomorphism in Bloch's theorem to be an *equivariant* isomorphism (i.e., by requiring it to commute with the group action), we get an action of S on $\Gamma(\mathcal{E})$ as well. Explicitly, the result of $\{ R \mid v \}$ acting on $\varphi \in \Gamma(\mathcal{E})$ is

$$\begin{aligned}
(\{ R \mid v \} \cdot \varphi)_{k_0}(x) &= \sum_{w \in \Pi} \exp(-i\langle k_0, w \rangle) \int_{X_\Pi} \varphi_k(R^{-1}(x + w) - R^{-1}v)\, \mathrm{d}k \\
&= \sum_{w \in \Pi} \exp(-i\langle k_0, w \rangle) \int_{X_\Pi} \varphi_k(R^{-1}x + R^{-1}w - R^{-1}v)\, \mathrm{d}k \\
&= \sum_{u \in \Pi} \exp(-i\langle k_0,\, Ru \rangle) \int_{X_\Pi} \varphi_k(R^{-1}x + u - R^{-1}v)\, \mathrm{d}k \\
&= \sum_{u \in \Pi} \exp(-i\langle k_0 \circ R,\, u \rangle) \int_{X_\Pi} \varphi_k(R^{-1}x + u - R^{-1}v)\, \mathrm{d}k \\
&= \varphi_{k_0 \circ R}(R^{-1}x - R^{-1}v).
\end{aligned}$$

Via the embedding $R \mapsto \{ R \mid 0 \}$, the action restricts to the P-action given by

$$(R \cdot \varphi)_k(x) = \varphi_{k \circ R}(R^{-1}x).$$

We can also interpret this action as one on the bundle $\mathcal{E}$ itself, instead of on the sections $\Gamma(\mathcal{E})$. Recall that the action of P on the Brillouin zone is given by $k \mapsto k \circ R^{-1}$ whenever $R \in P$. The P-action on $\Gamma(\mathcal{E})$ can be rewritten as

$$(R \cdot \varphi)_{k \circ R^{-1}} = \varphi_k \circ R^{-1}.$$

Notice that by precomposing with R^{-1}, the wave φ_k of momentum k turns into a wave of momentum $k \circ R^{-1}$. Thus, if we let P act on $\mathcal{E}$ by precomposition, this turns the Hilbert bundle $\mathcal{E}$ into a *P-equivariant Hilbert bundle*. By this we mean that

(i) the projection $\mathcal{E} \to X_\Pi$ is P-equivariant (i.e., commutes with the P-action);

(ii) the restriction $\mathcal{E}_k \to \mathcal{E}_{k\circ R^{-1}}$ is a linear isometry for all $R \in P$.

One quickly recognises this as the proper generalisation of an equivariant vector bundle — cf. Definition 2.1.

Remark 4.15. When S is not a semidirect product $P \ltimes \Pi$, the trouble lies in that P is no longer a subgroup of S. One may still regard P as a subset of S, but there will always be elements in P that multiply to a nontrivial translation from Π. Hence the S-action does not restrict nicely to a P-action. Instead the bundle gets a P-action that is twisted by a 2-cocycle, in a way similar to group actions being twisted by a quantum anomaly (see Definitions 3.9 and 3.10). This cocycle however has values that depend on the point $k \in X_\Pi$. For a motivation and precise treatment, see Stehouwer [39, §2.4, Ch. 3].

4.4. TIME-REVERSAL

So far we have only treated the spatial symmetries of crystals. When we consider a crystal as a subset of spacetime, it becomes apparent that a crystal can have additional symmetries. We do not consider relativistic effects, so we will model[2] spacetime by $\mathbb{E}^d \times \mathbb{E}^1$. Whereas a crystal in space consists of points, a crystal in spacetime consists of worldlines. Taking a slice of fixed time yields a crystal in the ordinary sense. The set of all spacetime symmetries of a crystal C, denoted by $\hat{S}(C)$, is called the **magnetic space group**.

A crystal should not change overtime, so all time-slices should be required to have the same space group. We impose this by requiring that a spacetime crystal should be invariant under a line of time translations. This line of time translations need not be the 'usual' one, i.e., time translations of the form $(0, \dots, 0, \Delta t)$. Formally, a line of time translations is the image of an injective group homomorphism

$$\mathbb{R} \longrightarrow \mathbb{R}^d \times \mathbb{R}\colon \quad t \longmapsto (f_1(t), \dots, f_d(t), t)$$

where $f_1, \dots, f_d$ are some homomorphisms. Importantly, every desired time-shift Δt can be accomplished, though it may require shifting in the spatial direction as well. Write U for this line of time-translations. Just as the lattice is a normal subgroup of the space group, so too U is a normal subgroup of $\hat{S}$. The quotient $\hat{S}/U$ corresponds to fixing a time-slice of the crystal. But it is not always equal to S, because $\hat{S}$ may contain time-reversing symmetries. If we quotient $\hat{S}/U$ out by the lattice Π of the crystal, we obtain a group called the **magnetic point group** of the crystal, which is labelled $\hat{P}$. Like the point group P, it is always a finite group. Its structure is turns out to be very simple: it is (isomorphic to) $P \oplus \mathbb{Z}_2$ when the crystal has time-reversal symmetry, and otherwise it is simply P. Essentially this is because we do not consider any relativistic effects; see Freed and Moore [11, §2.4] for the details.

[2] We do not use $\mathbb{E}^{d+1}$ here because the symmetries of spacetime are not allowed to mix space and time — after all, we do not work relativistically.

It is usually easier to forget the construction of the magnetic point group and simply use that it is either P or $P \oplus \mathbb{Z}_2$. That way one does not have to think about embedding the crystal in spacetime, but has only to decide whether time-reversal symmetry should be present. This is the route we will take.

Time-reversal always acts in the same way on the Brillouin zone. For it flips the direction of momenta, so on k-space it is given by $k \mapsto -k$. We previously discussed how this induces an involution on the Brillouin zone which we denoted by σ. Hence time-reversal acts via the involution σ on the Brillouin zone. On the Bloch bundle however its action is not uniquely determined: for instance, it might square to $+1$ or -1 on the fibres (see the discussion in Section 3.3).

4.5. PARTICLE-HOLE REVERSAL

The last crystal symmetry that we need to describe is hiding in the Hilbert space of an insulating crystal. Associated to each Bloch wave is an energy. Plotting the energies of all of the possible Bloch waves as a function of the momentum on the Brillouin zone gives the familiar picture of a *band structure*. Recall that a *band gap* is a range (i.e., interval) of energy values in which there are no bands. An insulator has an energy level called the *Fermi level* and a band gap around it, such that the states below the Fermi level are filled, and those above it are empty. The filled bands are called *valence bands*, and the empty ones are called *conduction bands*. If $\mathcal{E}$ the Bloch bundle for an insulator, this gives a direct sum decomposition of Hilbert bundles

$$\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+,$$

where $\mathcal{E}^-$ is the subbundle consisting of the valence bands, and $\mathcal{E}^+$ of the conduction bands. The band gap around the Fermi level is crucial here: this energy separation between the valence and conduction bands ensures that $\mathcal{E}^-$ and $\mathcal{E}^+$ form subbundles. The bundle $\mathcal{E}^-$ will always be assumed to have *finite rank*, or in other words, that it forms an actual vector bundle. This is justified by the physical requirement that there must be a ground state, meaning there can only be finitely many valence bands.

It is possible that an insulator is symmetric under the reversal of particles and holes. We briefly comment upon some motivation, for insulators that possess particle-hole symmetry are somewhat odd. The relativistic free particle described by the Dirac equation has this symmetry. Indeed, the Dirac equation is famed for having positive and negative energy solutions (see, e.g., Thomson [41, Ch. 4]). To remedy this, Dirac invented the *Dirac sea*: the filling up of all negative energy states. The creation of a particle is then the same as a particle moving from a negative-energy state to a positive-energy state. The resulting situation has particle-hole symmetry. We work non-relativistically, which we may view as a first-order approximation to the relativistic case. If our states are sufficiently close to the Fermi level, then this Fermi level behaves similarly to the zero-energy level of the Dirac sea. In principle it is possible, therefore, that a more complicated system should inherit this symmetry from the free situation. In fact, all symmetries of Altland–Zirnbauer classes have such a motivation: the free particle situation has these symmetries.

Henceforth we will denote particle-hole reversal by C, as agreed upon in Section 3.3. Recall also from Section 3.3 that this is an antiunitary operator that anticommutes with the Hamiltonian. We can now see an additional requirement on C: when acting on the Bloch bundle, it must reverse the valence and conduction bands in the sense that

$$C(\mathcal{E}^-) \subseteq \mathcal{E}^+ \quad \text{and} \quad C(\mathcal{E}^+) \subseteq \mathcal{E}^-.$$

By contrast, any map that does not reverse particles and holes must preserve this decomposition: if R preserves particles and holes, we instead have

$$R(\mathcal{E}^-) \subseteq \mathcal{E}^- \quad \text{and} \quad R(\mathcal{E}^+) \subseteq \mathcal{E}^+.$$

Since a symmetry either does or does not reverse particles and holes (mixing them is impossible), the decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ is an intrinsic property of the bundle $\mathcal{E}$. We shall say a symmetry **reverses** the decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ if the first equation above holds, and that it **preserves** the decomposition if the second one holds.

Remark 4.16. Since $\mathcal{E}^-$ always has finite rank by assumption, if there is a particle-hole reversing symmetry present, then $\mathcal{E}^+$ has finite rank as well. In that case the total Bloch bundle $\mathcal{E}$ is itself of finite rank, and hence forms a vector bundle.

Particle-hole reversal also flips the direction of momenta, since holes have opposite momentum compared to particles. Hence, similarly to time-reversal, it also acts as σ on the Brillouin zone. But, also like time-reversal, its action on the Bloch bundle is not uniquely determined.

4.6. Formalisation

Now that we have studied insulators, we are ready to give a mathematical formalisation. This formalisation should describe not just the insulator, but also the symmetries that the insulator has. Because we want to implement time-reversal and particle-hole symmetries, the natural candidate to describe these symmetries would be an extended quantum symmetry group (G, φ, τ, c) (see Definition 3.14). An insulator would then be described by an extended quantum system (see Definition 3.16). However, these concepts are too general. We are only interested in symmetry groups that describe crystal symmetry, and quantum systems which describe insulators. Chapter 3 on the other hand is only concerned with general quantum-mechanical symmetries. We shall therefore need to modify these definitions to describe the appropriate physical concepts. There are some rather technical requirements that we shall need later, but these we do not go into; see Freed and Moore [11, Def. 10.7, Hyp. 10.9] for more information.

4.6.1. Crystal symmetries

We shall write G for the group of all symmetries of a crystal. It should consist of the space group S of the crystal it describes, along with a possible combination of time-reversal and particle-hole reversal. But, importantly, it should not contain any symmetries besides these. We may phrase this requirement as follows. Recall from

Section 3.2.3 that $\theta := \varphi \cdot c$ tells if a symmetry reverses time or preserves it. Space group symmetries do not flip time or particles and holes, so θ and c restrict trivially to S. We impose that if a symmetry $g \in G$ satisfies $\theta(g) = c(g) = +1$, that it then must lie in S. Put mathematically, we require that the kernel of

$$\theta \times c \colon G \longrightarrow \{ \pm 1 \} \times \{ \pm 1 \}$$

is equal to S. The quotient G/S is then a finite subgroup of $\{ \pm 1 \} \times \{ \pm 1 \}$ that depends on what Altland–Zirnbauer class (see Section 3.3) the group belongs to.

But these requirements are still not enough. For just like the space group before, we would like to be able to work with the quotient G/Π instead of G (where $\Pi \subseteq S$ is the lattice). Thus Π should be a normal subgroup of all of G. Note that because S/Π and G/S are finite, the third isomorphism theorem automatically implies that G/Π is finite also. Both θ and c automatically descend to G/Π because they are trivial on S (and hence on Π too), but the anomaly τ need not. We therefore require that it does, meaning that τ comes from an anomaly τ' on G/Π:

$$\tau(g,h) = \tau'(\pi(g),\, \pi(h)) \quad \text{for all } g,h \in G,$$

where $\pi \colon G \to G/\Pi$ is the projection. To ease discussion, when a symmetry group is of this form, we shall say it is of *crystal type*.

Definition 4.17. Let C be a crystal and let S be the space group of this crystal. An extended quantum symmetry group (G, φ, τ, c) is said to be of **crystal type S** if the following conditions hold.

(i) The kernel of $\theta \times c \colon G \to \{ \pm 1 \} \times \{ \pm 1 \}$ is the space group S, where $\theta := \varphi \cdot c$.

(ii) The lattice $\Pi \subseteq S$ is a normal subgroup of G.

(iii) The anomaly τ comes from an anomaly τ' on G/Π in the sense that

$$\tau(g,h) = \tau'(\pi(g),\, \pi(h)) \quad \text{for all } g,h \in G,$$

where $\pi \colon G \to G/\Pi$ is the projection.

A group (G, φ, τ, c) is said to be of **crystal type** when it has the crystal type of a crystal.

Remark 4.18. Freed and Moore use a double prime to denote a quotient by the lattice: $G'' = G/\Pi$. Restricting to the kernel of $\theta \times c$ is denoted by a zero subscript: $G_0 = S$. In this notation, the point group P is G_0''.

Notice that all relevant information of the crystal can be recovered from its symmetry group. Most importantly, its Brillouin zone X_Π is defined only with reference to its lattice, which is contained in G. In this manner we simultaneously treat all crystals with space group S by abstracting to a symmetry group of crystal type S.

4.6.2. Band insulators

We write $\mathcal{H}$ for the Hilbert space of an insulator. As explained in Section 4.3, the space $\mathcal{H}$ should be isomorphic to the sections of the Bloch bundle $\mathcal{E}$. The Bloch bundle has a

decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ into valence and conduction bands, where $\mathcal{E}^-$ has finite rank. Notice that the decomposition of the Bloch bundle induces a decomposition $\Gamma(\mathcal{E}) = \Gamma(\mathcal{E}^-) \oplus \Gamma(\mathcal{E}^+)$. The isomorphism $\mathcal{H} \cong \Gamma(\mathcal{E})$ then yields a decomposition of the Hilbert space, $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$. Whether a symmetry reverses or preserves this decomposition is recorded by the homomorphism c. A quantum system of this form is called a *band insulator* by Freed and Moore [11, Def. 10.7].

Definition 4.19. Let (G, φ, τ, c) be an extended quantum symmetry group of crystal type. A **band insulator** is an extended quantum system $(\mathcal{H}, H, \rho)$ of this symmetry type along with a decomposition $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$, satisfying the following conditions.

(i) The Bloch sum gives an equivariant isomorphism between $\mathcal{H}$ and the sections of an equivariant Hilbert bundle $\mathcal{E}$ over X_Π.

(ii) Under the isomorphism $\mathcal{H} \cong \Gamma(\mathcal{E})$, the decomposition of $\mathcal{H}$ corresponds to a decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ of Hilbert bundles, where $\mathcal{E}^-$ has finite rank.

(iii) For all $g \in G$, the map $\rho(g)$ preserves the decomposition on $\mathcal{H}$ when $c(g) = +1$, and reverses it when $c(g) = -1$.

A (φ, τ)-twisted representation ρ satisfying Condition (iii) is called a **(φ, τ, c)-twisted representation** of G on the decomposed Hilbert space $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$.

As we pointed out in Remark 4.16, a system with particle-hole symmetry must have a finite-dimensional bundle of conduction bands. To ease the discussion, Freed and Moore call a band insulator of **type F** if the bundle of conduction bands is finite-dimensional, and of **type I** when it is infinite-dimensional. Thus any band insulator with a particle-hole reversing symmetry is of type F. Conversely, if an extended quantum symmetry group describes the symmetries of a type I insulator, then its homomorphism c must be the trivial map.

There is some ambiguity in the above definition. Namely, we have not specified precisely how the Bloch bundle $\mathcal{E}$ forms an equivariant Hilbert bundle. Of course, we have seen how it forms an S-equivariant bundle, and even a P-equivariant one when the crystal is symmorphic. But we have not commented on how to incorporate time-reversal or particle-hole reversal. An equivariant vector bundle is not the right concept, for there the group action is required to be complex-linear, whereas these symmetries are antilinear. Including these symmetries leads to a new type of vector bundle, and these bundles in turn make up a new type of K-theory: *twisted K-theory*.

5 TWISTED K-THEORIES

In Chapter 2 we met the G-equivariant K-group K_G, which classifies vector bundles with a group action. If we modify this group action to be the action of a quantum symmetry group (G, φ, τ), we get the *twisted G-equivariant K-group*, ${}^{\varphi}K_G^{\tau}$. Bundles in this group are the natural generalisation of the twisted representations from Definition 3.10. Physically speaking, the valence bands of an insulator naturally form a vector bundle of this kind. We can also generalise the K-group to an extended quantum symmetry group (G, φ, τ, c), yielding the *extended twisted G-equivariant K-group*, ${}^{\varphi}K_G^{\tau,c}$. The Bloch bundle $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ is naturally a bundle of this kind. These theories are treated in Sections 5.1 and 5.3, respectively. However, some subtleties in extended twisted K-theory require further algebraic preliminaries — these are treated in Section 5.2.

As our ultimate aim will be to compute some of these groups, we require some more basic tools. It turns out that the twisted K-theories of a point may be computed in an elegant algebraic way that generalises the representation ring defined in Example 1.3. This is described in Section 5.4.

Although these theories were originally defined by Freed and Moore [11, §7.4], their formulation is quite different from the way we have formulated K-theory thus far. Stehouwer [39, Ch. 3] gives a presentation that is much closer to ours. Our definitions in Sections 5.1 and 5.3 mostly agree with his, but have been simplified so as to only treat symmorphic crystals. This is because the theory to calculate the K-groups of nonsymmorphic crystals is not as far developed as the one for symmorphic crystals. When we deviate from either Freed and Moore or from Stehouwer, this is indicated.

5.1. TWISTED K-THEORY

In this section we treat quantum symmetry groups (G, φ, τ) with G *finite*. We shall henceforth phrase this condition by saying that (G, φ, τ) is finite. Like with all K-theories before, to define twisted K-theory we need to specify the category of spaces that it is defined on. But φ and τ have no effect on the action of G on the base space, so we can take the category of G-equivariant compact Hausdorff spaces described in Section 2.1.1. All of the modifications happen in the vector bundles.

Definition 5.1. Let (G, φ, τ) be a finite quantum symmetry group, and let X be a G-equivariant space. A **(φ, τ)-twisted G-equivariant vector bundle** over X is a complex vector bundle E over X, along with a collection of maps $\rho_g \colon E \to E$ for $g \in G$, satisfying the following conditions.

(i) The map ρ_1 is the identity map on E.

(ii) For all $g \in G$, we have $\pi(\rho_g(x)) = g \cdot \pi(x)$ for all $x \in E$.

(iii) For all $g \in G$, the map ρ_g is fibre-wise linear when $\varphi(g) = +1$, and fibre-wise antilinear when $\varphi(g) = -1$.

(iv) For all $g, h \in G$, we have $\rho_g \circ \rho_h = \tau(g,h) \cdot \rho_{gh}$.

Often we shall write ρ_g as multiplication by g when there is no confusion possible. Notice that the maps ρ_g do *not* turn E into a G-equivariant space, because $\rho_g \rho_h \neq \rho_{gh}$ in general. We may occasionally refer to this as a *twisted G-action*. When τ is trivial this is a G-action in the ordinary sense.

The motivating example for this definition is the vector bundle of valence bands from the previous chapter. We saw that it forms a P-equivariant bundle, but this did not include time-reversal, or an anomaly τ that might be present. The above definition is the natural notion that also includes possible time-reversal and an anomaly τ.

Recall from equivariant K-theory that, when a point $x \in X$ is fixed by the G-action, its fibre has a G-representation on it. Similarly, when $x \in X$ is fixed by G, then the fibre over x in a twisted bundle will have a (φ, τ)-twisted representation of G on it (see Definition 3.10). Or, more generally, any point $x \in X$ has a twisted representation of the stabiliser G_x on it, restricting φ and τ to $G_x \subseteq G$.

Remark 5.2. The requirement that τ is a unital 2-cocycle is crucial for this definition. The cocycle identity is required to make this definition compatible with associativity, and the unital property is required because we impose $\rho_1 = \mathrm{Id}_E$.

When E and F are two twisted equivariant vector bundles over X, a **homomorphism** from E to F is a vector bundle homomorphism that intertwines the twisted action of G. We define ${}^{\varphi}\mathrm{Vect}^{\tau}_G(X)$ to be the set of isomorphism classes of (φ, τ)-twisted G-equivariant vector bundles over X. The direct sum and pullback can be defined for twisted vector bundles too. In particular, the direct sum makes ${}^{\varphi}\mathrm{Vect}^{\tau}_G(X)$ a commutative semigroup.

Definition 5.3. Let (G, φ, τ) be a finite quantum symmetry group, and let X be a G-equivariant compact Hausdorff space. The **(φ, τ)-twisted G-equivariant K-group** of X, denoted by ${}^{\varphi}K^{\tau}_G(X)$, is the Grothendieck group of the semigroup ${}^{\varphi}\mathrm{Vect}^{\tau}_G(X)$.

Like with equivariant K-theory, this is a contravariant functor. However, there is one important difference: the twisted K-group does not form a ring. For the tensor product of two (φ, τ)-twisted bundles naturally forms a $(\varphi, \tau \cdot \tau)$-twisted bundle (where $\tau \cdot \tau$ denotes point-wise product). If τ is trivial we do get a ring structure on the twisted K-group, but because this does not happen in general we shall not use this structure.

Remark 5.4. The difference between φ and τ in the notation of the twisted K-group reflects that φ does not change under the tensor product, while τ does.

As with all K-theories before it, we have a variant for pointed spaces. When X is a pointed G-equivariant space, we have an equivariant map $\mathrm{pt} \hookrightarrow X$. This induces a

map ${}^{\varphi}K_G^{\tau}(X) \to {}^{\varphi}K_G^{\tau}(\mathrm{pt})$, the kernel of which we define to be ${}^{\varphi}\widetilde{K}_G^{\tau}(X)$. Because the map $\mathrm{pt} \hookrightarrow X$ has a left-inverse, we have a splitting

$$ {}^{\varphi}K_G^{\tau}(X) \cong {}^{\varphi}\widetilde{K}_G^{\tau}(X) \oplus {}^{\varphi}K_G^{\tau}(\mathrm{pt}). $$

Spheres become G-equivariant spaces under a trivial G-action. In this fashion, we can use the same definitions for the graded twisted K-groups as we did for all other K-theories:

$$ {}^{\varphi}\widetilde{K}_G^{\tau-n}(X) := {}^{\varphi}\widetilde{K}_G^{\tau}(\Sigma^n X) \qquad \text{when } X \text{ is pointed;} $$
$$ {}^{\varphi}K_G^{\tau-n}(X) := {}^{\varphi}\widetilde{K}_G^{\tau}(\Sigma^n(X^+)). $$

Instead of calling this theory "twisted equivariant K-theory," to ease discussion we will often simply refer to it as *twisted K-theory*. The axioms of Section 1.4 also apply here, mutatis mutandis: it is an equivariant cohomology theory. This is proved by Stehouwer [39, §3.8], but note that he calls this theory "topologically defined higher twisted K-theory." Twisted K-theory has an *eight*-fold periodicity, but as we will see it can be two-fold periodic in some cases. Like with previous K-theories, this periodicity allows us to extend the degrees to the integers, yielding ${}^{\varphi}K_G^{\tau+n}$ and ${}^{\varphi}\widetilde{K}_G^{\tau+n}$.

5.1.1. Special cases

Twisted K-theory generalises all previously seen K-theories.

Example 5.5 (K-theory). Take G, φ and τ to all be trivial. A (φ,τ)-twisted G-equivariant vector bundle is then simply a vector bundle. Therefore ${}^{\varphi}K_G^{\tau}(X)$ is simply $K(X)$. More generally, if G is any group and we take φ and τ trivial, then ${}^{\varphi}K_G^{\tau}(X)$ becomes $K_G(X)$. In all of these cases, the reduced groups coincide. So do the lower-degree groups, because they are defined via the same suspension.

In light of the above, we often omit trivial maps. For instance, when φ is trivial we will also write K_G^{τ} for ${}^{\varphi}K_G^{\tau}$. Note that this example shows that twisted K-theory can be two-fold periodic. However, the following example shows this is not always the case.

Example 5.6 (KR-theory). Take $G = \mathbb{Z}_2$, take $\varphi\colon G \to \{\pm 1\}$ to be the isomorphism, and take τ trivial. Then a (φ,τ)-twisted bundle is a vector bundle with a map T that is an involution (because τ is trivial) and which is antilinear on the fibres. Thus the twisted K-group ${}^{\varphi}K_G(X)$ is exactly the KR-group $KR(X)$.

More generally, take $G = H \oplus \mathbb{Z}_2$ for another group H, take φ the projection onto the second component, and again take τ trivial. Then ${}^{\varphi}K_G(X)$ is $KR_H(X)$, the H-equivariant KR-group from Section 2.2.3. Indeed, the condition that the H-action commutes with the Real involution is satisfied because G splits as a direct sum of H and $\mathbb{Z}_2$. Again the reduced and the lower-degree groups coincide.

Because (equivariant) KR-theory generalises (equivariant) KO-theory, so does twisted K-theory.

In addition to these examples, one special case is worth noting for later use. Consider $G = \mathbb{Z}_2$, with φ the isomorphism, but with τ such that $\tau(\bar{1}, \bar{1}) = -1$. A twisted bundle is then a bundle with a fibre-wise antilinear map ψ that squares to -1. The K-theory concerned with these types of bundles was already introduced by Dupont [8] and is called **symplectic K-theory**, although it has also become known as **Quaternionic K-theory**. We shall use the latter name and write it as $KQ(X)$. The name is inspired by the fact that a module over the quaternions $\mathbb{H}$ is the same as a complex vector space with an antilinear map that squares to -1. This map is to be thought of as multiplication by $j \in \mathbb{H}$. Indeed, multiplication by j is antilinear because $ij = -ji$, and it squares to -1 because $j^2 = -1$ in $\mathbb{H}$. The action of k is uniquely determined by that of j because $k = ij$. Hence an antilinear map squaring to -1 on a complex vector space is the same as an $\mathbb{H}$-module structure extending the complex vector space structure. As such, KQ-theory behaves similarly to KR-theory, but now with quaternionic fibres on points that are fixed by the involution.

The main result in KQ-theory is the following.

Theorem 5.7 (Dupont). *Let X be a Real compact Hausdorff space. Then for all $n \in \mathbb{N}_0$, we have a group isomorphism*

$$KQ^{-n}(X) \cong KR^{-n-4}(X).$$

In particular, $KQ(X) \cong KR^{-4}(X)$.

Proof. See Dupont [8], but note that he writes Ksp instead of KQ. ∎

Not only does this tell us how to compute the KQ-groups, but it also provides a different way to look at the group KR^{-4}. For example, we previously defined $KR^{-4}(\mathrm{pt})$ to be $\widetilde{KR}(S^4)$, which is hard (if not impossible) to visualise. On the other hand, the above discussion tells us that $KQ(\mathrm{pt})$ consists of quaternion-modules. By an argument analogous to the complex case in $K(\mathrm{pt})$, we can immediately see that $KQ(\mathrm{pt}) \cong \mathbb{Z}$. Of course, we already knew that $KR^{-4}(\mathrm{pt}) \cong \mathbb{Z}$ (see Example 2.11), but the method of calculation was not as insightful.

5.2. SUPERALGEBRA

Throughout this section, let F denote either $\mathbb{R}$ or $\mathbb{C}$ (although what we are about to define generalises to an arbitrary field). Recall that an ***F*-algebra** is a ring A together with a ring homomorphism $i\colon F \hookrightarrow Z(A)$ embedding F into the centre of A. This turns A into an F-vector space by, for $x \in A$ and $\lambda \in F$,

$$\lambda \cdot x := i(\lambda) \cdot x.$$

We shall often forget the embedding i and treat F as a subset of A. In short then, an F-algebra is a ring that is simultaneously an F-vector space. An **algebra homomorphism** is a ring homomorphism that commutes with the multiplication of F.

Inspired by the decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ into two parts, we can modify any algebraic structure by giving it such a decomposition. The resulting modification

will get the prefix 'super-'. This terminology comes from *supersymmetry*; it does not signify that there is anything particularly fascinating about the structure compared to its original counterpart. A basic example of this procedure is the following; much of the terminology associated to it carries over to other structures as well.

Definition 5.8. A **super vector space** is a vector space V together with a decomposition $V = V_0 \oplus V_1$ into subspaces V_0 and V_1. The subspace V_0 is called the **even** part of V, and V_1 the **odd** part. A linear map $L\colon V \to W$ between super vector spaces is called **even** when

$$L(V_0) \subseteq W_0 \quad \text{and} \quad L(V_1) \subseteq L(W_1),$$

and **odd** when

$$L(V_0) \subseteq W_1 \quad \text{and} \quad L(V_1) \subseteq L(W_0).$$

Remark 5.9. What we call 'super-' is also called '$\mathbb{Z}_2$-graded'.

Generalising the above, we also get a notion of a **super vector bundle**: a vector bundle E together with a decomposition $E = E_0 \oplus E_1$ into subbundles. The terminology for even and odd maps is analogous.

Definition 5.10. An ***F*-superalgebra** is an F-algebra A together with a decomposition $A = A_0 \oplus A_1$ into F-subalgebras A_0 and A_1, satisfying

$$A_i \cdot A_j \subseteq A_{i+j} \quad \text{with indices modulo 2.}$$

We write $|A|$ for the algebra underlying a superalgebra A. A **superalgebra homomorphism** between A and B is an algebra homomorphism $A \to B$ that maps A_0 to B_0 and A_1 to B_1.

The conditions on a superalgebra fit nicely with our terminology: it says that the product of two even elements is even, the product of two odd elements is even, and the product of an even and an odd element is odd. Notice that the unit element must always be even. Therefore, since A_0 is closed under scalar multiplication by F, the entire field F is contained in A_0. This motivates the following convention: throughout this work, $\mathbb{R}$ and $\mathbb{C}$ as superalgebras *always* have a **trivial decomposition**, meaning that everything is even. Even more so, $\mathbb{H}$ is also assumed to have trivial decomposition in this work.

We can consider modules over a superalgebra. The natural type of module itself also has a decomposition, in the following way.

Definition 5.11. Let A be a superalgebra. A **supermodule over *A*** is an A-module V, together with a decomposition $V = V_0 \oplus V_1$, such that multiplication by $x \in A$ is an even endomorphism of V when x is even, and an odd endomorphism of V when x is odd.

Notice that if A is an algebra, we can consider it as a superalgebra with trivial decomposition. In that case a supermodule V over the superalgebra A is the same as two modules over the algebra A. Indeed, all of A being even implies that all $x \in A$ restrict to V_0 and V_1 separately, so that V_0 and V_1 are both modules over A.

5.2.1. Tensor products

With rings, or even algebras, we can form the direct sum and tensor product. For superalgebras we can do the same. Let A and B be two F-superalgebras; their **direct sum** is the superalgebra with underlying algebra $A \oplus B$, under the decomposition

$$(A \oplus B)_0 := A_0 \oplus B_0 \quad \text{and} \quad (A \oplus B)_1 := A_1 \oplus B_1.$$

For the tensor product however, we have a question to answer: should elements from A and B commute in $A \otimes_F B$? There are two natural answers, but before we can introduce these, we need some terminology. An element x of a superalgebra A is called **homogeneous** when it lies in either A_0 or A_1, and is nonzero. When $x \in A$ is homogeneous, we write $|x|$ for its **parity**:

$$|x| := \begin{cases} 0 & \text{when } x \in A_0, \\ 1 & \text{when } x \in A_1. \end{cases}$$

This terminology applies to other super-structures in the same way. Notice that the homogeneous elements of a superalgebra span it as an F-vector space.

To define the tensor product of superalgebras, we first need a tensor product on super vector spaces. If V and W are F-super vector spaces, then we give $V \otimes_F W$ the decomposition

$$(V \otimes_F W)_0 := (V_0 \otimes_F W_0) \oplus (V_1 \otimes_F W_1),$$
$$(V \otimes_F W)_1 := (V_0 \otimes_F W_1) \oplus (V_1 \otimes_F W_0).$$

Definition 5.12. Let A and B be two F-superalgebras. The **(ungraded) tensor product** $A \otimes_F B$ of A and B is the F-superalgebra with $A \otimes_F B$ as underlying super vector space, with multiplication determined as follows. When $a, a' \in A$ and $b, b' \in B$ are homogeneous, we require

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'.$$

The **graded tensor product** $A \hat{\otimes}_F B$ of A and B is the superalgebra with the same underlying vector space, but with multiplication determined by, for $a, a' \in A$ and $b, b' \in B$ homogeneous,

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} \cdot (aa' \otimes bb').$$

The multiplication of the ungraded tensor product makes A and B commute with each other. The graded tensor product on the contrary makes homogeneous elements of A and B anticommute when both are odd, and commute otherwise. This commutation relation is also referred to as the **Koszul sign rule**.

Remark 5.13. When either A or B has a trivial decomposition, the graded and ungraded tensor product coincide.

Example 5.14. For illustrative purposes, we break our earlier convention (though we only do so here) and take the following nontrivial decomposition on $\mathbb{C}$ and $\mathbb{H}$:

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R},$$
$$\mathbb{H} = (\mathbb{R} \oplus k\mathbb{R}) \oplus (i\mathbb{R} \oplus j\mathbb{R}).$$

In other words, i and j are odd, while 1 and $k = ij$ are even. This makes $\mathbb{C}$ and $\mathbb{H}$ real superalgebras. We then have a natural isomorphism of real superalgebras

$$\mathbb{H} \cong \mathbb{C} \hat{\otimes}_{\mathbb{R}} \mathbb{C}.$$

Here the i in the second $\mathbb{C}$ is to be thought of as j. Indeed, i and j anticommute in $\mathbb{H}$, and in $\mathbb{C} \hat{\otimes}_{\mathbb{R}} \mathbb{C}$ we have

$$(i \otimes 1) \cdot (1 \otimes i) = i \otimes i \quad \text{while} \quad (1 \otimes i) \cdot (i \otimes 1) = -(i \otimes i).$$

5.2.2. Clifford algebras

Clifford algebras are generalisations of the complex numbers and quaternions. They pop up in various places in mathematics; in our case they will make an appearance in the K-theory of a point. Much can be said and proved about Clifford algebras, but our goal is to give a minimal overview of the results we will need going forward. Curious readers can consult Atiyah, Bott, and Shapiro [5, I] and Karoubi [23, III.3] for further study.

Definition 5.15. Let $p, q \in \mathbb{N}_0$. The **(p,q)-th real Clifford algebra**, denoted by $\mathrm{Cliff}_{p,q}$, is the real algebra generated by $p+q$ symbols $\gamma_1, \ldots, \gamma_{p+q}$ subject to the relations

$$\gamma_i \gamma_j = -\gamma_j \gamma_i \quad \text{when } i \neq j,$$
$$\gamma_i^2 = \begin{cases} -1 & \text{when } i = 1, \ldots, p, \\ +1 & \text{when } i = p+1, \ldots, p+q. \end{cases}$$

It becomes a real superalgebra when we make all the symbols $\gamma_1, \ldots, \gamma_{p+q}$ odd.

Remark 5.16. It is a matter of convention whether the first p elements square to $+1$ or -1. Our convention is most convenient for our intended use of Clifford algebras.

For small p and q we recover some familiar algebras. For instance, $\mathrm{Cliff}_{1,0}$ is the real algebra with one generator that squares to -1, which is $\mathbb{C}$. Analogously, $\mathrm{Cliff}_{2,0}$ is $\mathbb{H}$, where k is identified with $\gamma_1\gamma_2$. However, these identifications do not work as superalgebras, because of our convention that $\mathbb{C}$ and $\mathbb{H}$ have trivial decomposition. The correct statement is that $|\mathrm{Cliff}_{1,0}|$, the underlying real algebra, is isomorphic to $\mathbb{C}$, and analogously that $|\mathrm{Cliff}_{2,0}| \cong \mathbb{H}$. Although it requires more effort, all Clifford algebras can be computed in this fashion. The underlying algebras of some of these are given in Table 5.1. Note that we write $M_k(A)$ for $k \times k$ matrices over A. From this table,

n	$\lvert\mathrm{Cliff}_{n,0}\rvert$	$\lvert\mathrm{Cliff}_{0,n}\rvert$
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

Table 5.1: The first nine real Clifford algebras with the decomposition forgotten.

the others may be determined: the most striking feature of the Clifford algebras is that in a sense they repeat when either index is increased by eight.

Theorem 5.17 (Periodicity of Clifford algebras). *Let $p, q \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}_0$, we have real superalgebra isomorphisms*

$$\begin{aligned} \mathrm{Cliff}_{p+n,q+n} &\cong M_{2^n}(\mathrm{Cliff}_{p,q}), \\ \mathrm{Cliff}_{p+8,q} &\cong M_{16}(\mathrm{Cliff}_{p,q}), \\ \mathrm{Cliff}_{p,q+8} &\cong M_{16}(\mathrm{Cliff}_{p,q}). \end{aligned}$$

Proof. See Atiyah, Bott, and Shapiro [5, I] or Karoubi [23, III.3]. ∎

Remark 5.18. A matrix algebra $M_{2k}(A)$ over a superalgebra A is also a superalgebra, where block-diagonal matrices are even, and block off-diagonal matrices are odd (with blocks of size $k \times k$). This is what we take in the above proposition.

Remark 5.19. The resemblance between the periodicity of Clifford algebras and the periodicity of the double grading in KR-theory is no coincidence, but would take too long to do justice. See Atiyah [2, §4] or Atiyah and Segal [4, §8] for more information.

There is also a notion of a complex Clifford algebra.

Definition 5.20. Let $n \in \mathbb{N}_0$. The ***n*-th complex Clifford algebra**, denoted by $\mathbb{C}\mathrm{liff}_n$, is the complex algebra generated by n symbols $\gamma_1, \ldots, \gamma_n$ subject to the relations

$$\begin{aligned} \gamma_i \gamma_j &= -\gamma_j \gamma_i \quad \text{when } i \neq j, \\ \gamma_i^2 &= +1. \end{aligned}$$

It becomes a complex superalgebra when we make all the symbols $\gamma_1, \ldots, \gamma_n$ odd.

Standing out is the absence of negative squares. This is because we can change the squares at will: if $\gamma^2 = +1$, then $(i\gamma)^2 = -1$, because (by definition) $\mathbb{C}$ lies in the centre of a complex algebra. More generally, if $\gamma^2 = +1$, then $(z\gamma)^2 = z^2$ for any $z \in \mathbb{C}$. As

n	$\lvert\mathbb{C}\mathrm{liff}_n\rvert$
0	$\mathbb{C}$
1	$\mathbb{C} \oplus \mathbb{C}$
2	$M_2(\mathbb{C})$

Table 5.2: The first three complex Clifford algebras with the decomposition forgotten.

such, replacing $+1$ in the above definition with any nonzero complex number yields an isomorphic complex superalgebra.

Some of the algebras underlying the complex Clifford algebras are given in Table 5.2. The complex Clifford algebras also have a periodicity property: for all n, we have

$$\mathbb{C}\mathrm{liff}_{n+2} \cong M_2(\mathbb{C}\mathrm{liff}_n).$$

Here again the matrix algebra has the superalgebra structure described in Remark 5.18.

The complex Clifford algebras naturally arise as complexifications of real Clifford algebras.

Proposition 5.21. *Let $p, q \in \mathbb{N}_0$. We have an isomorphism of complex superalgebras*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cliff}_{p,q} \xrightarrow{\sim} \mathbb{C}\mathrm{liff}_{p+q}$$

determined by $z \otimes \gamma_i \mapsto z \cdot \gamma_i$.

Lastly, the graded tensor product naturally appears in Clifford algebras. For the generators of a Clifford algebra $\mathrm{Cliff}_{p,q}$ are imposed to anticommute, so we get an isomorphism

$$\mathrm{Cliff}_{p,q} \cong \underbrace{(\mathrm{Cliff}_{1,0} \hat{\otimes}_{\mathbb{R}} \cdots \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{1,0})}_{p \text{ times}} \hat{\otimes}_{\mathbb{R}} \underbrace{(\mathrm{Cliff}_{0,1} \hat{\otimes}_{\mathbb{R}} \cdots \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1})}_{q \text{ times}}. \tag{5.2.1}$$

By taking $p = 2$ and $q = 0$ we retrieve Example 5.14. For complex Clifford algebras we have a similar isomorphism:

$$\mathbb{C}\mathrm{liff}_n \cong \underbrace{\mathbb{C}\mathrm{liff}_1 \hat{\otimes}_{\mathbb{C}} \cdots \hat{\otimes}_{\mathbb{C}} \mathbb{C}\mathrm{liff}_1}_{n \text{ times}}.$$

5.3. Extended twisted K-theory

We now enrich the twisted K-theory from Section 5.1 to use an extended quantum symmetry group. Accordingly, we shall call the resulting theory *extended twisted K-theory*. Like with twisted K-theory, the base space can be left unmodified, so we shall also define this K-theory on the space of equivariant compact Hausdorff spaces. The type of vector bundle we consider is to be a generalisation of a (φ, τ, c)-twisted representation from Definition 4.19. Recall that (when phrased in our new terminology) this is a (φ, τ)-twisted representation on a super vector space such that multiplication by $g \in G$ is even when $c(g) = +1$, and odd when $c(g) = -1$.

Definition 5.22. Let (G, φ, τ, c) be a finite extended quantum symmetry group, and let X be a G-equivariant space. A **(φ, τ, c)-twisted G-equivariant super vector bundle** over X is a (φ, τ)-twisted G-equivariant vector bundle E that is simultaneously a super vector bundle $E = E_0 \oplus E_1$, such that ρ_g is an even map when $c(g) = +1$, and an odd map when $c(g) = -1$.

Like how the valence bands naturally form a twisted vector bundle, the Bloch bundle (if it has finite rank) is naturally a twisted super vector bundle. For depending on its value under c, a symmetry will either flip or preserve the decomposition $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ into valence and conduction bands (see Definition 4.19). The group action on a twisted super vector bundle captures precisely this behaviour.

A **homomorphism** of twisted super vector bundles is a super vector bundle homomorphism that intertwines the G-action. Like with twisted vector bundles from Section 5.1, the direct sum and pullback construction can also be done on these twisted super bundles. Here the direct sum is that of super vector bundles, i.e., on $E \oplus F$ we have the decomposition

$$(E \oplus F)_0 := E_0 \oplus F_0 \quad \text{and} \quad (E \oplus F)_1 := E_1 \oplus F_1.$$

By ${}^{\varphi}\mathrm{Vect}_G^{\tau,c}(X)$ we shall denote the set of isomorphism classes of twisted super vector bundles over X under *even* isomorphisms. This set is a commutative semigroup under the direct sum. One might expect the definition of the extended twisted K-group to be the Grothendieck group of ${}^{\varphi}\mathrm{Vect}_G^{\tau,c}(X)$. However, this definition does not yield a proper generalisation of the twisted K-group from Section 5.1. For we would like to retrieve ${}^{\varphi}K_G^{\tau}(X)$ when c is trivial. When c is trivial, a (φ, τ, c)-twisted super bundle E is the same as *two* (φ, τ)-twisted bundles, namely E_0 and E_1, the even and odd parts of E. What we will do instead is interpret the odd part as carrying a formal minus sign, meaning that that one should think of the super vector bundle $E_0 \oplus E_1$ as embodying $E_0 - E_1$. Reversing the even and odd parts should then yield an inverse. In this approach one has to be careful about the meaning of the symbol $\oplus$, for sometimes it refers to the direct sum of two super vector bundles, and other times it gives a decomposition into an even and odd part.

The formal definition involves a quotient of ${}^{\varphi}\mathrm{Vect}_G^{\tau,c}(X)$. A quotient of a commutative semigroup S is defined as follows. When $T \subseteq S$ is a subsemigroup (i.e., a subset closed under the operation), define an equivalence relation on S by

$$s \sim s' \iff \text{there exist } t, t' \in T \text{ such that } s + t = s' + t'.$$

We define $S/T := S/\!\sim$. It becomes a commutative semigroup by requiring the projection $S \twoheadrightarrow S/T$ to be a semigroup homomorphism. More details on this construction may be found in, e.g., Gomi [13, App. C] or Stehouwer [39, Lem B.18].

Definition 5.23. Let (G, φ, τ, c) be a finite extended quantum symmetry group, and let X be a G-equivariant compact Hausdorff space. A twisted super vector bundle E over X is called **super trivial** when it admits an automorphism $\psi \colon E \to E$ satisfying the following conditions.

(i) The map ψ is odd.

(ii) For all $g \in G$, the map ψ commutes with ρ_g when $c(g) = +1$, and anticommutes with it when $c(g) = -1$.

(iii) We have $\psi^2 = +1$.

Write ${}^{\varphi}\mathrm{Triv}_G^{\tau,c}(X)$ for the subsemigroup of ${}^{\varphi}\mathrm{Vect}_G^{\tau,c}(X)$ consisting of (classes of) super trivial bundles. The **(φ, τ, c)-twisted G-equivariant K-group** of X is the quotient

$$ {}^{\varphi}K_G^{\tau,c}(X) := \frac{{}^{\varphi}\mathrm{Vect}_G^{\tau,c}(X)}{{}^{\varphi}\mathrm{Triv}_G^{\tau,c}(X)}. $$

Remark 5.24. Freed and Moore [11, Def. 7.1] originally only required Conditions (i) and (ii). Gomi [13, §3.5] adds Condition (iii) but with $\psi^2 = -1$ instead. Stehouwer [39, §1.7] showed why imposing $\psi^2 = +1$ is necessary to reproduce previous physical results. More precisely, only this reproduces the $d = 0$ column from Table 6.1 in Section 6.3. See also Remark 6.17.

Remark 5.25. Putting c on the right side in the notation ${}^{\varphi}K_G^{\tau,c}$ reflects that c changes when taking a tensor product, just like τ is put on the right (cf. Remark 5.4).

Proposition 5.26. *Let (G, φ, τ, c) be a finite extended quantum symmetry group, and let X be a G-equivariant compact Hausdorff space. Then the extended twisted K-group ${}^{\varphi}K_G^{\tau,c}(X)$ forms an abelian group under the direct sum of twisted super vector bundles.*

Proof. See Freed and Moore [11, Lem. 5.12], which also applies to our case. ∎

Proposition 5.27. *Let (G, φ, τ) be a finite quantum symmetry group, and let X be a G-equivariant compact Hausdorff space. Let $c\colon G \to \{\pm 1\}$ be the trivial homomorphism. Then we have a group isomorphism*

$$ {}^{\varphi}K_G^{\tau,c}(X) \xrightarrow{\sim} {}^{\varphi}K_G^{\tau}(X)\colon \quad [E_0 \oplus E_1] \longmapsto [E_0] - [E_1]. $$

Proof. See Stehouwer [39, Lem. 3.34]. ∎

The reduced group is defined in the exact same manner as with twisted K-theory. The graded groups are defined in the same way as well, although the notation becomes cramped:

$$ {}^{\varphi}\widetilde{K}_G^{(\tau,c)-n}(X) := {}^{\varphi}\widetilde{K}_G^{\tau,c}(\Sigma^n X) \qquad \text{when } X \text{ is pointed;} $$
$$ {}^{\varphi}K_G^{(\tau,c)-n}(X) := {}^{\varphi}\widetilde{K}_G^{\tau,c}(\Sigma^n(X^+)). $$

Because we use the same suspensions to define the grading of extended twisted K-groups, by Proposition 5.27 these groups also reduce to the twisted K-groups when c is trivial. Extended twisted K-theory also satisfies the analogues of the axioms from Section 1.4, which is proved by Gomi [13, Thm. 3.11], although he gives a different construction. Like before, this makes it an equivariant cohomology theory. Extended twisted K-theory is *eight*-fold periodic (also proved by Gomi [13, Cor. 3.9]). Hence we can extend the grading to an arbitrary integer: ${}^{\varphi}K_G^{(\tau,c)+n}$ and ${}^{\varphi}\widetilde{K}_G^{(\tau,c)+n}$ with $n \in \mathbb{Z}$.

5.4. TWISTED REPRESENTATION RINGS

Back in Example 1.3, we defined the representation ring $R(G)$ of a group G. It naturally appeared in G-equivariant K-theory as the G-equivariant K-group of a point: $K_G(\mathrm{pt}) = R(G)$ (see Example 2.2). In equivariant K-theory it is possible to compute all the graded K-groups $K_G^{-n}(\mathrm{pt})$ of a point from this, as we did in Example 2.5. For (extended) twisted K-theory however, we require more tools. Certain generalisations of $R(G)$ naturally appear as the twisted K-theory of a point; we define these and then give tools for computing them. We only provide the minimal theory we need; Stehouwer [39, App. B and §1.7] gives a more detailed, self-contained treatment.

From elementary representation theory, it is known that a complex G-representation is equivalent to a module over the complex group algebra $\mathbb{C}[G]$. For twisted representations we have a similar correspondence, but now with the *twisted group algebra*.

Definition 5.28. Let (G, φ, τ, c) be a finite extended quantum symmetry group. The **(φ, τ)-twisted group algebra** of G, denoted by ${}^{\varphi}\mathbb{C}^{\tau}[G]$, is the real algebra generated by symbols e_g for every $g \in G$ and a formal imaginary unit i, subject to the relations

$$i^2 = -1, \qquad e_g \cdot e_h = \tau(g,h) \cdot e_{gh} \quad \text{and} \quad e_g \cdot i = \varphi(g) \cdot i \cdot e_g.$$

The **(φ, τ, c)-twisted group algebra** of G, denoted by ${}^{\varphi}\mathbb{C}^{\tau,c}[G]$, is the real superalgebra that has ${}^{\varphi}\mathbb{C}^{\tau}[G]$ as its underlying algebra, with the following grading. The element i is even, and an element e_g is even when $c(g) = +1$, and odd when $c(g) = -1$.

When no confusion is possible we will simply write g instead of e_g. Notice that the twisted group algebra is not always a complex algebra: $\mathbb{C}$ need not lie in the centre. It is a complex algebra if and only if φ is trivial. One quickly sees that a module over ${}^{\varphi}\mathbb{C}^{\tau}[G]$ is the same as a (φ, τ)-twisted representation of G. In the same way, a supermodule over ${}^{\varphi}\mathbb{C}^{\tau,c}[G]$ is the same as a (φ, τ, c)-twisted representation of G.

Remark 5.29. Similar to Remark 5.2, the cocycle identity of τ ensures that ${}^{\varphi}\mathbb{C}^{\tau}[G]$ is associative, and τ being unital implies that ${}^{\varphi}\mathbb{C}^{\tau}[G]$ has a unit element.

Instead of defining a more general representation ring for a group directly, we will define it for general superalgebras first. In the end we will take this algebra to be the twisted group algebra. This general representation ring has a definition similar to Definition 5.22. When A is a superalgebra, write $\mathrm{Mod}_s(A)$ for the set of isomorphism classes (under *even* isomorphisms) of left supermodules over A. This set forms a semigroup under the direct sum of supermodules. Recall that when A is a superalgebra with trivial decomposition, a supermodule over A is the same as two modules over the algebra A. Again the odd part is to be viewed as carrying a formal minus sign. Like with extended twisted K-theory, we take a quotient by 'trivial' supermodules to achieve this. This triviality condition is similar to the one in Definition 5.23.

Let A be a real superalgebra. A supermodule over the superalgebra $A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1}$ is the same as a supermodule over A with the choice of an odd automorphism that squares to $+1$, and that (anti)commutes with the even (odd) elements of A. Notice that we have an embedding of algebras $i_A \colon A \hookrightarrow A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1}$ mapping $x \in A$ to $x \otimes 1$.

Through restriction of scalars, this means a supermodule over $A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1}$ is also a module over A, by forgetting the extra automorphism. Hence we have a map

$$i_A^* \colon \mathrm{Mod}_s(A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1}) \longrightarrow \mathrm{Mod}_s(A).$$

The image of this map we shall take as the trivial modules. For now we shall be sloppy with notation and also write $\mathrm{Mod}_s(A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{0,1})$ for the image of the map i_A^*.

Definition 5.30. Let A be a real superalgebra and let $p, q \in \mathbb{N}_0$. The **representation ring in degree (p,q)** of A is the quotient

$$R^{p,q}(A) := \frac{\mathrm{Mod}_s(A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{p,q})}{\mathrm{Mod}_s(A \hat{\otimes}_{\mathbb{R}} \mathrm{Cliff}_{p,q+1})}.$$

We abbreviate $R^{0,0}(A)$ by $R(A)$.

Remark 5.31. Unlike its name would suggest, in general $R^{p,q}(A)$ does not form a ring, but only an abelian group. The name is merely there to emphasise that it generalises the representation ring from Example 1.3. The argument why it is an abelian group is entirely similar to the proof of Proposition 5.26.

If $A \to B$ is a superalgebra homomorphism, via restriction of scalars it induces a group homomorphism $R^{p,q}(B) \to R^{p,q}(A)$. This turns the representation rings into contravariant functors from superalgebras to abelian groups.

Definition 5.32. Let (G, φ, τ, c) be a finite extended quantum symmetry group, and let $p, q \in \mathbb{N}_0$. The **(φ, τ)-twisted representation ring in degree (p,q)** of G is

$${}^{\varphi}R^{\tau+(p,q)}(G) := R^{p,q}({}^{\varphi}\mathbb{C}^{\tau}[G]),$$

where ${}^{\varphi}\mathbb{C}^{\tau}[G]$ is regarded as a superalgebra with trivial decomposition. The **(φ, τ, c)-twisted representation ring in degree (p,q)** of G is

$${}^{\varphi}R^{(\tau,c)+(p,q)}(G) := R^{p,q}({}^{\varphi}\mathbb{C}^{\tau,c}[G]).$$

If we take $p = q = 0$, then the twisted representation ring ${}^{\varphi}R^{\tau,c}(G)$ is naturally the extended twisted K-group of a point: ${}^{\varphi}K_G^{\tau,c}(\mathrm{pt})$. Indeed, a supermodule over ${}^{\varphi}\mathbb{C}^{\tau,c}[G]$ is a (φ, τ, c)-twisted representation of G. A twisted super vector bundle over a point is the same as such a representation. Comparing Definitions 5.23 and 5.30, we see that we take the same quotient in both cases. Thus we conclude

$${}^{\varphi}R^{\tau,c}(G) = {}^{\varphi}K_G^{\tau,c}(\mathrm{pt}).$$

Notice that ${}^{\varphi}R^{(\tau,c)+(p,q)}(G)$ is the same as ${}^{\varphi}R^{\tau+(p,q)}(G)$ when c is trivial. In particular, by taking c trivial and $p = q = 0$ we see that ${}^{\varphi}R^{\tau}(G)$ is equal to ${}^{\varphi}K_G^{\tau}(\mathrm{pt})$. Taking an even more specific case by taking φ and τ trivial also, then ${}^{\varphi}R^{\tau}(G)$ agrees with our old definition of $R(G)$, because ${}^{\varphi}K_G^{\tau}$ reduces to K_G when φ and τ are trivial. As such, we may use the same notation $R(G)$ without ambiguity. In degree zero therefore, the twisted representation ring is the twisted K-group of a point. The reason we introduced the graded representation rings is because these account for the other degrees.

Theorem 5.33. *Let (G, φ, τ, c) be a finite extended quantum symmetry group. For all $p, q \in \mathbb{N}_0$, we have an isomorphism*

$$ {}^{\varphi}R^{(\tau,c)+(p,q)}(G) \cong {}^{\varphi}K_G^{(\tau,c)+(p-q)}(\mathrm{pt}). $$

Proof. See Stehouwer [39, p. 66], bearing in mind Remark 5.35. ∎

In light of this theorem, we will sometimes be sloppy and write R^{p-q} instead of $R^{p,q}$.

Remark 5.34. Different conventions and definitions can reverse the above relation. For instance, should we have chosen the opposite convention for Clifford algebras (see Remark 5.16), then this theorem would relate $R^{p,q}$ to K-theory in degree $q - p$. More fundamentally, if in Definition 5.30 we would divide out by an extra generator that squares to -1 instead of $+1$, the above would relate $R^{p,q}$ to K-theory in degree $q - p$. This is done by Atiyah, Bott, and Shapiro [5] and Gomi [13], for instance.

Remark 5.35. Appendix B in the work by Stehouwer [39] instead derives that $R^{p,q}$ is related to degree $q - p$ in K-theory. This is due to an error in the proof of Proposition B.23 in his work, which does not divide out by the appropriate type of modules. Explicit calculation using the theory of Atiyah, Bott, and Shapiro [5, §5] reproduces our result in the case of KO-theory.

5.4.1. Computation

Computing twisted representation rings is the same as computing the representation rings of the twisted group algebra. In our applications the twisted group algebras are easily determined, so the difficulty lies in computing the representation rings of these algebras. For us, the following two results will suffice.

Proposition 5.36. *Let A be a real superalgebra. Then for all $p, q, r, s \in \mathbb{N}_0$ we have*

$$ R^{p,q}(A \mathbin{\hat{\otimes}_{\mathbb{R}}} \mathrm{Cliff}_{r,s}) \cong R^{p+r,q+s}(A). $$

In particular, if A has trivial decomposition, we have

$$ R^{p,q}(A \otimes_{\mathbb{R}} \mathrm{Cliff}_{r,s}) \cong R^{p+r,q+s}(A). $$

Proof. The first isomorphism relies on the fact that

$$ \mathrm{Cliff}_{p,q} \mathbin{\hat{\otimes}_{\mathbb{R}}} \mathrm{Cliff}_{r,s} \cong \mathrm{Cliff}_{p+r,q+s}, $$

which follows at once from the isomorphism of Equation (5.2.1). The second is a consequence of Remark 5.13. ∎

Remark 5.37. The superalgebra structure of the Clifford algebras is crucial in the above proposition. In particular, a tensor product with $|\mathrm{Cliff}_{p,q}|$ (i.e., a Clifford algebra with *trivial* decomposition) does *not* shift the degree.

Theorem 5.38. *Let A be a real algebra. Suppose that for certain numbers $n_{\mathbb{R}}$, $n_{\mathbb{C}}$ and $n_{\mathbb{H}}$, we have an isomorphism between A and a direct sum of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$:*

$$A \cong \left(\bigoplus_{k=1}^{n_{\mathbb{R}}} M_{r_k}(\mathbb{R})\right) \oplus \left(\bigoplus_{k=1}^{n_{\mathbb{C}}} M_{s_k}(\mathbb{C})\right) \oplus \left(\bigoplus_{k=1}^{n_{\mathbb{H}}} M_{t_k}(\mathbb{H})\right).$$

Then we have an isomorphism

$$R^{p,q}(A) \cong (KO^{p-q}(\mathrm{pt}))^{n_{\mathbb{R}}} \oplus (K^{p-q}(\mathrm{pt}))^{n_{\mathbb{C}}} \oplus (KO^{p-q-4}(\mathrm{pt}))^{n_{\mathbb{H}}}.$$

Proof. See Stehouwer [39, App. B.3]. ∎

Note that the shifting of KO-theory by four is because of Theorem 5.7. To aid understanding we give two examples, the second of which will introduce notation to be used later.

Example 5.39. Let $G = \mathbb{Z}_2$, and let φ and τ be trivial. Writing $G = \{1, R\}$, we see that R and i commute, so the group algebra becomes a quotient of a polynomial ring:

$$\mathbb{C}[G] = \mathbb{R}[i, R] \,/ (i^2 + 1,\ R^2 - 1) \cong \mathbb{C}[R] \,/ (R^2 - 1) \cong \mathbb{C} \oplus \mathbb{C},$$

as follows by the Chinese remainder theorem. Therefore

$$R^{p,q}(G) \cong (K^{p-q}(\mathrm{pt}))^2.$$

Example 5.40. Again take $G = \mathbb{Z}_2$ with trivial τ, but this time with nontrivial φ. If we write $G = \{1, T\}$, this means T and i anticommute. The twisted group algebra ${}^{\varphi}\mathbb{C}[G]$ therefore cannot be a quotient of a polynomial ring: a polynomial ring is always commutative. For simplicity however we would like to use a similar notation. By the expression

$$\mathbb{R}[i, T] \,/ (i^2 = -1,\ T^2 = +1,\ Ti = -iT)$$

we shall denote the real algebra generated by symbols i and T subject to the relations $i^2 = -1$, $T^2 = +1$, and $Ti = -iT$. This we recognise as $|\mathrm{Cliff}_{1,1}|$, identifying i with γ_1 and T with γ_2. By Theorem 5.17 this is $M_2(\mathbb{R})$, so

$${}^{\varphi}R^{p,q}(G) \cong KO^{p-q}(\mathrm{pt}).$$

6 TOPOLOGICAL PHASES

Topological phases have proved to be a very interesting area of research. They were first encountered in the *integer quantum Hall effect*, the effect very briefly sketched in the introduction to this thesis. It turns out that a similar effect can arise without a magnetic field: the *quantum spin Hall effect*. In fact, this effect can only arise in the absence of a magnetic field. This made it the first example of a *symmetry-protected* topological phase. We describe these effects in Sections 6.1 and 6.2, respectively. More details may be found in, e.g., Hasan and Kane [15] and Moore [33].

As topological phases are still relatively new and are very hard to create in general, we cannot yet be fully sure about their possible applications. Currently, among other things, it is expected that they can be used to create *Majorana particles*. This would be a first step toward creating a *topological quantum computer*, which is particularly well protected against errors. See Moore [33] for some further information. But instead of going into the applications of these effects, we use them as an introduction to the general concept of a topological phase. This prepares us to then in Sections 6.3 and 6.4 treat this concept mathematically, which will be the basis for Part III.

Throughout this section, and indeed this whole work, we only treat non-interacting electrons in the non-relativistic case. Interacting theories would require a very different approach — see for instance Kruthoff [27].

6.1. THE INTEGER QUANTUM HALL EFFECT

Suppose that we set up a plane that electrons are restricted to move through, with a constant current $\mathbf{I}$ flowing in a fixed direction. We may choose this plane to be the xy-plane, with the current $\mathbf{I}$ flowing in the x-direction. Now suppose we turn on a homogeneous magnetic field $\mathbf{B}$ perpendicular through this plane (i.e., in the z-direction). Because of the Lorentz force $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$ on the electrons, a voltage will appear in the y-direction. This is called the *Hall effect*. The setup is depicted in Figure 6.1.

We can express this behaviour in a different way. The current $\mathbf{I}$ causes a nonzero current density $\mathbf{J}$ to appear, which everywhere points in the y-direction with constant magnitude. This current also gives rise to an electric field $\mathbf{E}$. Recall that Ohm's law may be stated as $\mathbf{J} = \sigma \cdot \mathbf{E}$, where σ is the conductivity (see, e.g., Griffiths [14, §7.1]). This law still holds in our setup, but the magnetic field requires σ to be a matrix instead of a scalar:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix}.$$

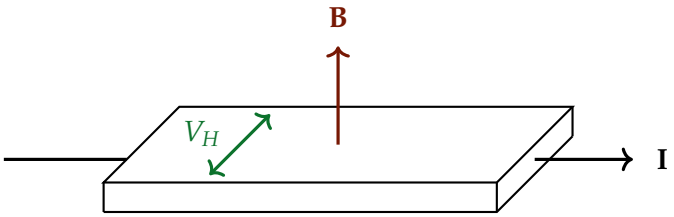


Figure 6.1: The setup of the quantum Hall effect. A current **I** runs through the plate in the x-direction, and a magnetic field **B** runs through it in the z-direction. This causes a voltage, here denoted by V_H, to appear over the y-direction.

The magnetic field causes the current to bend in the x-direction; the conductivity σ_{xy} measures this effect. With this in mind, one can see that the relations $\sigma_{yx} = -\sigma_{xy}$ and $\sigma_{yy} = \sigma_{xx}$ follow from rotational invariance.

But this classical effect is not what we are after. More interesting things occur when we lower the temperature to near absolute zero, and greatly increase the strength of the magnetic field. In 1980, von Klitzing [25] discovered that this causes the conductivity σ_{xy} to be quantised as

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \cdot \nu \quad \text{where } \nu \in \mathbb{Z}. \tag{6.1.1}$$

Here ν depends on the magnitude of the magnetic field **B**. This cannot be explained classically; it is a quantum-mechanical phenomenon. Accordingly it became known as the **integer quantum Hall effect** (henceforth IQHE). Yet it is not a typical quantisation that one might normally encounter in quantum mechanics: the conductivity σ_{xy} is a *global* property of the material. This is in stark contrast to quantities like the energy of a single particle in a harmonic oscillator, which are very microscopic. For its discovery, von Klitzing was awarded the 1985 Nobel Prize in Physics.

For a long time the IQHE was used to very precisely measure the fine-structure constant $\alpha = e^2/4\pi\varepsilon_0\hbar c$. However, its curiosities go further than mere experimental use: it can be elegantly described by *topology*, or more specifically, by using *vector bundles*. This made the IQHE the first example of a so-called *topological phase* of matter, starting a whole new area of physics, both experimental and theoretical. It is the theoretical sides that we shall focus on. We do simplify for the sake of brevity; most importantly, we do not discuss the role of disorder, which does play an important role. Our presentation is heavily based on the lecture notes by Tong [43], with an emphasis on the topological aspects. A much more detailed exposition can be found in these notes if so desired.

6.1.1. Informal approach

Before we go into the mathematics of the IQHE, there is a picture that gives some intuition about the effect. The magnetic field makes the electrons spin in circles, say counterclockwise. The electrons however cannot move past the edge, which results in them bouncing back into the material. The result is that the electrons move in one direction along the edge, giving a one-way conducting edge — see Figure 6.2.

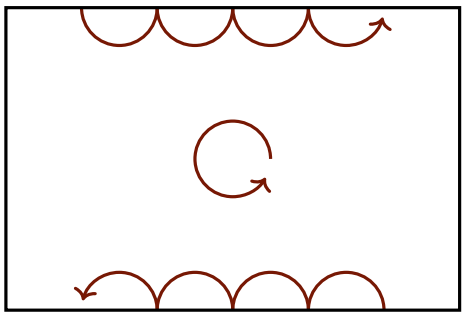


Figure 6.2: An informal picture of the conducting edge of the IQHE. Viewing the plate from above, the magnetic field causes the electrons to move in circular orbits as depicted. On the edge these orbits cannot be 'completed', so the electrons skip to the next circular orbits.

6.1.2. The Berry connection

Let us suppose that we perturb a Hamiltonian (and its eigenstates) slowly overtime, eventually returning to our original Hamiltonian after some time T. The word 'slowly' here means that an energy eigenstate of the Hamiltonian at time zero will retain its energy eigenvalue overtime. The state itself however can acquire a nontrivial phase. Some of this phase is due to time-evolution, but crucially this is not all of the phase that a state acquires as we perturb the Hamiltonian. In addition to this phase from time-evolution, a *Berry phase* is given to the state. This Berry phase can be computed as

$$\exp\left(-\int_0^T \langle\psi(t)|\dot{\psi}(t)\rangle \,\mathrm{d}t\right).$$

Instead of viewing the Hamiltonian as being perturbed overtime, we can think of it as taking a path through some parameter-space. Since we assume we eventually end up at our original Hamiltonian again, this forms a closed path in this parameter-space. The coordinates of this parameter space we shall write as λ_a. Then

$$|\dot{\psi}(t)\rangle = \sum_a \partial_a|\psi(\lambda)\rangle \cdot \dot{\lambda}_a$$

by the chain rule (with ∂_a denoting the partial derivative with respect to λ_a). If we define a vector $\mathcal{A}$ with components $\mathcal{A}_a := i\langle\psi|\partial_a|\psi\rangle$, then the Berry phase becomes

$$\exp\left(i\int_0^T \sum_a \mathcal{A}_a\dot{\lambda}_a \,\mathrm{d}t\right) = \exp\left(i\oint_C \mathcal{A}\right),$$

where C is the (closed) path taken through parameter-space. The quantity $\mathcal{A}$ is called the **Berry connection**. It is not our intention to develop the mathematics of connections, or more precisely of *gauge theory*. Much more in-depth material can be found in Nakahara [35], but we briefly point out only what we need. It turns that the Berry connection $\mathcal{A}$ behaves rather similarly to the vector potential A^μ in electromagnetism.

More specifically, in electromagnetism on Minkowski spacetime, one defines the field strength $F^{\mu\nu} := \partial^\nu A^\mu - \partial^\mu A^\nu$. By analogy we can also define

$$\mathcal{F}_{ab}(\lambda) := \partial_b \mathcal{A}_a - \partial_a \mathcal{A}_b,$$

which is now called the **curvature** of $\mathcal{A}$. This can be used to give an alternate computation for the Berry phase,

$$\exp\left(i \oint_C \mathcal{A}\right) = \exp\left(i \int_S \mathcal{F}\right)$$

with S a surface enclosed by the closed path C.

The curvature $\mathcal{F}$ has one special property. When S is any closed surface, the quantity

$$C := \frac{1}{2\pi} \int_S \mathcal{F}$$

is called a **Chern number**, and is always an *integer*. We shall not go into why this is, as it would require the detailed treatment of connections we intend to avoid.

6.1.3. The TKNN invariant

The integer appearing in Equation (6.1.1) is best understood in terms of vector bundles. Let us assume that the plane that the electrons move through in the IQHE is an insulating crystal. This means we can apply the theory from Chapter 4 to describe the system. The momentum of the electrons can be viewed as lying in the Brillouin zone. Because the material is two-dimensional, this is a 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, as we showed in Proposition 4.8. The valence bands form a vector bundle $\mathcal{E}^-$ over the Brillouin zone consisting of Bloch waves (see Definitions 4.9 and 4.13). We treat non-interacting systems only, so the states in this vector bundle are filled up according to the Pauli exclusion principle. As such it makes sense to describe the system by this vector bundle.

Notice that a valence band is a one-dimensional subbundle of $\mathcal{E}^-$. A band can be parametrised by a function $\mathbb{T}^2 \to \mathcal{E}^-$, assigning to each momentum $k \in \mathbb{T}^2$ a wave function ψ_k. Using this we define a Berry connection on this band (as a function of $k \in \mathbb{T}^2$):

$$\mathcal{A}_a(k) := i\langle \psi_k | \partial_a | \psi_k \rangle,$$

with corresponding curvature

$$\mathcal{F}_{ab} = \partial_b \mathcal{A}_a - \partial_a \mathcal{A}_b,$$

with ∂_a being the partial derivative with respect to k_a. From the Berry connection we can derive whether this band is a trivial vector bundle or not. For associated to it is the Chern number

$$C = \frac{1}{2\pi} \int_{\mathbb{T}^2} \mathcal{F}$$

by integrating over the Brillouin zone. If the bundle is trivial, this number turns out to be zero; equivalently, a bundle with nonzero Chern number is nontrivial. What is

more, if two bundles of equal rank have a different Chern number, then they are not isomorphic.[1] The Chern number of vector bundles is additive under direct sum, so the Chern number of $\mathcal{E}^-$ is simply the sum over all the Chern numbers of the valence bands within it.

Theorem 6.1. *The conductivity σ_{xy} in the IQHE can be expressed as*

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \sum_{\alpha} C_\alpha,$$

*with α running over all valence bands, and with C_α denoting the Chern number belonging to band α. The total Chern number $\sum_\alpha C_\alpha$ is called the **TKNN invariant** of the system.*

Proof. See Thouless, Kohmoto, Nightingale, and den Nijs [42]. ∎

Note that the sum in the TKNN invariant is finite because there are only finitely many valence bands. Thus, the conductivity σ_{xy} tells us whether the valence bands form a nontrivial vector bundle over the Brillouin zone. This is very surprising: while the Chern number is an exact integer quantity, computed from the abstract notion of a vector bundle, the conductivity σ_{xy} is a real-world quantity, dependent upon many intricate details in the system.

If we slowly perturb the system without closing the gap, this does not change the TKNN invariant. This is not overly surprising: it is an integer quantity, so changing it would require a discontinuity to occur somewhere. More formally, the Chern number of two isomorphic vector bundles are the same. Thus a perturbation which does not close the gap and which does not change the vector bundle structure of the valence bands preserves the TKNN invariant. What this means in practise is that a system with nontrivial TKNN invariant is incredibly stable under perturbations and impurities. This rigidity motivates the following terminology.

Definition 6.2. Two insulators are said to be in the same **topological phase** when they can be continuously deformed into one another without closing the gap. An insulator is in a **trivial topological phase** when it is in the same topological phase as the vacuum.

In practise this definition is too strict. For instance, by the above definition an insulator with ten valence bands cannot be in the same topological phase as one with ten million valence bands, even if they both have a trivial vector bundle structure. Often the above is modified by allowing a number of trivial valence bands to be added without changing the topological phase. In that case the TKNN invariant exactly classifies the topological phase an insulator is in: the Chern number of a trivial vector bundle is zero, so the Chern number does not change when we add trivial bands.

6.2. The Quantum Spin Hall Effect

Crucial to the IQHE is the presence of the magnetic field. One might wonder if it is at all possible to get similar effects in the absence of a magnetic field. More generally,

[1] This is not true in general: only in specific cases such as the 2-torus does this hold.

we could ask if there are *time-reversal invariant* systems that exhibit similar topological behaviour. This is a generalisation of our original question because magnetic fields break time-reversal symmetry: the force $\mathbf{F}$ on a particle of charge q in a magnetic field $\mathbf{B}$ depends on the velocity $\mathbf{v}$ of the particle, according to

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Of course, velocities are odd under time-reversal: reversing time flips the sign. Thus magnetic fields break time-reversal symmetry. From this we can quickly see that the conductivity σ_{xy} is also odd under time-reversal. In other words, a time-reversal invariant system must have zero Hall conductivity, and is therefore topologically trivial.

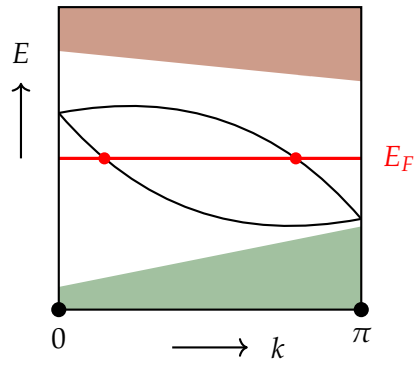
Contrary to what the previous discussion might suggest, it is still possible to create a system with nontrivial topological behaviour that is time-reversal invariant. Key to this idea is to use the *spin* of the electrons. Informally, through spin-orbit coupling one can place two IQH systems in the same place, with spin-up electrons moving one way, and spin-down electrons the other way. The edge would then be conducting, due to the two-way moving electrons. This had been theorised for a while, but it was not clear why such a system would be at all stable like the IQH states. These doubts started to disappear in 2005 when Kane and Mele [21, 22] gave a more realistic model, which was not dependent upon keeping the spin-up and spin-down electrons separate.

In this model the idea of having two IQH systems still holds true to some extent. If we label the Chern numbers of these two systems by n_1 and n_2, then the Chern number of the total state is $n_1 + n_2$. We must have $n_1 + n_2 = 0$ because the system is time-reversal invariant: the conductivity $\sigma_{xy} = (n_1 + n_2) \cdot e^2/2\pi\hbar$ must be zero. But the *difference* $n_1 - n_2$ can be nonzero. This turns out to always be even, so it is more interesting to consider $\frac{1}{2}(n_1 - n_2)$. In and of itself it is not invariant under deformations, but its *parity* is: the residue class

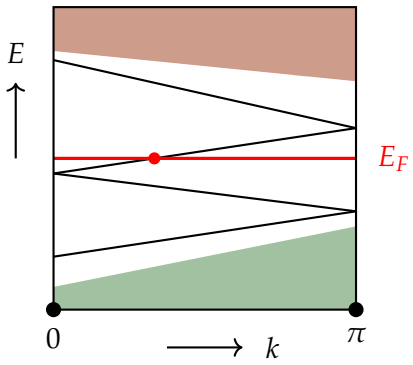
$$\tfrac{1}{2}(n_1 - n_2) \bmod 2$$

is invariant under deformations that do not close the gap and which do not reverse time. It is called the **Kane–Mele invariant**. As such, a system is either in a trivial or in a unique nontrivial phase. This effect became known as the quantum spin Hall effect (henceforth QSHE). The unique nontrivial phase is now called a **topological insulator**. Such a state was first observed experimentally in 2007 by König et al. [26].

Take particular note of the fact that this nontrivial insulator is trivial in the IQHE, because its Chern number (and hence its conductivity) is zero. This means that we can deform this state, without closing the gap, to a trivial insulator. As such there does not seem to be any topological behaviour at all, seemingly contradicting our previous comments. What this shows is that it is crucial that we consider only deformations that preserve the time direction: only when considering these deformations is there a distinction between the topological insulator and a normal insulator. Because of this, the topological insulator is called a **symmetry-protected topological phase**: a topological phase which is only definable when restricting to gap-preserving deformations that also respect certain symmetries. In this case, we say that the QSHE is protected by time-reversal symmetry.



(a) An even number of crossings of the Fermi level.



(b) An odd number of crossings of the Fermi level.

Figure 6.3: The difference between a usual and a topological insulator, simplified to a one-dimensional picture. Pictured is the band structure of an insulator, with valence bands at the bottom (shaded green) and conduction bands at the top (shaded red). At time-reversal invariant points (here the points $k = 0$ and $k = \pi$), we have a Kramers degeneracy according to Proposition 6.3. At other points this degeneracy is lifted. This is illustrated by the black lines coming together at the time-reversal invariant points, and splitting apart at the others. Either these lines cross the Fermi level E_F (the light-red horizontal line) an even number of times, or an odd number of times. These situations are depicted in Figures 6.3a and 6.3b, respectively.

6.2.1. Kramers pairs

Let us describe why the parity of $\frac{1}{2}(n_1 - n_2)$ is topologically invariant.

Proposition 6.3 (Kramers). *Let V be a complex vector space, and let $T\colon V \to V$ be an antilinear map such that $T^2 = -1$. If $v \in V$ is a nonzero eigenvector of T, then v and Tv are linearly independent.*

Proof. It suffices to prove that Tv is not a scalar multiple of v. Suppose that it is, i.e., suppose that $Tv = z \cdot v$ for some $z \in \mathbb{C}$. Then

$$-v = T^2 v = T(z \cdot v) = \bar{z} \cdot Tv = \bar{z}z \cdot v = |z|^2 \cdot v,$$

which means that $|z|^2 = -1$, an impossibility. ∎

Now suppose that k is a time-reversal invariant momentum. Then in the vector bundle of valence bands, if we have an eigenstate $|\psi\rangle$ of T in the fibre over k, then $T|\psi\rangle$ is a different state. Because time-reversal T commutes with the Hamiltonian, $|\psi\rangle$ and $T|\psi\rangle$ have the same energy. In other words, we have a two-fold degeneracy. The states $|\psi\rangle$ and $T|\psi\rangle$ form what is called a **Kramers pair**.

On momenta that are not time-reversal invariant, the degeneracy will be lifted. These states cross the Fermi level $n_1 - n_2$ times in total. Because time-reversal acts on the Brillouin torus $\mathbb{T}^2$ as $(k_x, k_y) \mapsto (-k_x, -k_y)$, we can consider one half of the

Brillouin zone only. In this half, we therefore have $\frac{1}{2}(n_1 - n_2)$ crossings of the Fermi level. Figure 6.3 depicts this situation in a simplified one-dimensional case. Plainly, the number of crossings is either even or odd. By deforming the Hamiltonian in a time-reversal symmetric way, we can add or subtract an even number of crossings, but not an odd number. Note that we therefore cannot change the parity of the number of crossings, at least not without closing the gap or breaking time-reversal symmetry. These are precisely the conditions that distinguish the topological insulator from an ordinary one: a topological insulator has an odd number of band crossings.

6.2.2. Three dimensions

Both the IQHE and the QSHE describe two-dimensional systems. Ever since the discovery of the IQHE, people have searched for an analogous situation in three dimensions. This turned out not to exist: the only three-dimensional analogue of the IQHE that one could make consists of many two-dimensional IQH states stacked on top of one another. People were hesitant to call this a three-dimensional topological phase, as there was nothing truly three-dimensional about it. Worse, it did not have the same stability of the normal IQHE.

The QSHE on the other hand does generalise to three-dimensions, which was theorised by Fu, Kane, and Mele [12] in 2007. Such a system is described by four $\mathbb{Z}_2$-valued invariants $(\nu_0, \nu_1, \nu_2, \nu_3)$, but only the first of these is truly three-dimensional. It is appropriately called the **Fu–Kane–Mele invariant**, and an insulator with nontrivial invariant is appropriately called a *three-dimensional topological insulator*. To communicate that ν_0 is a truly three-dimensional property whereas the others are two-dimensional, this ν_0 is called a **strong invariant**, and the others are called **weak invariants**. This terminology is general: for instance, the three-dimensional IQHE consists only of weak invariants, because there is no truly three-dimensional invariance at play. A system with $\nu_0 = 0$, i.e., a system with only nontrivial weak invariants, is not as stable as the strong topological insulator with $\nu_0 = 1$.

6.3. KITAEV'S PERIODIC TABLE

Thus far, in order to see if a system is in a nontrivial topological phase, we considered the valence bands as a vector bundle and used various means of distilling it down into a number. This number, called an *invariant*, then tells us what phase the system is in. In order to generalise more easily, we abstract away from this perspective and shall instead interpret the vector bundle *itself* as the invariant. More precisely, the K-group of the Brillouin zone consists of (isomorphism classes of) vector bundles over the Brillouin zone. Hence the valence bands determine an element of this K-group, and it is this element that we shall view as the invariant. In this way the K-group plays the role of the group of all invariants. If this point of view is to be at all useful, it should coincide with the invariants found previously. For example, the K-group we associate to the IQHE should be isomorphic to $\mathbb{Z}$, because we already know that the IQHE is classified by an integer: the Chern number.

Class \ d	0	1	2	3	4	5	6	7
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 6.1: Kitaev's periodic table of topological insulators and superconductors. An entry gives the group of invariants in a dimension d (indicated by the column) protected by the symmetries of an Altland–Zirnbauer class (indicated by the row). The table is periodic in the dimension with period 8, so the entries for dimensions higher than 7 can be deduced from the ones listed here.

Kitaev [24] was the first person to take this approach. In 2009 he showed that K-theory does indeed generalise the invariants we have seen above. However, what truly made it powerful (and hence interesting) was how it could be used to predict many more, previously unknown invariants. He summarised his findings in a periodic table, as follows. Given a dimension d and an Altland–Zirnbauer class, he computed all the different the topological phases protected by the symmetries in that class.[2] The resulting periodic table is given in Table 6.1. Striking is the fact that the table depends only on the dimension *modulo* 8. This makes Kitaev's table a truly 'periodic' table, as opposed to the more commonly known periodic table of elements. As Ryu et al. [36, p. 13] note, dimensions higher than three can also be of physical relevance: a parameter space can increase the dimension beyond three, even though the spatial spatial dimension is at most three. One example of this is the four-dimensional QHE theorised by Zhang and Hu [44] in 2001.

The table contains both the IQHE and the QSHE. Recall from Table 3.1 that class A is the class without any symmetries. In two dimensions this has a $\mathbb{Z}$-invariant, corresponding to the integers in the IQHE. Indeed, the IQHE is not protected by any symmetries. Class AII has time-reversal satisfying $T^2 = -1$, and it has a $\mathbb{Z}_2$-invariant in two dimensions. This is the QSHE: it is described by the spin of electrons, and because electrons are fermions, we have $T^2 = -1$. Notice how this table also predicts that the QSHE generalises to three dimensions (class AII also has a $\mathbb{Z}_2$-invariant there), while the IQHE does not (class A has only trivial phases in three dimensions). One also sees that it only lists the strong invariants: the three-dimensional QSHE has four $\mathbb{Z}_2$-invariants, but only one of these is strong.

[2] Here a zero-dimensional system is a system where the total Hilbert space is finite-dimensional.

Those with an eye for patterns may have spotted that both the K-theory and the KO-theory of a point appear in this table. In dimension zero, classes A and AIII contain the K-theory of a point, and the other classes contain the KO-theory of a point. Higher dimensions shift these groups. The distinction between A and AIII and the rest is no accident: these two classes do not have antilinear symmetries. An antilinear symmetry essentially reduces a complex vector bundle to a real one (as in Proposition 2.6), so only classes A and AIII have a complex K-group, while the others have a KO-group.

We shall only explain the calculation of the phases in class A. Note again that this contains the IQHE. A more complicated story is needed for the other classes; we will treat these later (see Corollary 6.16).

Proposition 6.4. *The row belonging to class A in Kitaev's periodic table is given by $\mathbb{Z}$ if d is even, and 0 when d is odd.*

Proof. Since we do not consider any sort of lattice symmetries, the Brillouin zone becomes simply $\mathbb{R}^d$, with d the dimension of the system. We have to compute $K(\mathbb{R}^d)$. This immediately raises questions because $\mathbb{R}^d$ is not compact, while our development of K-theory uses only compact spaces. Recall that, in our definition of a topological phase, we did not want the dimension of the bundle to have any influence on the phase of a system. We can accomplish this by adding a 'point at infinity' to $\mathbb{R}^d$, and then taking the reduced K-group of the resulting space. The d-sphere $\mathbb{S}^d$ is this space, which motivates the definition $K(\mathbb{R}^d) := \widetilde{K}(\mathbb{S}^d)$. By definition of lower-degree K-theory this is $K^{-d}(\mathrm{pt})$, so we conclude

$$K(\mathbb{R}^d) = K^{-d}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & d \text{ even}, \\ 0 & d \text{ odd}. \end{cases}$$

■

Remark 6.5. It turns out this is a general construction: given a locally compact Hausdorff space X, we can construct its *one-point compactification* X^+. As a set this is X with one new point, and there is a unique topology that makes it compact and makes it have X as a subspace (see, e.g., Munkres [34, §29]). We could define $K(X)$ to be $\widetilde{K}(X^+)$. In case X is already compact, X^+ is simply adding a disjoint point to X, so this would extend our definition. In fact, this gives a different perspective on suspensions because we have a (pointed) homeomorphism $\Sigma^n(X^+) \cong (X \times \mathbb{R}^n)^+$ when X is locally compact.

As elegant as it is, this periodic table is far from the whole story. One of its bigger shortcomings is that it it only deals with symmetries from the Altland–Zirnbauer classes. The most natural next step is to include crystal symmetries. Although Kitaev commented upon including lattice symmetries, he did not discuss any further possible crystal symmetries.

6.4. Freed and Moore's classification

In 2013, a few years after Kitaev, Freed and Moore [11] created a mathematical framework for classifying topological phases. They went further than Kitaev by including general crystal symmetries, but they also formalised the concept of a topological phase.

With this, the correspondence between topological phases and a K-group of the Brillouin zone can be proved rather than motivated. This formalisation is what we have been working towards in the previous chapters, so we give a more detailed exposition of their classification. Even so, there are details we omit for the sake of simplicity.

An insulator is described mathematically by what we called a band insulator (see Definition 4.19). Two insulators are supposed to be in the same topological phase if we can continuously deform one into the other without closing the band gap. Since we wish to speak of symmetry-protected phases, this deformation should respect the symmetries that we are considering. Instead of considering only symmetries associated to Altland–Zirnbauer classes, we also incorporate crystal symmetries. We argued in Section 4.6 that symmetries of this form are described by an extended quantum symmetry group of crystal type (see Definition 4.17). In this mathematical language, the appropriate notion of deformation is the following.

Definition 6.6. Let (G, φ, τ, c) be an extended quantum symmetry group of crystal type. Let $(\mathcal{H}_0, H_0, \rho_0)$ and $(\mathcal{H}_1, H_1, \rho_1)$ be two band insulators with this symmetry group. The two systems are called **homotopic** if there exists a Hilbert bundle $\mathcal{H}_\bullet \to [0,1]$ over $[0,1]$, along with:

- a decomposition $\mathcal{H}_\bullet = \mathcal{H}_\bullet^- \oplus \mathcal{H}_\bullet^+$;
- a continuous family of operators $H_\bullet \colon \mathcal{H}_\bullet \to \mathcal{H}_\bullet$, and a continuous family of maps $\rho_\bullet \colon G \to \mathcal{H}_\bullet$,

satisfying the following conditions.

(i) Every fibre $\mathcal{H}_s$ for $s \in [0,1]$ with the induced decomposition forms a band insulator $(\mathcal{H}_s, H_s, \rho_s)$ with symmetry group (G, φ, τ, c).

(ii) The fibres over 0 and 1 recover the systems $(\mathcal{H}_0, H_0, \rho_0)$ and $(\mathcal{H}_1, H_1, \rho_1)$, respectively.

Notice that Condition (i) implies that the deformation does not close the gap. In particular, the deformation restricts to yield separate deformations of $\mathcal{H}_i^-$ and $\mathcal{H}_i^+$. Symmetry-protection is implemented by the requirement that each fibre should have (G, φ, τ, c) as its symmetry group.

Remark 6.7. We have simplified the above definition by not explaining the word 'continuous' as it applies to the families $H_\bullet$ and $\rho_\bullet$. Although crucial for a mathematically formal proof, a more detailed treatment does not aid general understanding. One may find the details in Freed and Moore [11, App. D].

In mathematics, a way to formally define a property is to model it as a set consisting of all objects that are intended to satisfy the property. This is how we formalise the concept of a topological phase.

Definition 6.8. Let (G, φ, τ, c) be an extended quantum symmetry group of crystal type. A homotopy-equivalence class of band insulators with this symmetry group

is called a **topological phase protected by (G, φ, τ, c)**. The set of all such topological phases is denoted by

$$\mathrm{TP}(G, \varphi, \tau, c).$$

If c is trivial, we denote this set by $\mathrm{TP}(G, \varphi, \tau)$.

To aid the calculation of topological phases, we turn this set into a commutative semigroup by the direct sum of (super) Hilbert spaces. As we have discussed previously, the Hilbert space of a band insulator is to be viewed as the space of one-particle states, because we assume the electrons are non-interacting. Two band insulators are then coupled to one quantum system by taking the direct sum of the Hilbert spaces, for the direct sum then describes one particle which can live in either insulator. The one-particle nature is crucial here, otherwise the natural combination of two systems would be the *tensor product* of the Hilbert spaces. Although this motivates the semigroup structure physically, for us it is only a tool for calculating topological phases.

Sadly, topological phases as defined above do not perfectly correspond to a K-group of the Brillouin zone. We have to alter the set TP, although the method depends on whether we work with type F or type I insulators. This distinction is essentially the same as the one between twisted and extended twisted K-theory from Chapter 5.

Definition 6.9. Let (G, φ, τ, c) be an extended quantum symmetry group of crystal type. For type F insulators, define the group of **reduced topological phases** as the quotient of $\mathrm{TP}(G, \varphi, \tau, c)$ by the elements possessing an odd automorphism squaring to 1. We denote this group by $\mathrm{RTP}(G, \varphi, \tau, c)$. For type I insulators, define the group of reduced topological phases as the Grothendieck group of $\mathrm{TP}(G, \varphi, \tau)$. We denote this group by $\mathrm{RTP}(G, \varphi, \tau)$.

At the time of writing there does not seem to be a physical motivation for passing to reduced topological phases. It is not even clear that this cruder invariant is able to detect the phases we have encountered thus far. Freed and Moore [11, §11] have proved that it does detect the Kane–Mele invariant, among others.

Remark 6.10. Like in Chapter 5, we have followed the convention of Stehouwer instead of following Freed and Moore; see Remark 5.24.

In the end, these definitions and requirements allow us to calculate the reduced topological phases by the twisted K-theory of the Brillouin zone. This is the fundamental theorem for topological phases protected by crystal symmetry. The divide between twisted K-theory and extended twisted K-theory appears here as well.

Theorem 6.11 (Freed and Moore). *Let (G, φ, τ, c) be an extended quantum symmetry group of symmorphic crystal type. For band insulators of type F, we have an isomorphism*

$$\begin{aligned} \mathrm{RTP}(G, \varphi, \tau, c) &\xrightarrow{\sim} {}^{\varphi}K^{\tau', c}_{G/\Pi}(X_\Pi) \\ [(\mathcal{H}, H, \rho)] &\longmapsto [\mathcal{E}^- \oplus \mathcal{E}^+]. \end{aligned}$$

For band insulators of type I, we instead have an isomorphism

$$\begin{aligned} \mathrm{RTP}(G, \varphi, \tau) &\xrightarrow{\sim} {}^{\varphi}K^{\tau'}_{G/\Pi}(X_\Pi) \\ [(\mathcal{H}, H, \rho)] &\longmapsto [\mathcal{E}^-]. \end{aligned}$$

Proof. See Freed and Moore [11, §10.3–4], bearing in mind our modifications to extended twisted K-theory from Chapter 5 (see Remark 5.24). ∎

Remark 6.12. For nonsymmorphic crystals there is an analogous result. What changes in that case is the cocycle τ', as sketched in Remark 4.15.

This theorem is our motivation for developing K-theory for compact spaces and finite groups only. It formally shows why only the valence bands played a role in the IQHE and the QSHE: these are type I insulators, and in that case the bundle of valence bands alone determines the topological phase of the system. Because extended twisted K-theory reduces to twisted K-theory when c is trivial (Proposition 5.27), the distinction between type I and type F can often be forgotten in computations.

Notably absent from this result is a distinction between weak and strong invariants (see Section 6.2.2). Rather, the twisted K-group gives all the invariants of the system, including the dimension of the bundle (which we argued is often too strong an invariant). Freed and Moore do not discuss weak and strong invariants, and neither does there seem to be a straightforward mathematical formalisation of the concept of a weak and strong invariant. One is left to study the situation at hand in order to determine which parts of the K-group describe weak and strong invariants.

6.4.1. Special cases

With this theorem in hand, some special cases are worth pointing out.

Example 6.13 (Class A). If $G = S$, so φ and τ are trivial, then $G/\Pi = P$, and φ and τ restrict trivially to it. In Example 5.5 we argued that in this case the twisted K-group ${}^{\varphi}K_P^{\tau'}(X_\Pi)$ is simply $K_P(X_\Pi)$. So when only treating crystal symmetry, the (reduced) topological phases protected by G are classified by the P-equivariant K-group $K_P(X_\Pi)$.

Example 6.14 (Class AI). When G also contains time-reversal symmetry $T^2 = +1$, then G/Π is the magnetic point group, which we know to be $P \oplus \mathbb{Z}_2$ in this case (see Section 4.4). Time-reversal is anti-unitary, so φ is the projection onto the second component. Because $T^2 = +1$ there is no further twisting from τ (or τ'). The corresponding K-group is ${}^{\varphi}K_{P\oplus\mathbb{Z}_2}(X_\Pi)$, and from Example 5.6 we know this to be equivariant KR-theory: $KR_P(X_\Pi)$.

Example 6.15 (Class AII). Again take G to be S along with time-reversal, but now with $T^2 = -1$. From Section 5.1.1 we know that this is (equivariant) KQ-theory, which by Theorem 5.7 is simply KR-theory shifted by four: $KR_P^{-4}(X_\Pi)$.

This third example illustrates that going to a different Altland–Zirnbauer class can be accounted for by a degree-shift in K-theory. This is also what happened in Kitaev's periodic table. Under some additional requirements, it also holds for all symmetry classes in Freed and Moore's classification.

Corollary 6.16. *Let (G, φ, τ, c) be an extended quantum symmetry group of symmorphic crystal type. Denote by A the image of the map $\theta \times c \colon G \to \{\pm 1\} \times \{\pm 1\}$, where $\theta := \varphi \cdot c$. Suppose that G/Π splits as a direct sum $G/\Pi \cong P \oplus A$ and that τ restricts trivially to S. Then we have the following table for the reduced topological phases according to the Altland–Zirnbauer class of (G, φ, τ, c).*

Class	*A*	*AIII*	*AI*	*BDI*	*D*
${}^{\varphi}K^{\tau',c}_{G/\Pi}(X_\Pi)$	$K_P(X_\Pi)$	$K_P^{-1}(X_\Pi)$	$KR_P(X_\Pi)$	$KR_P^{-1}(X_\Pi)$	$KR_P^{-2}(X_\Pi)$

Class	*DIII*	*AII*	*CII*	*C*	*CI*
${}^{\varphi}K^{\tau',c}_{G/\Pi}(X_\Pi)$	$KR_P^{-3}(X_\Pi)$	$KR_P^{-4}(X_\Pi)$	$KR_P^{-5}(X_\Pi)$	$KR_P^{-6}(X_\Pi)$	$KR_P^{-7}(X_\Pi)$

In all of these cases, the Real involution on X_Π is given by the involution $\sigma \colon k \mapsto -k$.

Proof. The proof given by Freed and Moore [11, Cor. 10.25] still applies if one uses our relation between Clifford algebras and K-theory instead (see Theorem 5.33). ∎

Remark 6.17. This result relies on the modification we made to extended twisted K-theory as compared to Freed and Moore — see Remarks 5.24 and 5.34. Because Freed and Moore have a different definition of trivial bundles, they have the degrees in KR-theory running the other direction, which does not reproduce Kitaev's results.

Remark 6.18. In the above corollary we have omitted the distinction between type I and type F insulators from Theorem 6.11. To reintroduce it, note that a type I insulator can only be of class A, AI or AII: these are the only classes without particle-hole reversing symmetries. These are exactly the cases covered in Examples 6.13–6.15.

It is this corollary that is responsible for the seemingly odd ordering of the Altland–Zirnbauer classes. The appearance of the graded K-groups here gives a physical explanation for why we introduced them back in Chapters 1. One can think of this corollary as a generalisation of Kitaev's periodic table, though strictly speaking it is not: the periodic table did not take into account crystal or even lattice symmetry. Even so, the resemblance is striking: classes A and AIII are also classified by complex K-theory, while the others have a real K-theory group (more precisely, a Real one). In zero dimensions the Brillouin zone is a point, so in that case this is a true generalisation of Kitaev's results.

It is important to point out that the assumptions made in this corollary are not very general. Any situation where the point group has a nontrivial cocycle τ' on it is excluded. This means, for example, that a fermionic system with a 180-degree rotation R is not described by this corollary, because then $R^2 = -1$, which requires τ' to be nontrivial.

Part III.

Calculations

7 ONE-DIMENSIONAL CLASSIFICATION

Research into one-dimensional topological phases has been done, but surprisingly, at the time of writing no classification using the theory of Freed and Moore seems to have been given yet. In this chapter we present precisely that, classifying the non-interacting, non-relativistic, one-dimensional topological phases. Our results are summarised in Table 7.1.

In two and three dimensions, mathematically rigorous calculations tend to use an advanced tool called a *spectral sequence*. In contrast, as we will illustrate, one-dimensional cases are all computable using the more elementary tools presented earlier in this work. This is especially important for cases involving particle-hole symmetry, because a spectral sequence has not yet been developed for these cases. The reason for this simplicity is two-fold: there are very few point groups in one dimension (all of which are very small), and the one-dimensional Brillouin zone is a circle (which is much easier to work with than higher-dimensional tori).

Despite this simplicity, the classification as a whole does require all the K-theory tools we have developed up to this point. As a side effect, this makes it a good didactic tool: it is simple enough to be accessible to beginners, yet the amount of K-theory involved encourages and motivates a further study of (twisted) K-theory.

7.1. PRELIMINARIES AND ASSUMPTIONS

According to Theorem 6.11, the (reduced) topological phases protected by (G, φ, τ, c) are classified by the K-group

$$ {}^{\varphi}K^{\tau',c}_{G/\Pi}(X_\Pi), $$

under the assumption that the crystal underlying G is symmorphic. To calculate this K-group, in principle one therefore needs to know two things: the group G/Π with its twistings, and the Brillouin zone X_Π. More precisely, one needs to know the (G/Π)-equivariant Brillouin zone: the action of G/Π on X_Π does impact the K-group. As such, to classify all possible one-dimensional phases, one needs to know all possibilities for G/Π and for X_Π. (We assume for the moment that all one-dimensional crystals are symmorphic — see Proposition 7.1.) One cannot describe the latter without knowing the former, so we have to study the group G/Π first.

Because only G/Π appears in the K-group, throughout this chapter we shall denote by G the group that we previously denoted by G/Π in Chapter 6. As such we shall also write τ instead of τ'. To know this group G, we need to know the point group

P, and how this point group 'sits' inside G. The (only) assumptions we make in our classification are the following.

(i) The group G splits as a direct sum $G \cong P \oplus A$.

(ii) The group A comes from an Altland–Zirnbauer class, i.e., it is of the from $\mathbb{Z}_2^k$ for a certain k, with its twistings being one of the possibilities from Table 3.1.

Physically, Assumption (i) means that point group symmetries commute with time-reversal and particle-hole reversal. Assumption (ii) in particular means that time-reversal, particle-hole reversal and chiral symmetry all commute with each other and that they all square to a complex phase. Although one can imagine physical systems that do not satisfy these assumptions, in calculations of topological phases they are commonly assumed. A fuller classification would have to determine all physically sensible alternatives to these assumptions. Take particular note of the fact that Assumption (i) is one of the conditions of Corollary 6.16.

In order to know all possible forms of (G, φ, τ, c) under these assumptions, it suffices to determine all possible one-dimensional point groups. Indeed, we already know all possible forms of A and its twistings.

7.1.1. The point group

One-dimensional Euclidean space has few symmetries: only translations and reflections. In particular, $O(1) = \{ \pm 1 \}$ consists only of the identity and a reflection around a fixed origin. Since the point group P of a one-dimensional crystal is a subgroup of $O(1)$ (see Section 4.2), this means P is either trivial or $\mathbb{Z}_2$. From this we can verify that all one-dimensional crystals are in fact symmorphic.

Proposition 7.1. *Let $S \subseteq \mathbb{E}(1)$ be the space group of a one-dimensional crystal. Then we have an isomorphism $S \cong P \ltimes \Pi$, with $\Pi \subseteq S$ the lattice and $P = S/\Pi$ the point group.*

Proof. If P is trivial, the proposition is trivial, so assume that $P = \{ \pm 1 \}$. This means there is a point $x_0 \in \mathbb{E}^1$ such that the reflection R around x_0 is an element of S. Because $R^2 = 1$, we have a homomorphism $i \colon P \to S$ given by sending -1 to R. Notice that $\pi \circ i = \mathrm{Id}_P$, where π denotes the canonical projection $S \twoheadrightarrow P$. For any $s \in S$, the element

$$f_0(s) := s \cdot (i \circ \pi)(s)^{-1}$$

lies in $\Pi \subseteq S$; after all, it lies in the kernel of π:

$$\pi(s \cdot (i \circ \pi)(s)^{-1}) = \pi(s) \cdot (\pi \circ i \circ \pi)(s)^{-1} = \pi(s) \cdot \pi(s)^{-1} = 1.$$

The map i defines an action of P on Π, namely conjugation by $i(p)$ when $p \in P$. Now define

$$f \colon S \longrightarrow P \ltimes \Pi \colon \quad s \longmapsto \{ \pi(s) \mid f_0(s) \}.$$

It is a trivial verification that this map is an isomorphism. ∎

Remark 7.2. We have proved a general fact in group theory: if N is a normal subgroup of G such that the projection $G \twoheadrightarrow G/N$ admits a right-inverse which is also a group homomorphism, then $G \cong N \rtimes G/N$.

Henceforth we write R for the generator of P. This map R must be a unitary map, as all point group symmetries are unitary. However, in principle it could square to -1. Up to isomorphism[1] the cases $R^2 = +1$ and $R^2 = -1$ are the only possible ones. In conclusion therefore, the point group is of one of the following types:

(i) trivial point group, abbreviated by $R^2 = 0$;

(ii) $P \cong \mathbb{Z}_2$ with trivial twisting, abbreviated by $R^2 = +1$;

(iii) $P \cong \mathbb{Z}_2$ with nontrivial cocycle τ, abbreviated by $R^2 = -1$.

As in Section 3.3, the abbreviation $R^2 = 0$ does not mean that we have a group acting by the zero map, but only serves as an abbreviation. Combining these with the ten Altland–Zirnbauer classes, we get a total of 30 possible symmetry groups.

Only the cases $R^2 = 0$ and $R^2 = +1$ have trivial twisting of the point group. Combining this with Assumption (i) from above, we see that in these cases we can use Corollary 6.16 to compute the topological phases. When $R^2 = -1$ this does not hold.[2]

7.1.2. The Brillouin zone

The Brillouin zone X_Π is the quotient $\mathbb{R}^*/\Pi^*$ of reciprocal space by the reciprocal lattice (see Definition 4.7). Without loss of generality we may take Π to be $\mathbb{Z}$. Identifying $\mathbb{R}^*$ with $\mathbb{R}$ via the standard inner product, then Π^* can be naturally identified with $2\pi\mathbb{Z}$. Hence the Brillouin zone is the circle $\mathbb{R}/2\pi\mathbb{Z}$.

If the point group has a reflection in it, it acts on the Brillouin zone as $k \mapsto -k$, which reflects the circle as illustrated in Figure 7.1. As noted in Section 4.2.1, this map we denote by σ. In Sections 4.4 and 4.5 we explained that both time-reversal and particle-hole reversal also act by σ on the Brillouin zone. Notice that this implies that chiral symmetry, being the product of these two, acts trivially on the Brillouin zone. Because these are the only the symmetries that can arise in one dimension, we thus have a full description of the equivariant Brillouin zone.

7.1.3. Mayer–Vietoris

For most of the thirty cases, we will use the Mayer–Vietoris exact sequence to calculate the corresponding K-group of the Brillouin zone. We divide the circle into two parts, A and B, as indicated in Figure 7.2. Notice that A and B are symmetric around the reflection axis of σ. Therefore, for all possible G, the space A is G-equivariantly homotopy equivalent to $\{\, a \,\}$, with a as indicated in Figure 7.1. Similarly B is equivariantly homotopy equivalent to $\{\, b \,\}$, with b indicated in Figure 7.1. Both of these spaces therefore

[1] For a formal explanation, one needs to develop Remark 3.8 further; see, e.g., Stehouwer [39, App. A].

[2] In the case $R^2 = -1$, the lower-degree KR-groups do not agree with the ones we compute in Section 7.4. This demonstrates that the condition on τ cannot be omitted from Corollary 6.16.

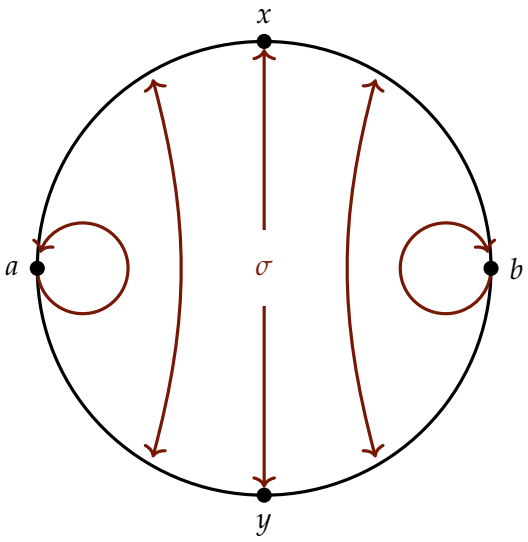


Figure 7.1: The action of the involution σ on the one-dimensional Brillouin zone, viewed as a circle. It fixes the points labelled a and b, and reverses those labelled x and y.

have the K-theory of a point, which we can calculate using the twisted representation rings from Section 5.4.

The intersection $A \cap B$ is equivariantly homotopy equivalent to $\{ x, y \}$. This space also has the K-theory of a point, but with a different group: the stabiliser of x. Throughout this chapter we denote this stabiliser by H. Notice then that $\{ x, y \} \cong G/H$ as G-equivariant spaces.

Proposition 7.3. *Let (G, φ, τ, c) be a finite extended quantum symmetry group. Let $H \subseteq G$ be a subgroup. Give G/H the discrete topology. Then for all $n \in \mathbb{N}_0$, we have an isomorphism*

$$ {}^{\varphi}K_G^{(\tau,c)-n}(G/H) \cong {}^{\varphi}K_H^{(\tau,c)-n}(\mathrm{pt}). $$

For $n = 0$, this isomorphism is given by $[E] \mapsto [E_H]$.

Proof. See Stehouwer [39, Lem. 3.72]. ∎

Remark 7.4. Note that when we pass to twisted H-equivariant twisted K-theory in the above, we have to restrict φ, τ and c to H. Because this would clutter the notation we have left this implicit, which we will do in the remainder of this chapter also.

In light of the above, the Mayer–Vietoris exact sequence takes the form

$$ \begin{array}{ccccccc} \cdots & \xrightarrow{\delta} & {}^{\varphi}K_G^{(\tau,c)-n}(\mathbb{S}^1) & \longrightarrow & {}^{\varphi}K_G^{(\tau,c)-n}(\mathrm{pt}) \oplus {}^{\varphi}K_G^{(\tau,c)-n}(\mathrm{pt}) & \longrightarrow & {}^{\varphi}K_H^{(\tau,c)-n}(\mathrm{pt}) \\ & & & & & & \downarrow{\scriptstyle \delta} \\ \cdots & \xleftarrow{\delta} & {}^{\varphi}K_H^{(\tau,c)-n+1}(\mathrm{pt}) & \longleftarrow & {}^{\varphi}K_G^{(\tau,c)-n+1}(\mathrm{pt}) \oplus {}^{\varphi}K_G^{(\tau,c)-n+1}(\mathrm{pt}) & \longleftarrow & {}^{\varphi}K_G^{(\tau,c)-n+1}(\mathbb{S}^1). \end{array} $$

Theorem 5.33 tells us that we can compute the K-groups of a point using the twisted representation ring of the group. This motivates the following general procedure to

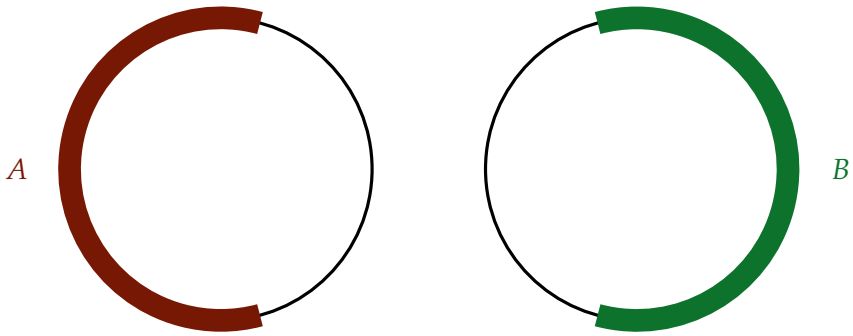


Figure 7.2: The definition of the closed subsets A and B of the circle. Both sets are symmetric under reflection over the x-axis.

compute ${}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1)$. First one computes the twisted group algebras of H and G. The K-theories of a point can then be derived from these. Afterwards one computes some of the maps in the Mayer–Vietoris sequence in case this is needed to give a unique answer. As we will see, this last step is not always needed, and sometimes we can even get around using the Mayer–Vietoris sequence altogether.

7.2. TRIVIAL POINT GROUP

Previously we argued why we can use Corollary 6.16 in the case $R^2 = 0$. This now means that, to classify all topological phases protected by this point group, we have to compute

$$K^{-q}(\mathbb{S}^1) \quad \text{and} \quad KR^{-q}(\mathbb{S}^1) \qquad \text{for all } q.$$

This is especially simple because the circle is itself a suspension. The arguments for this are very similar to the ones in Section 1.5. In that section we already stated that $K(\mathbb{S}^1) \cong \mathbb{Z}$. Notice that $\widetilde{K}^{-1}(\mathbb{S}^1) = K^{-2}(\mathrm{pt}) \cong \mathbb{Z}$, so by Proposition 1.20 we find

$$K^{-1}(\mathbb{S}^1) \cong \widetilde{K}^{-1}(\mathbb{S}^1) \oplus K^{-1}(\mathrm{pt}) \cong \mathbb{Z} \oplus 0 = \mathbb{Z}.$$

In short, we have

$$K(\mathbb{S}^1) \cong \mathbb{Z} \quad \text{and} \quad K^{-1}(\mathbb{S}^1) \cong \mathbb{Z}.$$

For the KR-groups, the Real involution on $\mathbb{S}^1$ is given by σ, as noted in Corollary 6.16. The action of σ is to mirror the circle. In the language of Section 2.2.2, this makes the circle equal to $\mathbb{S}^{1,1}$. Notice that this is $\Sigma^{1,0}(\mathrm{pt}^+)$. The proof of Proposition 1.20 immediately generalises to KR-theory, so for all q we have an isomorphism

$$KR^{-q}(\mathbb{S}^{1,1}) \cong \widetilde{KR}^{-q}(\mathbb{S}^{1,1}) \oplus KR^{-q}(\mathrm{pt}).$$

The reduced KR-group of $\mathbb{S}^{1,1}$ is a KR-group of a point:

$$\widetilde{KR}^{-q}(\mathbb{S}^{1,1}) = \widetilde{KR}^{-q}(\Sigma^{1,0}(\mathrm{pt}^+)) = KR^{-q+1}(\mathrm{pt}).$$

The KR-theory of a point is the KO-theory of a point (see Example 2.11), so

$$KR^{-q}(\mathbb{S}^{1,1}) \cong KO^{-q+1}(\mathrm{pt}) \oplus KO^{-q}(\mathrm{pt}).$$

By plugging in the KO-theory of a point from Equation (1.6.1) we find the following table for the KR-theory of $\mathbb{S}^{1,1}$.

q	0	1	2	3	4	5	6	7
$KR^{-q}(\mathbb{S}^{1,1})$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0

7.3. UNTWISTED REFLECTION

When $R^2 = +1$, we have that $P \cong \mathbb{Z}_2$. Again, as explained above, in the case $R^2 = +1$ we can also use Corollary 6.16, meaning we have to compute the P-equivariant K- and KR-theory of the circle:

$$K_P^{-q}(\mathbb{S}^1) \quad \text{and} \quad KR_P^{-q}(\mathbb{S}^1) \qquad \text{for all } q.$$

However, the previous suspension argument breaks down because the circle has a nontrivial group action. We shall therefore resort to using the Mayer–Vietoris sequence as described above.

Complex K-theory. Notice that the stabiliser of the point x on the circle is trivial, meaning $A \cap B$ has the normal K-theory of a point. Therefore, as outlined above, we need the $\mathbb{Z}_2$-equivariant K-theory and the normal K-theory of a point. The K-theory of a point is given by Equation (1.5.1). For the $\mathbb{Z}_2$-equivariant K-theory of a point we may use the twisted representation rings from Section 5.4: twisted K-theory reduces to ordinary K-theory when the twists are trivial. This was actually already done in Example 5.39, which found

$$R^{-q}(P) \cong (K^{-q}(\mathrm{pt}))^2 \cong \begin{cases} \mathbb{Z}^2 & q \text{ even}, \\ 0 & q \text{ odd}. \end{cases}$$

Because equivariant K-theory is two-fold periodic (Axiom (7) in Section 1.4, see also Section 2.1), the Mayer–Vietoris sequence becomes periodic also:

$$\begin{array}{ccccc} K_P(\mathbb{S}^1) & \longrightarrow & K_P(A) \oplus K_P(B) & \longrightarrow & K_P(A \cap B) \\ \uparrow & & & & \downarrow \\ K_P^{-1}(A \cap B) & \longleftarrow & K_P^{-1}(A) \oplus K_P^{-1}(B) & \longleftarrow & K_P^{-1}(\mathbb{S}^1). \end{array}$$

Notice that only the vertical maps change the degree. The map $K_P(A \cap B) \to K_P^{-1}(\mathbb{S}^1)$ decreases it instead of increasing it because of the identification of degrees $+1$ and -1. Plugging in what we have found, this reduces to

$$\begin{array}{ccccc} K_P(\mathbb{S}^1) & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \xrightarrow{\psi} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & K_P^{-1}(\mathbb{S}^1). \end{array}$$

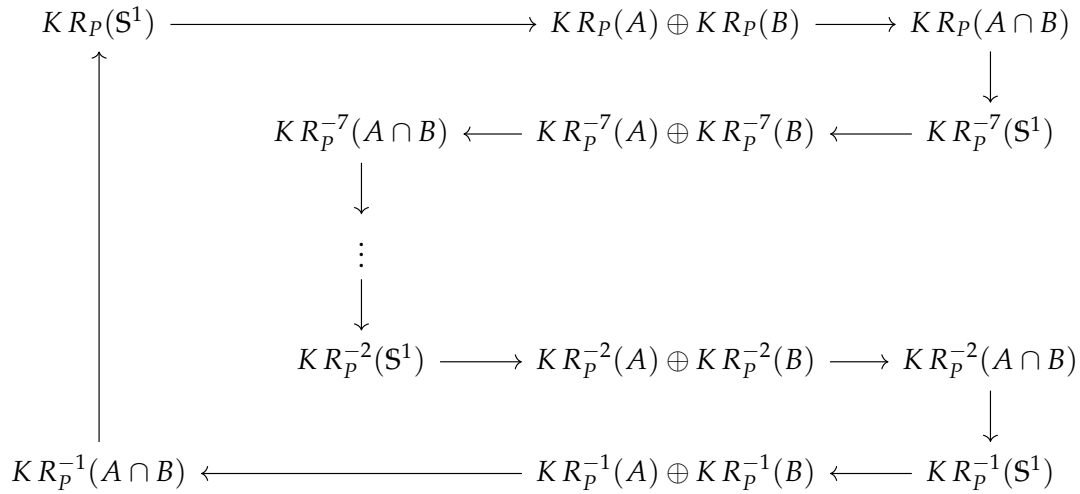
We need to calculate the map ψ under the above isomorphisms. This case is simple enough that we can do this as follows. The first component of $\mathbb{Z}^2$ under the isomorphism $K_P(A) \cong \mathbb{Z}^2$ represents the dimension of the bundle over A, and similarly for B. Recall from Section 1.4 that the map ψ is induced as follows. The inclusion map $A \cap B \hookrightarrow A$ induces the restriction map $K_P(A) \to K_P(A \cap B)$, and similarly for $K_P(B) \to K_P(A \cap B)$. The map ψ is the difference of these two maps. The isomorphism $K_P(A \cap B) \cong K(\mathrm{pt})$ from Proposition 7.3 is by taking the fibre over the point x. We know that the isomorphism $K(\mathrm{pt}) \cong \mathbb{Z}$ is given by the dimension of the bundle. Therefore ψ is given by

$$\psi\colon \mathbb{Z}^2 \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z}\colon \quad (a,b,c,d) \longmapsto a - c.$$

From this we finally conclude

$$K_P(\mathbb{S}^1) \cong \mathrm{Ker}(\psi) \cong \mathbb{Z}^3 \quad \text{and} \quad K_P^{-1}(\mathbb{S}^1) \cong \mathrm{Coker}(\psi) = 0.$$

KR-theory. Here we can use an analogous computation, with the only complication that we have an eight-fold periodicity instead of a two-fold one. Therefore the Mayer–Vietoris sequence becomes much longer, but is still of a similar form.



Here too only the vertical maps change the degree, with only one decreasing it because of the identification of degrees -7 and $+1$.

Before we continue, we compute the KR-theories of A, B and $A \cap B$. For A and B this is the $\mathbb{Z}_2$-equivariant KR-theory of a point. This is, in particular, a twisted K-group (see Example 5.6), so we can use the method of twisted group algebras from Section 5.4. Writing the Real involution as T, notice that the stabiliser of x is $H = \{1, RT\}$. Because it plays this special role, it is convenient to abbreviate $S := RT$. Notice that S is antilinear and that $S^2 = +1$. The corresponding twisted group algebra of G is then

$$\begin{aligned}
{}^{\varphi}\mathbb{C}[G] &= \mathbb{R}[i,R,S] \,/ (i^2 = -1,\ S^2 = +1,\ R^2 = +1,\ Si = -iS,\ Ri = iR,\ RS = SR) \\
&\cong |\mathrm{Cliff}_{1,1}| \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R}) \\
&\cong M_2(\mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R}) \\
&\cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R}),
\end{aligned}$$

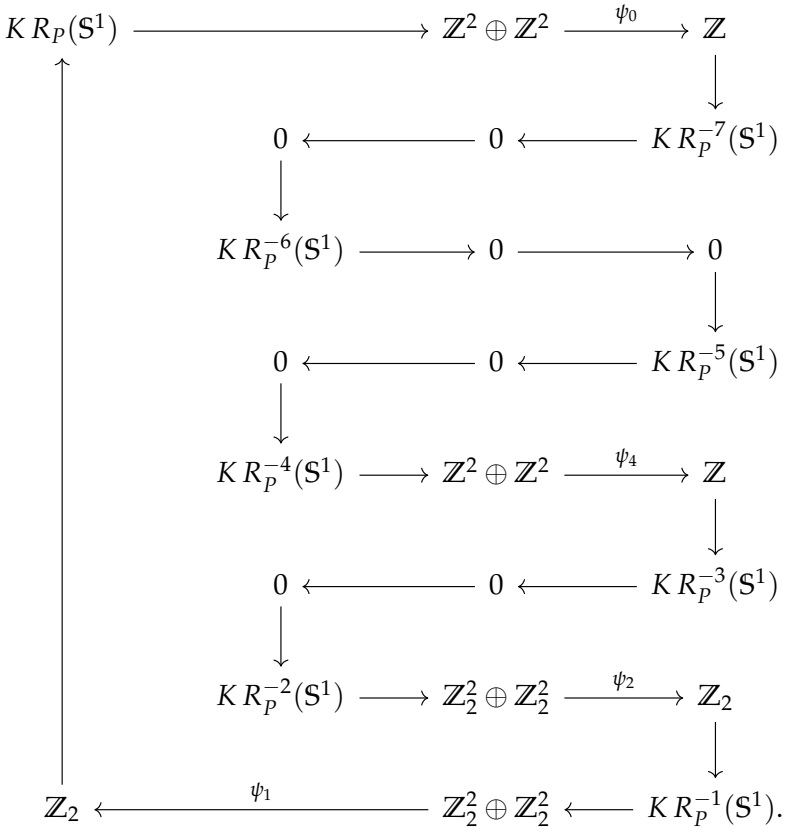
where we use i and S to generate $|\mathrm{Cliff}_{1,1}|$ and R to generate $\mathbb{R} \oplus \mathbb{R}$ (by the Chinese remainder theorem). From this we deduce

$$KR_P^{-q}(\mathrm{pt}) \cong R^{-q}(M_2(\mathbb{R}) \oplus M_2(\mathbb{R})) \cong (KO^{-q}(\mathrm{pt}))^2.$$

Because $H = \{1, RT\}$, the twisted group algebra ${}^{\varphi}\mathbb{C}[H]$ of H is $M_2(\mathbb{R})$, the first component of ${}^{\varphi}\mathbb{C}[G]$. Therefore we have

$$KR_P^{-q}(A \cap B) \cong KO^{-q}(\mathrm{pt}).$$

Plugging these groups into the Mayer–Vietoris exact sequence yields



Two of the KR-groups of the circle we can immediately read off:

$$KR_P^{-6}(\mathbb{S}^1) = 0 \quad \text{and} \quad KR_P^{-5}(\mathbb{S}^1) = 0.$$

To determine the remaining ones, we need expressions for the maps ψ_0, ψ_1, ψ_2 and ψ_4 indicated in the diagram. Because we used twisted group algebras, we can compute all of these simultaneously. For we know that ${}^{\varphi}\mathbb{C}[G]$ is $M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$, and that ${}^{\varphi}\mathbb{C}[H]$ is simply the first component $M_2(\mathbb{R})$ of this. On twisted representation rings therefore, the inclusion map ${}^{\varphi}\mathbb{C}[H] \hookrightarrow {}^{\varphi}\mathbb{C}[G]$ induces the projection onto the first component. Notice that the maps $KR_P^{-q}(A) \to KR_P^{-q}(A \cap B)$ and $KR_P^{-q}(B) \to KR_P^{-q}(A \cap B)$ both equal this induced map. This means that ψ_0 and ψ_4, under these isomorphisms, are given by

$$\mathbb{Z}^2 \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z}\colon \quad (a, b, c, d) \longmapsto a - c,$$

and that ψ_1 and ψ_2 are given by

$$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2 : \quad (a,b,c,d) \longmapsto a - c.$$

Computing the remaining KR-groups is now an easy exercise in exact sequences, resulting in the following table.

q	0	1	2	3	4	5	6	7
$KR_P^{-q}(\mathbb{S}^1)$	$\mathbb{Z}^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	0	$\mathbb{Z}^3$	0	0	0

7.4. TWISTED REFLECTION

Corollary 6.16 does not apply in the case $R^2 = -1$, so all K-groups have to be computed separately. Before we do so, let us comment upon some general trends.

Because the reflection R squares to -1, it generates an algebra isomorphic to $\mathbb{C}$. The grading here agrees as well, since R is even (point group symmetries do not reverse particles and holes) and because of our convention that all elements of $\mathbb{C}$ are even. Note that R commutes with all other symmetries, and also with i. Finally, notice that $G/H \cong \{ 1, R \}$. These considerations imply that the superalgebra ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[G]$ will always be the complexification of the superalgebra ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[H]$ with R acting as the imaginary unit. We will see this more concretely below. Because of this relationship, it is most fruitful to first determine ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[H]$, and afterwards find ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[G]$ by taking the tensor product with $\mathbb{C}$.

A particle-hole reversing symmetry will give rise to an odd element in ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[G]$. Thus only in symmetry classes with particle-hole reversing symmetries do we have to worry about the superalgebra structure of ${}^{\varphi}\mathbb{C}^{\tau,\mathcal{C}}[G]$. If we can recognise the tensor product with a Clifford algebra in this, then this will result in a degree shift according to Proposition 5.36. Classes without particle-hole symmetry will not have any degree shift take place (cf. Remark 5.37).

We have organised the different classes here in order of increasing complexity. Some steps or details will repeat themselves; we shall omit these if they have been explained before.

Classes A and AIII. These classes are actually the same as their $R^2 = +1$ counterpart, because there are no antilinear symmetries present. Formally, define $\tilde{R} := iR$, then we have $\tilde{R}^2 = i^2 \cdot R^2 = +1$. Using $\tilde{R}$ instead of R gives isomorphic groups, even in the presence of chiral symmetry (which is complex-linear). Thus the situations are the same, in particular yielding the same invariants.

Class AI. In addition to R, in this class there is one additional symmetry: time-reversal T with $T^2 = +1$. Remember that time-reversal is antilinear. Notice that $H = \{ 1, RT \}$. If we write $S := RT$, then S is antilinear and $S^2 = -1$. Therefore we have

$${}^{\varphi}\mathbb{C}^{\tau}[H] \cong \mathbb{R}[i,S]/(i^2 = S^2 = -1,\ Si = -iS) \cong \mathbb{H},$$

implying

$$ {}^{\varphi}K_G^{\tau-q}(A \cap B) \cong {}^{\varphi}K_H^{\tau-q}(\mathrm{pt}) \cong R^{-q}(\mathbb{H}) \cong KQ^{-q}(\mathrm{pt}) \cong KO^{-q-4}(\mathrm{pt}). $$

Notice that T and R together generate the same elements as S and R. We may therefore without harm work with S instead of T for ${}^{\varphi}\mathbb{C}^{\tau}[G]$ also:

$$ \begin{aligned} {}^{\varphi}\mathbb{C}^{\tau}[G] &= \mathbb{R}[i,R,S] \,/ (i^2 = R^2 = S^2 = -1,\ Si = -iS,\ Ri = iR,\ RS = SR) \\ &\cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau}[H] \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}). \end{aligned} $$

From this we conclude

$$ {}^{\varphi}K_G^{\tau-q}(\mathrm{pt}) \cong {}^{\varphi}R^{\tau-q}(G) \cong R^{-q}(M_2(\mathbb{C})) \cong K^{-q}(\mathrm{pt}). $$

Part of the Mayer–Vietoris exact sequence now reads

$$ 0 \longrightarrow {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \longrightarrow \mathbb{Z}^2 \xrightarrow{\ \psi\ } \mathbb{Z}. $$

It is not hard to see that ψ is not the zero map. Hence its kernel is isomorphic to $\mathbb{Z}$, making us conclude

$$ {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \cong \mathbb{Z}. $$

Class AII. This class is similar to the previous one, only having $T^2 = -1$ instead. Again writing $S := RT$, we now have $S^2 = +1$, so

$$ {}^{\varphi}\mathbb{C}^{\tau}[H] = \mathbb{R}[i,S]/(i^2 = -1,\ S^2 = +1,\ Si = -iS) \cong |\mathrm{Cliff}_{1,1}| \cong M_2(\mathbb{R}), $$

so that

$$ {}^{\varphi}K_G^{\tau-q}(A \cap B) \cong {}^{\varphi}R^{\tau-q}(H) \cong R^{-q}(M_2(\mathbb{R})) \cong KO^{-q}(\mathrm{pt}). $$

Similarly to the previous case, the twisted group algebra of G is

$$ \begin{aligned} {}^{\varphi}\mathbb{C}^{\tau}[G] &= \mathbb{R}[i,R,S]/(i^2 = R^2 = -1,\ S^2 = +1,\ Si = -iS,\ Ri = iR,\ RS = SR) \\ &\cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau}[H] \cong M_2(\mathbb{C}), \end{aligned} $$

so we find

$$ {}^{\varphi}K_G^{\tau-q}(\mathrm{pt}) \cong K^{-q}(\mathrm{pt}). $$

Accordingly, the K-group of the circle fits into the exact sequence

$$ 0 \longrightarrow \mathbb{Z}_2 \longrightarrow {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \longrightarrow \mathbb{Z}^2 \xrightarrow{\ \psi\ } \mathbb{Z}. $$

Again ψ is nonzero, so its kernel is isomorphic to $\mathbb{Z}$. Therefore ${}^{\varphi}K_G^{\tau}(\mathbb{S}^1)$ fits into a short exact sequence

$$ 0 \longrightarrow \mathbb{Z}_2 \longrightarrow {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \longrightarrow \mathbb{Z} \longrightarrow 0, $$

which must split because $\mathbb{Z}$ is free, ergo

$$ {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}_2. $$

Class C. Here we have particle-hole reversal C that squares to -1. Particle-hole reversal is odd and antilinear. If we write $U := RC$, then $H = \{1, U\}$. As $U^2 = +1$, we therefore have

$$^{\varphi}\mathbb{C}^{\tau,c}[H] = \mathbb{R}[i, U]/(i^2 = -1,\ U^2 = +1,\ Ui = -iU),$$

where U is odd and i is even. If i were odd, we could recognise a Clifford algebra here. We can make it odd as follows: define $j := iU$, then j is odd and $j^2 = iUiU = -i^2U^2 = +1$. Notice that j and U anticommute and generate the same elements as i and U do. Consequently we may without harm replace i with j. Together these generate $\mathrm{Cliff}_{0,2}$, so by Proposition 5.36,

$$^{\varphi}K_G^{(\tau,c)-q}(A \cap B) \cong {}^{\varphi}R^{(\tau,c)-q}(H) \cong R^{-q}(\mathrm{Cliff}_{0,2}) \cong R^{-q-2}(\mathbb{R}) \cong KO^{-q-2}(\mathrm{pt}).$$

We once again find that $^{\varphi}\mathbb{C}^{\tau,c}[G]$ is simply the complexification of $^{\varphi}\mathbb{C}^{\tau,c}[H]$:

$$\begin{aligned} {}^{\varphi}\mathbb{C}^{\tau,c}[G] &= \mathbb{R}[i, R, U]\,/(i^2 = R^2 = -1,\ U^2 = +1,\ Ui = -iU,\ Ri = iR,\ RU = UR) \\ &\cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau,c}[H] \cong \mathbb{C}\mathrm{liff}_2\,. \end{aligned}$$

Note that this is an isomorphism of superalgebras because R is an even element. This means that

$$^{\varphi}R^{(\tau,c)-q}(G) \cong R^{-q}(\mathbb{C}\mathrm{liff}_2) \cong R^{-q-2}(\mathbb{C}) \cong K^{-q-2}(\mathrm{pt}) \cong K^{-q}(\mathrm{pt}),$$

and plugging these results into the Mayer–Vietoris exact sequence, we find

$$0 \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow \mathbb{Z}^2 \xrightarrow{\ \psi\ } \mathbb{Z}_2.$$

Whatever ψ may be, its kernel is always isomorphic to $\mathbb{Z}^2$; this means

$$^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \cong \mathbb{Z}^2.$$

Class D. This is similar to the previous class, but now we have $C^2 = +1$. Hence, using the same procedure as before,

$$\begin{aligned} {}^{\varphi}\mathbb{C}^{\tau,c}[H] &\cong \mathbb{R}[i, U]/(i^2 = U^2 = -1,\ Ui = -iU) \cong \mathrm{Cliff}_{2,0}, \\ {}^{\varphi}\mathbb{C}^{\tau,c}[G] &\cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau,c}[H] \cong \mathbb{C}\mathrm{liff}_2\,. \end{aligned}$$

Therefore we get

$$\begin{aligned} {}^{\varphi}K_G^{(\tau,c)-q}(A \cap B) &\cong R^{-q}(\mathrm{Cliff}_{2,0}) \cong KO^{-q+2}(\mathrm{pt}) \cong KO^{-q-6}(\mathrm{pt}), \\ {}^{\varphi}K_G^{(\tau,c)-q}(\mathrm{pt}) &\cong R^{-q}(\mathbb{C}\mathrm{liff}_2) \cong K^{-q}(\mathrm{pt}). \end{aligned}$$

The Mayer–Vietoris sequence reads

$$0 \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

which means

$$^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \cong \mathbb{Z}^2.$$

Class BDI. This class has both time-reversal and particle-hole symmetry, with $T^2 = C^2 = +1$. It is the easiest to work with $U := RC$, and $S := iTC$, and $j := iRC$. Then U, S and j together generate ${}^{\varphi}\mathbb{C}^{\tau,c}[H]$, whereas R, U and S and j generate ${}^{\varphi}\mathbb{C}^{\tau,c}[G]$. Note that U, S, and j are all odd, and that $U^2 = S^2 = j^2 = -1$. Lastly, these three all anticommute with each other. Therefore

$$\begin{aligned}{}^{\varphi}\mathbb{C}^{\tau,c}[H] &= \mathbb{R}[j, U, S] \,/ (j^2 = U^2 = S^2 = -1,\ Sj = -jS,\ Uj = -jU,\ SU = -US) \\ &\cong \mathrm{Cliff}_{3,0}\,.\end{aligned}$$

Analogously to previous cases, because R commutes with everything and $R^2 = -1$, we have

$${}^{\varphi}\mathbb{C}^{\tau,c}[G] \cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau,c}[H] \cong \mathbb{C}\mathrm{liff}_3\,.$$

This means that

$$\begin{aligned}{}^{\varphi}K_G^{(\tau,c)-q}(A \cap B) &\cong R^{-q}(\mathrm{Cliff}_{3,0}) \cong KO^{-q+3}(\mathrm{pt}) \cong KO^{-q-5}(\mathrm{pt}), \\ {}^{\varphi}K_G^{(\tau,c)-q}(\mathrm{pt}) &\cong R^{-q}(\mathbb{C}\mathrm{liff}_3) \cong K^{-q+3}(\mathrm{pt}).\end{aligned}$$

As such, Mayer–Vietoris reads

$$0 \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow 0,$$

so we conclude

$${}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) = 0.$$

Class DIII. With $T^2 = -1$ and $C^2 = +1$, the first calculations in this class are very similar to the previous one. We again write $U := RC$, and $S := iTC$, and $j := iRC$. The only difference compared to class BDI is that we now have $U^2 = j^2 = -1$ and $S^2 = +1$. As such,

$$\begin{aligned}{}^{\varphi}\mathbb{C}^{\tau,c}[H] &\cong \mathrm{Cliff}_{2,1}, \\ {}^{\varphi}\mathbb{C}^{\tau,c}[G] &\cong \mathbb{C} \otimes_{\mathbb{R}} {}^{\varphi}\mathbb{C}^{\tau,c}[H] \cong \mathbb{C}\mathrm{liff}_3,\end{aligned}$$

so that

$$\begin{aligned}{}^{\varphi}K_G^{(\tau,c)-q}(A \cap B) &\cong R^{-q}(\mathrm{Cliff}_{2,1}) \cong KO^{-q+1}(\mathrm{pt}), \\ {}^{\varphi}K_G^{(\tau,c)-q}(\mathrm{pt}) &\cong R^{-q}(\mathbb{C}\mathrm{liff}_3) \cong K^{-q+1}(\mathrm{pt}).\end{aligned}$$

Mayer–Vietoris now reads

$$\mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z} \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow 0.$$

We can no longer uniquely determine ${}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1)$ from the shape of this sequence alone, so we have to compute the map ψ. Recall that ψ is induced as follows. For each q, the inclusion $A \cap B \hookrightarrow A$ induces a map ${}^{\varphi}K_G^{(\tau,c)-q}(A) \to {}^{\varphi}K_G^{(\tau,c)-q}(A \cap B)$, and similarly for $A \cap B \hookrightarrow B$. The map ψ is the difference of the induced maps belonging to $q = 1$. But both of these maps in degree $-q$ equal the map

$$R^{-q+1}(\mathbb{C}) \longrightarrow R^{-q+1}(\mathbb{R})$$

induced by the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. The degree shift of $+1$ is because ${}^{\varphi}\mathbb{C}^{\tau,c}[H] \cong \mathrm{Cliff}_{2,1}$. For $q = 1$ this is $R(\mathbb{C}) \to R(\mathbb{R})$. Notice that this is simply the map $K(\mathrm{pt}) \to KO(\mathrm{pt})$ given by sending a complex vector space to its underlying real vector space. As such it is multiplication by 2 as a map $\mathbb{Z} \to \mathbb{Z}$. So ψ is given by $(a, b) \mapsto 2(a - b)$, meaning that it has cokernel $\mathbb{Z}_2$, so

$$ {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \cong \mathbb{Z}_2. $$

Class CI. Now that $T^2 = +1$ and $C^2 = -1$, the same arguments as above apply, only with some differing minus signs. Still writing $U := RC$, and $S := iTC$, and $j := iRC$, we have $U^2 = S^2 = j^2 = +1$. This means

$$ \begin{aligned} {}^{\varphi}\mathbb{C}^{\tau,c}[H] &\cong \mathrm{Cliff}_{0,3}, \\ {}^{\varphi}\mathbb{C}^{\tau,c}[G] &\cong \mathbb{C}\mathrm{liff}_3; \\ {}^{\varphi}K_G^{(\tau,c)-q}(A \cap B) &\cong KO^{-q-3}(\mathrm{pt}), \\ {}^{\varphi}K_G^{(\tau,c)-q}(\mathrm{pt}) &\cong K^{-q-3}(\mathrm{pt}). \end{aligned} $$

Plugging this into Mayer–Vietoris yields

$$ \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z} \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow 0. $$

Like before, the maps induced by $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$ in degree $-q$ both equal the map

$$ R^{-q-3}(\mathbb{C}) \longrightarrow R^{-q-3}(\mathbb{R}) $$

induced by $\mathbb{R} \hookrightarrow \mathbb{C}$. Using the theory of Atiyah, Bott, and Shapiro [5, §5] one may explicitly calculate that this map is the identity map $\mathbb{Z} \to \mathbb{Z}$ when $q = 1$. The map ψ is therefore given by $(a, b) \mapsto a - b$, meaning it is surjective, so

$$ {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) = 0. $$

Class CII. In this class, $T^2 = C^2 = -1$. In the exact same manner, we have

$$ \begin{aligned} {}^{\varphi}\mathbb{C}^{\tau,c}[H] &\cong \mathrm{Cliff}_{1,2}, \\ {}^{\varphi}\mathbb{C}^{\tau,c}[G] &\cong \mathbb{C}\mathrm{liff}_3; \\ {}^{\varphi}K_G^{(\tau,c)-q}(A \cap B) &\cong KO^{-q-1}(\mathrm{pt}), \\ {}^{\varphi}K_G^{(\tau,c)-q}(\mathrm{pt}) &\cong K^{-q-1}(\mathrm{pt}). \end{aligned} $$

Mayer–Vietoris now reads

$$ \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z}_2 \longrightarrow {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \longrightarrow 0. $$

One may compute the map ψ using the same arguments as in class CI, in which case one finds that it is $(a, b) \mapsto \overline{a - b}$. Hence we have

$$ {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) = 0. $$

7.5. Discussion

Our approach to the classification demonstrates that different methods give different insights into the groups. In the case $R^2 = 0$, each entry is given by the K-group of a point, plus the K-group of a point shifted by one:

$$K^{-q}(\mathrm{pt}) \oplus K^{-q-1}(\mathrm{pt}) \quad \text{or} \quad KR^{-q}(\mathrm{pt}) \oplus KR^{-q-1}(\mathrm{pt}).$$

This is reminiscent of the use of an equivariant splitting by Stehouwer [39, §4.2, §4.5]. By this method he retrieves the Fu–Kane–Mele invariant (see Section 6.2.2) as

$$KR^{-4}(\mathbb{T}^3) \cong KR^{-4}(\mathrm{pt}) \oplus (KR^{-3}(\mathrm{pt}))^3 \oplus (KR^{-2}(\mathrm{pt}))^3 \oplus KR^{-1}(\mathrm{pt}) \cong \mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2.$$

This motivates identifying the K-group of a point that has the highest shift as a strong invariant, and the others as weak invariants (see Section 6.2.2 for a definition of weak and strong invariants). If we apply this to our case $R^2 = 0$, then the strong invariants here recover the one-dimensional invariants from Kitaev's periodic table (Table 6.1).

7.5.1. Band structure combinatorics

However, because we could not use a similar suspension argument for the other point group types, we cannot distinguish the weak and strong invariants as easily. We can take the reduced K-theory in each entry, but this does not seem to always remove all local invariants. For instance, in class A with point group $R^2 = +1$, we have $\mathbb{Z}^3$ as the group of invariants. The K-group of a point we calculated to be $\mathbb{Z}^2$ in that case, so the reduced K-group of the circle here is $\mathbb{Z}$. Yet this $\mathbb{Z}$ arises because of the second fixed point of the circle under the reflection. Indeed, a fixed point will have a representation of $\mathbb{Z}_2$ on its fibre, and we calculated that $R(\mathbb{Z}_2) \cong \mathbb{Z}^2$, where the first component indicates the dimension. We have two fixed points on the circle, which might suggest an invariant of $\mathbb{Z}^4$. Yet to ensure that the vector bundle has the same dimension everywhere, the representations must have the same dimension, yielding a group of $\mathbb{Z}^3$, as our method found as well. So in this case there do not seem to be any strong or non-local invariants, despite the reduced K-theory being nonzero.

This slightly different look at class A with $R^2 = +1$ is an application of the method called **band structure combinatorics** introduced by Kruthoff et al. [28]. In this method, first the fixed points of (subgroups of) the group G are identified. Each point has a representation of its stabiliser on its fibre, and compatibility conditions then give the total group of invariants. This method only applies for class A, and is proved to be correct in dimension one and two.[3] We already verified that their method agrees with ours for $R^2 = +1$, and it is an easy exercise to see that it also does in $R^2 = 0$. (Remember that $R^2 = +1$ and $R^2 = -1$ are the same case in class A — see Section 7.4.)

Later, Kruthoff et al. [29] extended this method to class AI and AII, i.e., by including time-reversal, either $T^2 = +1$ or $T^2 = -1$, respectively. They in particular give an interpretation of this method using *transition functions*, an interpretation which is particularly close to our method. In this approach, we cut up the circle into two lines,

[3] Whether it is always correct in three dimensions will be discussed in the next chapter.

give each of these lines a vector bundle of the same rank, and glue them back together using a transition function. This function is a map $\{x, y\} \to U(n)$, with x and y as indicated in Figure 7.1, with n the rank bundles over the two lines, and $U(n)$ the $n \times n$ unitary matrices. This map should be symmetric under the symmetries we consider, in this case T and R. Whenever two transition maps are equivariantly homotopic, the resulting bundles over the circle are isomorphic.

We can already see the striking similarities with the Mayer–Vietoris approach. For the kernel of the map

$$ {}^{\varphi}K_G^{\tau}(A) \oplus {}^{\varphi}K_G^{\tau}(B) \longrightarrow {}^{\varphi}K_G^{\tau}(A \cap B) $$

consists precisely of pairs of bundles over A and B which can be glued together. Indeed, if two vector bundles restrict to the same bundle over $A \cap B$, then these come from a vector bundle over $\mathbb{S}^1$. However, in general ${}^{\varphi}K_G^{\tau}(\mathbb{S}^1)$ does not equal this kernel, for we have an exact sequence

$$ {}^{\varphi}K_G^{\tau-1}(A \cap B) \longrightarrow {}^{\varphi}K_G^{\tau}(\mathbb{S}^1) \longrightarrow {}^{\varphi}K_G^{\tau}(A) \oplus {}^{\varphi}K_G^{\tau}(B) \longrightarrow {}^{\varphi}K_G^{\tau}(A \cap B). $$

So if ${}^{\varphi}K_G^{\tau-1}(A \cap B) \neq 0$, there does not need to be a unique way to glue two vector bundles over A and B together. This reflects the fact that not all transition functions are homotopic.

7.5.2. General weak and strong invariants

The above consideration actually gives us a method to distinguish between weak and strong invariants in all one-dimensional cases. For in all cases, we got a splitting

$$ \begin{aligned} {}^{\varphi}K_G^{\tau,c}(\mathbb{S}^1) \cong\ & \operatorname{Ker}({}^{\varphi}K_G^{\tau,c}(A) \oplus {}^{\varphi}K_G^{\tau,c}(B) \to {}^{\varphi}K_G^{\tau,c}(A \cap B)) \\ & \oplus \operatorname{Coker}({}^{\varphi}K_G^{(\tau,c)-1}(A) \oplus {}^{\varphi}K_G^{(\tau,c)-1}(B) \to {}^{\varphi}K_G^{(\tau,c)-1}(A \cap B)). \end{aligned} \tag{7.5.1} $$

The kernel part is the same as a consistent assignment of invariants to the points a and b. The cokernel is a global invariant: it measures how the two 'halves' of a vector bundle are glued together. In other words, the kernel consists of weak invariants, and the cokernel of strong invariants. Because we used the Mayer–Vietoris approach for both $R^2 = +1$ and $R^2 = -1$, this allows one to quickly recognise the strong invariants in all of these cases. Doing this, one finds that the $R^2 = +1$ case has only weak invariants, and for the $R^2 = -1$ case we have a strong $\mathbb{Z}_2$-invariant in classes AII and DIII. By using the Mayer–Vietoris approach for the $R^2 = 0$ cases, one can see that this method of differentiating weak and strong invariants yields the same distinction as the method described above for the $R^2 = 0$ case.

It should be noted that in this case, the Mayer–Vietoris approach is also very similar to the approach using spectral sequences. In fact, it seems to even coincide with the spectral sequence approach. The splitting from Equation (7.5.1) is the same type of splitting obtained there. We do not go into detail about this here, but note that in the next chapter we will use a simplified version of the spectral sequence.

$P \setminus$ Class	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
$R^2 = \quad 0$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$R^2 = +1$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	0	$\mathbb{Z}^3$	0	0	0
$R^2 = -1$	$\mathbb{Z}^3$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^2$	0

Table 7.1: The classification of one-dimensional, non-interacting, non-relativistic topological phases using the theory of Freed and Moore. Each entry gives the twisted K-group corresponding to the invariants protected by an Altland–Zirnbauer class (given by the column) and a point group P of a one-dimensional crystal (given by the row). Here $R^2 = 0$ denotes the trivial point group, $R^2 = +1$ the point group consisting of only a reflection R, and $R^2 = -1$ the point group with a reflection R that squares to -1 on the quantum level. If in the row for $R^2 = 0$ the reduced K-group is taken, one retrieves the one-dimensional invariants of Kitaev (see Table 6.1). These invariants are also the strong invariants for the case $R^2 = 0$. In $R^2 = +1$ all invariants are weak, and in $R^2 = -1$ only the $\mathbb{Z}_2$-invariants appearing in classes AII and DIII are strong.

8 SPACE GROUP *F*222

In Section 7.5, we discussed a method to calculate topological phases in class A called *band structure combinatorics*, introduced by Kruthoff et al. [28] in 2017. We remarked that in one and two dimensions, this method is proved to yield the correct result. In three dimensions there is a subtlety, which is illustrated by the three-dimensional space group *F*222. In 2018, Shiozaki et al. [38, p. 25] found that it has $\mathbb{Z}^7 \oplus \mathbb{Z}_2$ as its group of invariants. Band structure combinatorics does not detect this particular $\mathbb{Z}_2$-invariant. Nonetheless, some doubt hangs over this result because McAlister [32, §5.6] instead found $\mathbb{Z}^7$, although he used a completely different method. Moreover, Shiozaki et al. do not give many details on their computation.

In this chapter we verify the result of Shiozaki et al. in more detail. We use the Atiyah–Hirzebruch spectral sequence, which was particularly simplified by Stehouwer et al. [40, §4.3] for class A invariants. With this simplification, we do not have to make explicit mention of spectral sequences, but can instead use only *Bredon cohomology* to arrive at our result. In Section 8.1 we describe this theory, but in a drastically simplified version for the sake of brevity. To use this theory, one needs to know the representation rings of all the subgroups of the point group, and one needs to study the Brillouin zone in further detail. We study these things in Sections 8.3 and 8.4, respectively. In Section 8.5 we compute the K-group of the Brillouin zone of *F*222.

8.1. BREDON EQUIVARIANT COHOMOLOGY

Bredon equivariant cohomology is a method of studying equivariant spaces by studying points only. More precisely, it studies *G-equivariant CW complexes* by considering *G-equivariant points*. We discuss the latter concept first.

We know from Example 2.2 that $K_G(\mathrm{pt}) \cong R(G)$. For the space G/H of cosets of H (under the discrete topology) we have an analogous result, as follows.

Proposition 8.1. *Let G be a finite group and let H be a finite subgroup of G. Give G/H the discrete topology. Then we have a ring isomorphism*

$$K_G(G/H) \xrightarrow{\sim} R(H): \quad [E] \longmapsto [E_{1\cdot H}].$$

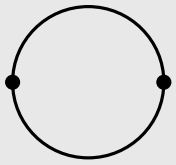
Proof sketch. If V is a complex H-representation, then the group H acts on the space $G \times V$ by

$$(h, (g, v)) \longmapsto (h \cdot g,\ h \cdot v).$$

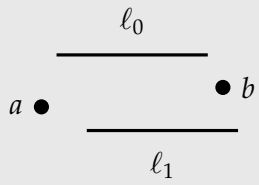
The quotient $(G \times V)/H$ is a G-equivariant vector bundle over G/H. The assignment $[V] \mapsto [(G \times V)/H]$ is a two-sided inverse to the given map. ■

This motivates one to think of G/H not as a collection of points, but as one 'equivariant point': a point with a G-action on it. The stabiliser of the equivariant point G/H is then H. Briefly said, G-equivariant CW complexes are spaces that are built out of equivariant *cells*. To grasp this concept, first consider the following example.

Example 8.2. The circle can be made by taking two points, and gluing two lines between these points, as follows.



A formal way to see this is as follows. First, define $X_0 = \{ a, b \}$ to be a discrete set on two points. Then we consider the disjoint union $X_0 \sqcup \ell_0 \sqcup \ell_1$ of X_0 with two lines ℓ_0 and ℓ_1.



The quotient of this space identifying a with the left endpoints of both ℓ_0 and ℓ_1, and b with the other endpoints of ℓ_0 and ℓ_1, is (homeomorphic to) the circle.

We can turn the circle into a $\mathbb{Z}_2$-equivariant space by the mirroring described in Figure 7.1. This group action behaves well with this decomposition into points and lines: it keeps the points a and b fixed, while it reverses the lines ℓ_0 and ℓ_1.

We generalise the idea presented in this example. The ***n*-cell** $\mathbb{D}^n$ is the topological space

$$\mathbb{D}^n := \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}.$$

More concretely, for $n = 0$ this is a point, for $n = 1$ this is a line, and for $n = 2$ this is a disk. We have $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$, the $(n-1)$-sphere. A **CW complex** is a topological space X that is constructed as follows. One starts with a discrete set of points X_0. One forms X_{n+1} by gluing a number of $(n+1)$-cells along their boundary to X_n. The smallest n for which $X = X_n$ is called the **dimension** of the CW complex.[1] If each cell is given an orientation,[2] the CW complex is called **oriented**. More information, along with a more elaborate definition of CW complexes, may be found in, e.g., Hatcher [16, Ch. 0, App.]. Notice that every n-cell is *contractible*, so in particular has the K-theory of a point.

[1] Infinite-dimensional CW complexes are also possible. As a set these are simply $\bigcup_n X_n$; their topology is described by, e.g., Hatcher [16, Ch. 0]. They will not appear in this work.

[2] An orientation of a point (i.e., a 0-cell) is a sign, $+1$ or -1.

Similar to equivariant points, the G-equivariant space

$$(G/H) \times \mathbb{D}^n,$$

with G acting *trivially* on the component $\mathbb{D}^n$, should be thought of as an *equivariant n-cell*. This space is equivariantly homotopy equivalent to G/H, because G acts trivially on $\mathbb{D}^n$ and because $\mathbb{D}^n$ is contractible. In particular, the space has the G-equivariant K-theory of G/H. A ***G*-equivariant CW complex** is an oriented CW complex where the group action comes from using equivariant n-cells in the construction of the complex, and where the orientations are compatible with the G-action. A more detailed definition of a G-equivariant CW complex is given by Bredon [6, §1.1] and is summarised by Stehouwer [39, App. C.2].

In a certain sense therefore, a G-equivariant CW complex is built out of equivariant points only, because equivariant n-cells have the same homotopy type as equivariant points. But this does not mean the K-theory of an equivariant CW complex is the direct sum of the K-theory of its cells, as even Example 8.2 illustrates. For the K-group of the circle is $\mathbb{Z}$ (see Section 1.5), whereas if we sum the K-groups of the cells in the circle (of which there are four), we would get $\mathbb{Z}^4$. The reason it is not this simple is because there are 'gluing conditions': a bundle over the circle must have constant rank, so we cannot independently assign $\mathbb{Z}$ to each of the cells. The formal tool that incorporates these gluing conditions is *Bredon equivariant cohomology*. We have simplified the following definitions to only apply only to our case; in particular, we only discuss finite *abelian* groups. The original definition was given by Bredon [6]; for a concise summary, see Stehouwer [39, App. C.3].

Definition 8.3. Let G be a finite abelian group and let X be a G-equivariant CW complex. Denote by $C_n(X)$ the set of (non-equivariant) n-cells of X. An ***n*-dimensional *G*-cochain** is a map

$$f \colon C_n(X) \longrightarrow \bigsqcup_{H \subseteq G} R(H),$$

with H running over all subgroups of G, satisfying the following conditions.

(i) For all $\sigma \in C_n(X)$, we have $f(\sigma) \in R(G_\sigma)$, with G_σ the stabiliser of σ.

(ii) For all $g \in G$ and $\sigma \in C_n(X)$, we have $f(g \cdot \sigma) = f(\sigma)$.

We write $C^n_G(X)$ for the set of n-dimensional G-cochains on X.

Remark 8.4. Condition (ii) has been changed the most compared to Bredon's original definition. Our modification requires G to be abelian: this guarantees that σ and $g \cdot \sigma$ have the same stabiliser.

Notice that every $C^n_G(X)$ forms an abelian group under the operation $(f + f')(\sigma) := f(\sigma) + f'(\sigma)$. We have a cochain complex[3]

$$0 \longrightarrow C^0_G(X) \xrightarrow{d} C^1_G(X) \xrightarrow{d} \cdots \xrightarrow{d} C^n_G(X) \xrightarrow{d} C^{n+1}_G(X) \xrightarrow{d} \cdots,$$

[3] I.e., a chain complex where the differentials *increase* the degree instead of decreasing it.

with differential given as follows. First, note that if an n-cell τ lies on the boundary of an $(n+1)$-cell σ, then we have $G_\sigma \subseteq G_\tau$. Thus we have a restriction map $R(G_\tau) \to R(G_\sigma)$ induced by restricting representations. We denote the image of an element $V \in R(G_\tau)$ under this map by $V\big|_{G_\sigma}$. If $f \in C_G^n(X)$, then $df \in C_G^{n+1}(X)$ is given by

$$(df)(\sigma) := \sum_{\tau \in C_n(X)} [\tau : \sigma] \cdot f(\tau)\big|_{G_\sigma}. \tag{8.1.1}$$

This lies in $R(G_\sigma)$ and is hence well-defined. Here the symbol $[\tau : \sigma]$ is 0 if τ does not lie on the boundary of σ, and otherwise is ± 1 depending on the orientation of the cell σ. For instance, if $\ell = \mathbb{D}^1$ is the 1-cell with standard orientation, then $[+1 : \ell] = +1$ and $[-1 : \ell] = -1$. If f is then a zero-dimensional G-cochain, we have in this case

$$(df)(\ell) = f(+1)\big|_{G_\ell} - f(-1)\big|_{G_\ell}.$$

It is quickly verified that these differentials are group homomorphisms. We have $d^2 = 0$ because the boundary of a boundary is zero.

Definition 8.5. Let G be a finite abelian group and let X be a G-equivariant CW complex. The n-th cohomology group of the above cochain complex is called the **Bredon G-equivariant cohomology group in degree n** of X, and is denoted by $H_G^n(X)$. In other words, it is the quotient

$$H_G^n(X) := \frac{\operatorname{Ker}(d\colon C_G^n(X) \to C_G^{n+1}(X))}{\operatorname{Im}(d\colon C_G^{n-1}(X) \to C_G^n(X))}.$$

Remark 8.6. Bredon's definition does not use the representation rings, but rather uses an arbitrary functor F that assigns abelian groups to equivariant points. In that case the Bredon cohomology groups are denoted by $H_G^n(X, F)$. As such, what we denote by $H_G^n(X)$ is usually denoted by $H_G^n(X, \mathcal{R}_G)$. Here $\mathcal{R}_G$ denotes the functor that maps G/H to $R(H)$; its action on maps is described by Stehouwer [39, §3.9]. Because we do not describe this more general theory, we have modified the notation accordingly.

In contrast to the naive approach from before, Bredon cohomology does allow us to compute K-groups. This is proved using the Atiyah–Hirzebruch spectral sequence, resulting in the following.

Theorem 8.7 (Class A, 3D). *Let G be an abelian group, and let X be a three-dimensional G-equivariant CW complex. Then we have an isomorphism*

$$K_G(X) \cong H_G^0(X) \oplus H_G^2(X).$$

Proof. See Stehouwer et al. [40, §4.3]. ∎

Remark 8.8. In the above isomorphism, the group $H_G^0(X)$ captures zero-dimensional invariants, whereas $H_G^2(X)$ captures two-dimensional invariants. For $C_G^n(X)$ consists of maps that assign representations to n-cells, and $H_G^n(X)$ is a subquotient of this group. Note that this splitting is very similar to the splitting discussed in Section 7.5.2 and summarised in Equation (7.5.1).

8.2. THE GROUP *F*222

Up to isomorphism, there are 230 different three-dimensional space groups (see Definition 4.4 for the definition of a space group). One of these is $F222$, the 22nd in the International Table for Crystallography [20]. It is a symmorphic space group, generated by the translations

$$u_1 := \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad u_2 := \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad u_3 := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix},$$

and rotations

$$R_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_y := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_z := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In other words, R_x is the rotation of 180 degrees around the x-axis, and similarly for R_y and R_z. The point group is thus $\{ 1, R_x, R_y, R_z \}$, which is the Klein four-group V_4 (notice that $R_z = R_x R_y = R_y R_x$). This explains the '222' in the name $F222$: around the x, y and z-axes we each have a rotation of order 2. The lattice Π spanned by $\{ u_1, u_2, u_3 \}$ is face-centred cubic (FCC), illustrated in Figure 8.1a. This explains the 'F' in the name $F222$.

8.3. REPRESENTATION RINGS

We shall henceforth denote the point group of $F222$ by G. It has five different subgroups: the trivial ones $\{ 1 \}$ and G, and

$$H_x := \{ 1, R_x \}, \quad H_y := \{ 1, R_y \}, \quad H_z := \{ 1, R_z \}.$$

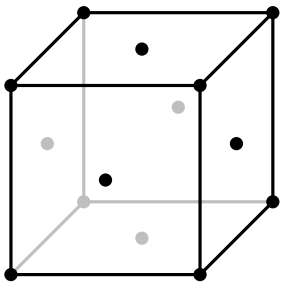
In Bredon cohomology, we see the representation rings of subgroups of G appear, as well as restriction maps between them. We therefore determine all of these for the group G. The quickest method to this end is to compute the representation rings using group algebras. We have

$$\mathbb{C}[G] = \frac{\mathbb{C}[R_x, R_y, R_z]}{(R_x^2 - 1,\ R_y^2 - 1,\ R_z - R_x R_y)} \cong \mathbb{C}^4,$$

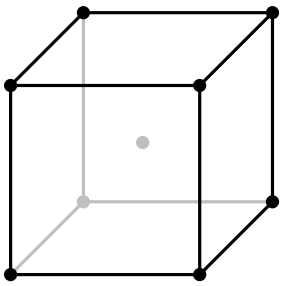
as follows from the Chinese remainder theorem. Hence $R(G) \cong \mathbb{Z}^4$ by Theorem 5.38. It also shows that $R(H_x) \cong R(H_y) \cong R(H_z) \cong \mathbb{Z}^2$ by considering the corresponding subalgebras of $\mathbb{C}[G]$. However, the restriction maps are different:

$$\begin{aligned} \pi_x \colon R(G) &\longrightarrow R(H_x) \colon \quad (a,b,c,d) \longmapsto (a,b), \\ \pi_y \colon R(G) &\longrightarrow R(H_y) \colon \quad (a,b,c,d) \longmapsto (a,c), \\ \pi_z \colon R(G) &\longrightarrow R(H_z) \colon \quad (a,b,c,d) \longmapsto (a,d). \end{aligned}$$

Lastly, we have $R(\{ 1 \}) \cong \mathbb{Z}$, with all restriction maps being the projection onto the first component. In the remainder of this chapter we will view all the representation rings under the above isomorphisms.



(a) A cube representing a face-centred cubic (FCC) lattice. Lattice points are found on the corners of the cube, and in the middle of each face.



(b) A cube representing a body-centred cubic (BCC) lattice. Lattice points are found on the corners of the cube, and in the middle of the body of the cube.

Figure 8.1: The cubes representing the FCC and BCC lattices. In both cases, the entire lattice can be reconstructed by filling all of space by repeating the cube.

8.4. The Brillouin zone

Recall from Definition 4.7 that the Brillouin zone X_Π is the quotient of $(\mathbb{R}^3)^*$ by the *reciprocal* lattice Π^* (i.e., the set $\mathrm{Hom}(\Pi, 2\pi\mathbb{Z})$ of group homomorphisms from Π to $2\pi\mathbb{Z}$). To make later discussion easier, we identify $(\mathbb{R}^3)^*$ and $\mathbb{R}^3$ via

$$\mathbb{R}^3 \xrightarrow{\sim} (\mathbb{R}^3)^* : \quad x \longmapsto 2\pi \cdot \langle x, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the default inner product on $\mathbb{R}^3$. The factor 2π appears here to prevent it from appearing anywhere else in this chapter. The reciprocal lattice Π^* is the lattice spanned by linear functionals f_1, f_2, f_3 satisfying $f_i(u_j) = 2\pi\delta_{ij}$. Thus, under our identification, these are vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $\langle v_i, u_j \rangle = \delta_{ij}$. This is the same as finding the rows of the inverse of the matrix[4]

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

which are quickly found to be

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

The lattice $\Pi' \subseteq \mathbb{R}^3$ spanned by $\{ v_1, v_2, v_3 \}$ is body-centred cubic (BCC), as illustrated in Figure 8.1b. Note however that the cube shown there has sides of length 2, not 1.

Recall from Section 4.2.1 that we defined the action of a point group P on the Brillouin zone by, for $k \in X_\Pi$ and $R \in P$,

$$(R, k) \longmapsto (R^*)^{-1}(k),$$

[4]The formulas for reciprocal lattice vectors commonly used in condensed matter literature (see, e.g., Hook and Hall [19, Eq. (11.9)]) describe precisely this, up to a factor of 2π.

with R^* denoting the dual map. Under our identification $(\mathbb{R}^3)^* \cong \mathbb{R}^3$ the dual map of a matrix is the same as the transpose of the matrix. The matrices R_x, R_y and R_z are all symmetric and their own inverse. Therefore the action on $\mathbb{R}^3$ through the isomorphism $(\mathbb{R}^3)^* \cong \mathbb{R}^3$ is given by the same matrices. In other words, the action of the point group G on reciprocal space is also by rotations over 180 degrees around the x, y and z-axes.

8.4.1. CW structure

We give the Brillouin zone the structure of a G-equivariant CW complex. To make things easier, we work with the quotient $\mathbb{R}^3/\Pi'$ (with Π' as above), which we previously argued is equivariantly isomorphic to $(\mathbb{R}^3)^*/\Pi^*$. First of all, notice that every point in the quotient $\mathbb{R}^3/\Pi'$ is equivalent to a point in the cube shown in Figure 8.1b. Thus we only have to consider that region. Notice however that the unit cube is not a primitive unit cell[5] because of the lattice point in the middle of the cube. This means that every point in $\mathbb{R}^3$ is also equivalent to a point in the left-half of this cube (i.e., all points with $0 \leq x < 1$). Notice that this has volume 4 (bear in mind that the cube in Figure 8.1b has side-length 2).

Our CW structure will be based around the unit cube highlighted red in Figure 8.2a. We denote this volume by $\mathcal{V}$. In Figure 8.2b we label the vertices of this smaller cube:

$$\Gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Notice that the points Γ, A, B and M are all fixed by all elements of G. For example, we have

$$R_x B = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = B,$$

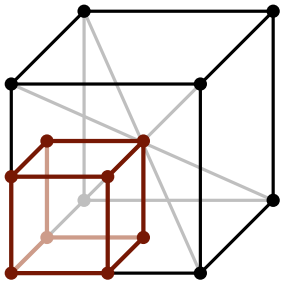
because the vector $[0\ 2\ 0]$ lies in the lattice Π' (it is $v_1 + v_3$). In the terminology of Section 8.1, they are all equivariant points with stabiliser G.

In Figure 8.2b we label the edges of $\mathcal{V}$. There are twelve lines in total, but these make up six equivariant lines. Indeed, a and a' make up the same equivariant line, for R_y reverses them. Notice that a has $H_x = \{ 1, R_x \}$ as its stabiliser, so $\{ a, a' \}$ is the entire orbit of a. A similar consideration applies to the other lines. Notice that in Figure 8.2b we have given these lines an orientation that is compatible with the action of G.

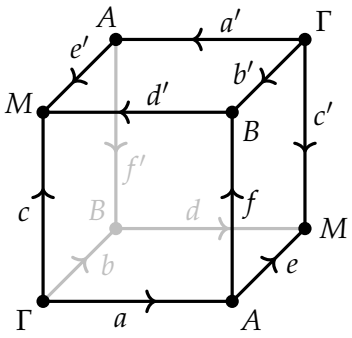
Figure 8.2c labels the faces of $\mathcal{V}$. Notice that all of the faces have trivial stabiliser. The notation is similar to the 1-cells; for instance, α and α' lie in the same orbit since $\alpha' = R_z \cdot \alpha$, so they are part of the same equivariant 2-cell. However, Figure 8.2c does not show $R_x \cdot \alpha$ or $R_y \cdot \alpha$, but it is easy to see where they must lie in Figure 8.2a.

Lastly, $\mathcal{V}$ also has trivial stabiliser. Its orbit therefore consists of four 3-cells, which make up a single equivariant 3-cell. As $\mathcal{V}$ has volume 1, its orbit therefore has total volume 4. By previous comments, this means its orbit is the entire Brillouin zone.

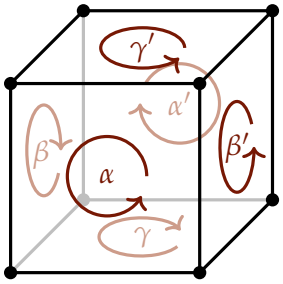
[5] In more mathematical terminology, a primitive unit cell is a *fundamental domain* for the lattice.



(a) The unit cube $\mathcal{V}$, highlighted in red, in the BCC lattice. By letting the point group G act on $\mathcal{V}$, it fills the entire cube depicted here up to lattice vectors from Π'.



(b) The unit cube $\mathcal{V}$ with labelled vertices and edges. Vertices with the same label are the same point up to a lattice vector from Π'. The arrows on the edges indicate their orientation. Edges whose names differ only by a prime (e.g., a and a') lie in the same orbit under the action of the point group G.



(c) The unit cube $\mathcal{V}$ with labelled faces. The arrows on the faces indicate the orientation. Faces whose names differ only by a prime (e.g., α and α') lie in the same orbit under the action of the point group G.

Figure 8.2: The CW structure of the Brillouin zone of $F222$. In total it has four points, twelve lines, twelve faces, and four volumes. These make up four equivariant points, six equivariant lines, three equivariant faces, and one equivariant volume.

8.5. THE CALCULATION

We compute the phases protected by $F222$ in class A, i.e., without any other symmetries. If we write G for the point group of $F222$ and X for the Brillouin zone of $F222$, then from Example 6.13 we see that this amounts to computing $K_G(X)$. The Brillouin zone is isomorphic to the G-equivariant CW complex discussed in Section 8.4.1 (which we will also denote by X). By Theorem 8.7 we can therefore find its G-equivariant K-group by computing the the Bredon cohomology of this CW complex. This is the cohomology of the cochain complex

$$0 \longrightarrow C_G^0(X) \xrightarrow{d} C_G^1(X) \xrightarrow{d} C_G^2(X) \xrightarrow{d} C_G^3(X) \longrightarrow 0. \tag{8.5.1}$$

Notice that $C_G^4(X) = 0$ because X has no 4-cells (X is three-dimensional). All groups and differentials have to be determined in order to find the relevant cohomology groups. We begin with the groups $C_G^n(X)$.

$C_G^0(X)$. We have four G-equivariant points in X, all of which have G as their stabiliser. The representation ring $R(G)$ is $\mathbb{Z}^4$. This means there are four independent options for $f(\Gamma)$ for a 0-cochain, namely $e_i \in \mathbb{Z}^4$ for $i = 1, 2, 3, 4$, and similarly for A, B, and M. We denote by

$$\pi_\Gamma^{(i)} \quad \text{for } i = 1, 2, 3, 4$$

the equivariant 0-cochain that maps Γ to $e_i \in R(G)$, and maps all other 0-cells to $0 \in R(G)$. Thus $C_G^0(X)$ is isomorphic to

$$\langle \pi_\Gamma^{(1)}, \pi_\Gamma^{(2)}, \dots, \pi_A^{(1)}, \dots, \pi_B^{(1)}, \dots, \pi_M^{(1)}, \dots, \pi_M^{(4)} \rangle_{\mathbb{Z}} \cong \mathbb{Z}^{16}.$$

Here $\langle\ \rangle_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-linear span.

$C_G^1(X)$. We have six G-equivariant lines, all with $\mathbb{Z}_2$ as stabiliser (although not the same $\mathbb{Z}_2$ as a subgroup of G). For instance, the line a has stabiliser H_x. Since $R(H_x) \cong \mathbb{Z}^2$, this means there are two independent possibilities to assign to a, namely e_1 and e_2 in $R(H_x)$. We denote by

$$\lambda_a^{(i)} \quad \text{for } i = 1, 2$$

the equivariant 1-cochain that maps a to $e_i \in R(H_x)$, maps the other line in the orbit of a to e_i also, and maps all other 1-cells to zero. Similarly, b has H_y as stabiliser, but now we write

$$\lambda_b^{(i)} \quad \text{for } i = 1, 3$$

for the 1-cochain that maps b to e_1 and e_2 in $R(H_y)$, respectively. This is to reflect the fact that the restriction map $\pi_y \colon R(G) \to R(H_y)$ projects onto the first and *third* component (see Section 8.3). For the other 1-cells we use similar notation, with $i = 1, 4$ for lines with stabiliser H_z. In this way $C_G^1(X)$ is isomorphic to

$$\langle \lambda_a^{(1)}, \lambda_a^{(2)}, \lambda_b^{(1)}, \lambda_b^{(3)}, \dots \rangle_{\mathbb{Z}} \cong \mathbb{Z}^{12}.$$

$\boldsymbol{C^2_G(X)}$**.** We have three G-equivariant faces, all with trivial stabiliser. Write φ_α for the 2-cochain that maps α to $1 \in \mathbb{Z}$, maps all faces in the orbit of α to 1 also, and maps the other faces to 0. We define φ_β and φ_γ similarly. Then we have

$$C^2_G(X) \cong \langle \varphi_\alpha, \varphi_\beta, \varphi_\gamma \rangle_{\mathbb{Z}} \cong \mathbb{Z}^3.$$

$\boldsymbol{C^3_G(X)}$**.** There is only one G-equivariant face, which has trivial stabiliser. Therefore

$$C^3_G(X) \cong \mathbb{Z}.$$

Knowing the groups, we determine the differentials between them.

$\boldsymbol{C^0_G(X) \to C^1_G(X)}$**.** Looking at the orientation given in Figure 8.2b, we see that $\partial a = A - \Gamma$. By the definition of the differential in Equation (8.1.1), this means that

$$(d\pi^{(i)}_\Gamma)(a) = \pi^{(i)}_\Gamma(A)\Big|_{G_a} - \pi^{(i)}_\Gamma(\Gamma)\Big|_{G_a} = -\pi^{(i)}_\Gamma(\Gamma)\Big|_{G_a} = \begin{cases} -(1,0) & i = 1, \\ -(0,1) & i = 2, \\ (0,0) & i = 3,4. \end{cases}$$

Here we used that $G_a = H_x$ and that the restriction map $\pi_x \colon R(G) \to R(H_x)$ is the projection onto the first two components. Similarly we have

$$(d\pi^{(i)}_\Gamma)(b) = \begin{cases} -(1,0) & i = 1, \\ -(0,1) & i = 3, \\ (0,0) & i = 2,4; \end{cases} \qquad (d\pi^{(i)}_\Gamma)(c) = \begin{cases} -(1,0) & i = 1, \\ -(0,1) & i = 4, \\ (0,0) & i = 2,3; \end{cases}$$

and since Γ does not lie on the boundary of d, e or f, we have

$$(d\pi^{(i)}_\Gamma)(d) = (d\pi^{(i)}_\Gamma)(e) = (d\pi^{(i)}_\Gamma)(f) = (0,0).$$

The images of these 1-cells uniquely determine any 1-cochain. This makes us conclude

$$d\pi^{(1)}_\Gamma = -(\lambda^{(1)}_a + \lambda^{(1)}_b + \lambda^{(1)}_c),$$
$$d\pi^{(2)}_\Gamma = -\lambda^{(2)}_a, \qquad d\pi^{(3)}_\Gamma = -\lambda^{(3)}_b, \qquad d\pi^{(4)}_\Gamma = -\lambda^{(4)}_c.$$

In the exact same way we find

$$d\pi^{(1)}_A = \lambda^{(1)}_a - \lambda^{(1)}_e - \lambda^{(1)}_f, \quad d\pi^{(2)}_A = \lambda^{(2)}_a, \quad d\pi^{(3)}_A = -\lambda^{(3)}_e, \quad d\pi^{(4)}_A = -\lambda^{(4)}_f;$$
$$d\pi^{(1)}_B = \lambda^{(1)}_b - \lambda^{(1)}_d + \lambda^{(1)}_f, \quad d\pi^{(2)}_B = -\lambda^{(2)}_d, \quad d\pi^{(3)}_B = \lambda^{(3)}_b, \quad d\pi^{(4)}_B = \lambda^{(4)}_c;$$
$$d\pi^{(1)}_M = \lambda^{(1)}_d + \lambda^{(1)}_e + \lambda^{(1)}_c, \quad d\pi^{(2)}_M = \lambda^{(2)}_d, \quad d\pi^{(3)}_M = \lambda^{(3)}_e, \quad d\pi^{(4)}_M = \lambda^{(4)}_c.$$

$C^1_G(X) \to C^2_G(X)$. Using the orientations indicated in Figure 8.2c, we see that

$$\begin{aligned}\partial\alpha &= a + f + d' - c,\\ \partial\beta &= b - f' + e' - c,\\ \partial\gamma &= a + e - d - b.\end{aligned}$$

Recall that $G_\alpha = \{\,1\,\}$. This means that

$$(d\lambda_a^{(i)})(\alpha) = \lambda_a^{(i)}(a)\Big|_{G_\alpha} = \begin{cases} 1 & i = 1,\\ 0 & i = 2;\end{cases}$$

and analogously

$$(d\lambda_a^{(i)})(\beta) = 0, \qquad (d\lambda_a^{(i)})(\gamma) = \begin{cases} 1 & i = 1,\\ 0 & i = 2.\end{cases}$$

Therefore $d\lambda_a^{(1)} = \varphi_\alpha + \varphi_\gamma$ and $d\lambda_a^{(2)} = 0$. Similarly, for $d\lambda_d^{(i)}$ we find that

$$(d\lambda_d^{(i)})(\alpha) = \lambda_d^{(i)}(d')\Big|_{G_\alpha} = \begin{cases} 1 & i = 1,\\ 0 & i = 2;\end{cases} \qquad (d\lambda_d^{(i)})(\gamma) = \begin{cases} -1 & i = 1,\\ 0 & i = 2.\end{cases}$$

Notice that this first equality relies on Condition (ii) in Definition 8.3, which implies that all 1-cochains assign the same value to d and d'. We conclude that

$$d\lambda_d^{(1)} = \varphi_\alpha - \varphi_\gamma \quad \text{and} \quad d\lambda_d^{(2)} = 0.$$

In the exact same manner we find

$$d\lambda_b^{(1)} = \varphi_\beta - \varphi_\gamma, \quad d\lambda_c^{(1)} = -\varphi_\alpha - \varphi_\beta, \quad d\lambda_e^{(1)} = \varphi_\beta + \varphi_\gamma, \quad d\lambda_f^{(1)} = \varphi_\alpha - \varphi_\beta,$$

and the differential of all other generators are zero.

$C^2_G(X) \to C^3_G(X)$. The volume $\mathcal{V}$ pictured in Figure 8.2 has boundary

$$\alpha - \alpha' + \beta - \beta' + \gamma - \gamma'.$$

Notice that any 2-cochain must have the same value at α as at α' because they lie in the same orbit, and similarly for the other faces. So if φ is a 2-cochain, then $d\varphi$ is zero on α, β, and γ. But this implies that $d\varphi$ is zero. Therefore the differential $C^2_G(X) \to C^3_G(X)$ is the zero map.

$H^0_G(X)$. From the form of the cochain complex in Equation (8.5.1), we see that the zeroth Bredon cohomology group is the kernel of the differential $C^0_G(X) \to C^1_G(X)$. Our computations above show that the image of of this differential is spanned by

$$\begin{gathered}\lambda_a^{(2)}, \lambda_d^{(2)}, \lambda_b^{(3)}, \lambda_e^{(3)}, \lambda_c^{(4)}, \lambda_f^{(4)},\\ \lambda_a^{(1)} + \lambda_b^{(1)} + \lambda_c^{(1)}, \quad \lambda_d^{(1)} + \lambda_e^{(1)} + \lambda_f^{(1)}, \quad \lambda_a^{(1)} - \lambda_e^{(1)} - \lambda_f^{(1)}.\end{gathered}$$

These elements are also independent over $\mathbb{Z}$. Therefore the image of the differential is isomorphic to $\mathbb{Z}^9$. Previously we argued that $C^0_G(X) \cong \mathbb{Z}^{16}$. Because $\mathbb{Z}$ is a free group, this means that the kernel of this differential is isomorphic to $\mathbb{Z}^{16-9} = \mathbb{Z}^7$, so

$$H^0_G(X) \cong \mathbb{Z}^7.$$

$\boldsymbol{H^2_G(X)}$. Because the differential $C^2_G(X) \to C^3_G(X)$ is zero, we have

$$H^2_G(X) = \frac{C^2_G(X)}{\mathrm{Im}(d\colon C^1_G(X) \to C^2_G(X))}.$$

Abbreviate $I := \mathrm{Im}(d\colon C^1_G(X) \to C^2_G(X))$. Above we argued that $C^2_G(X) \cong \mathbb{Z}^3$ with $\varphi_\alpha, \varphi_\beta, \varphi_\gamma$ as generators. Under this isomorphism, we see from our earlier calculations that I is spanned over $\mathbb{Z}$ by

$$(1,0,1), \quad (0,1,-1), \quad (1,0,-1), \quad (0,1,1), \quad (1,-1,0).$$

Notice that $T := \langle (2,0,0), (0,2,0), (0,0,2) \rangle_{\mathbb{Z}}$ is therefore contained in I. The quotient $H^2_G(X)$ therefore factorises through T by the third isomorphism theorem:

$$\frac{\mathbb{Z}^3}{I} \cong \frac{\mathbb{Z}^3/T}{I/T} \cong \frac{\mathbb{Z}_2^3}{\langle (\bar{1},\bar{1},\bar{0}),\ (\bar{1},\bar{0},\bar{1}) \rangle_{\mathbb{Z}_2}}.$$

Notice that the group homomorphism

$$f\colon \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2\colon \quad (a,b,c) \longmapsto a+b+c$$

is surjective and has $\langle (\bar{1},\bar{1},\bar{0}),\ (\bar{1},\bar{0},\bar{1}) \rangle_{\mathbb{Z}_2}$ as its kernel. The first isomorphism theorem therefore implies

$$H^2_G(X) \cong \mathbb{Z}_2.$$

By Theorem 8.7, we have thus proved the following.

Theorem 8.9 (*F*222, class A, [38]). *Write G for the point group of F222, and write X for the Brillouin zone of F222. Then we have an isomorphism*

$$K_G(X) \cong \mathbb{Z}^7 \oplus \mathbb{Z}_2.$$

8.6. Discussion

As we noted at the beginning of this chapter, McAlister [32] found the group $\mathbb{Z}^7$ instead. Notable is the fact that he uses C*-algebras and their K-theory to find this result. We do not discuss this approach in this work, and hence must leave an analysis of this disagreement to future work.

8.6.1. The nontrivial phase

Our approach does not merely give us the resulting K-group, but also provides some insight into what vector bundles represent the nontrivial element of this $\mathbb{Z}_2$ part. Physically this is interesting because it provides a starting point for further describing this phase. To describe it, consider the quotient map

$$C^2_G(X) \longrightarrow H^2_G(X).$$

We argued above why $C_G^2(X) \cong \mathbb{Z}^3$. The three components here describe something about the vector bundles on the faces α, β and γ, respectively. To properly understand what they describe, we use the take a closer look at the description given by Stehouwer et al. [40, App. A.3].

Notice that, because these faces have trivial stabilisers, we no longer have to deal with the group G when restricting to one of these faces. Using the theory from Section 1.5, one may find that

$$K(\mathbb{S}^2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The first component describes the dimension; the second component describes the *degree* of vector bundles over the sphere (see, e.g., Hatcher [17, pp. 22–24] for a further explanation). Now notice that a vector bundle over one of the faces α, β and γ yields a vector bundle over the sphere $\mathbb{S}^2$ by collapsing the boundary of the face to a single point.] In this fashion we can talk about the degree of a vector bundle over one of these faces. As explained by Stehouwer et al. [40, App. A.3], the components of $\mathbb{Z}^3 \cong C_G^2(X)$ give the degrees of the vector bundles over α, β and γ.

Under the above isomorphisms, the quotient map $C_G^2(X) \to H_G^2(X)$ is

$$\mathbb{Z}^3 \longrightarrow \mathbb{Z}_2 \colon \quad (a, b, c) \longmapsto \overline{a + b + c}.$$

Thus, in order to be in the nontrivial phase, the sum of these degrees should be odd.

8.6.2. Possible cause of torsion

It has been suggested that the torsion arising in this K-group is due to $\mathbb{RP}^2$ appearing in the Brillouin zone. We do not verify this, but let us briefly summarise the results of Shiozaki et al. [38, p. 25]. They state that the boundary of the volume $\mathcal{V}$ has a boundary homeomorphic to $\mathbb{RP}^2$. They then use this to give a formula for the $\mathbb{Z}_2$-invariant in this case, which in our labelling from Figure 8.2 is

$$(-1)^\nu = \exp\left(\int_{a+e-c'} \operatorname{Tr} \mathcal{A} - \tfrac{1}{2} \int_{\beta-\alpha-\gamma} \operatorname{Tr} \mathcal{F} \right),$$

with $\nu \in \mathbb{Z}_2 = \{0, 1\}$ the invariant, and with $\mathcal{A}$ and $\mathcal{F}$ the Berry connection and its curvature, respectively.

CONCLUSION

We have described the basics of topological K-theory, as well as the twisted variant introduced by Freed and Moore. We discussed the basics of topological insulators, focusing on how K-theory naturally appears in the study of the free-fermion version. This is summed up in the main result of Freed and Moore, that (reduced) topological phases are classified by a suitable twisted K-group. Finally, we have applied this theory in numerous cases, which is the most important part of this thesis.

ONE DIMENSION

In Chapter 7 we classified all one-dimensional topological phases. At the base of this classification lies the computation of the twisted K-theories of a point. This we did by using the (twisted) group algebras and (twisted) representation rings outlined in Section 5.4. Note that these K-groups are simply zero-dimensional topological phases. Thus the theory outlined in this thesis on twisted representation rings suffices to compute zero-dimensional topological phases.

To go from zero-dimensional to one-dimensional topological phases, an additional tool is required. We took this tool to be the Mayer–Vietoris exact sequence, except when a suspension argument would provide a shortcut. The only difficulties that arise in this method is the need to compute the maps in the sequence. By using twisted group algebras we resolved this difficulty also. Thus we have illustrated how the Mayer–Vietoris exact sequence suffices to compute these one-dimensional topological phases. More strongly, for the circle, the Mayer–Vietoris approach coincides with the spectral sequence approach.

There is still room for future work in this area. On the mathematical side, one could consider generalising the assumptions we listed in Section 7.1. Physically it would be interesting to compare the phases we found with previously found results, be they theoretical or experimental. But more importantly, one should find a way to explicitly calculate what phase a given system is in. We outlined such a method for the IQHE in Section 6.1, which was the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}^2}\sum_{\alpha}\mathcal{F}_{\alpha},$$

with α running over the valence bands, and $\mathcal{F}_{\alpha}$ being the curvature of the Berry connection of the valence band α. When similar expressions are found for the (strong) one-dimensional topological invariants, our results can be compared to the existing condensed matter literature much more effectively. Additionally, one could try to find

explicit Hamiltonians that represent these different phases. For Kitaev's periodic table this can be done (see, e.g., Kruthoff [27, §2.1.2]), but for Freed and Moore's classification no general procedure exists.

THREE DIMENSIONS

In Chapter 8 we gave a more detailed version of the calculation done by Shiozaki et al. [38] about $F222$. We found that their result, the group $\mathbb{Z}^7 \oplus \mathbb{Z}_2$, is indeed correct. For this we used the Atiyah–Hirzebruch spectral sequence, which for this particular case reduces to Bredon cohomology. This makes our calculation uninteresting as far as spectral sequence are concerned: in this case (3D in class A) it always gives a unique answer. In classes with time-reversal symmetry (i.e., AI and AII) this is already no longer the case, even in two dimensions, as discussed in detail by Stehouwer et al. [40].

This method of computation has the advantage of giving some insight into what vector bundle represents the nontrivial phases. In Section 8.6.1 we briefly described this vector bundle. We then briefly commented on the formula that Shiozaki et al. [38] found to detect if a system is in this nontrivial phase, but we did not verify it, nor connect it with our description of the phase.

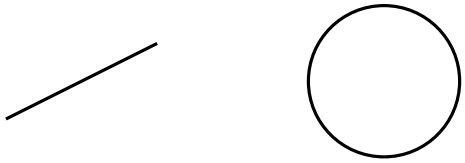
POPULAR SUMMARY (DUTCH)

In 1980 werd een experiment gedaan waarbij men een totaal nieuwe soort isolator ontdekte. Het is een soort isolator die je niet in het alledaagse leven tegen zal komen: om het te maken moest er gewerkt worden bij bijna 0 K. Maar nutteloos is het ook niet: lange tijd werd het gebruikt om natuurconstanten heel precies te meten, en ze blijken nu zelfs nuttig te zijn voor het maken van quantum computers. Deze isolatoren worden ook wel *topologische fasen* genoemd. Waar deze naam vandaan komt is het makkelijkst te begrijpen als we snappen wat een *fase* is.

Fase-overgangen komen op vele plaatsen tevoorschijn in de natuurkunde. Het simpelste voorbeeld is dat van water. Water heeft drie fasen: vast, vloeibaar, en gas. Als je water genoeg verhit (of afkoelt) kan je het water van de ene fase naar de andere over laten gaan. Zo'n overgang noemen we een *fase-overgang*. Verhitten of afkoelen zal niet altijd een fase-overgang veroorzaken: immers, water van 10 °C naar 20 °C verhitten zal niet zo veel veranderen. Je weet dat je een fase-overgang hebt gemaakt als er fundamenteel iets verandert.

Het blijkt dat zoiets ook kan gebeuren tussen isolatoren. In het experiment van 1980 werd een isolator gemaakt die op zijn rand altijd geleidt, ook al isoleert de binnenkant. Hier stopte het vreemde gedrag niet: als de isolator in twee stukken zou worden gehakt, zou het oppervlak dat je daarmee maakt ook spontaan gaan geleiden! Dit deed natuurkundigen denken aan een fase-overgang: als je van deze vreemde isolator naar een 'gewone' isolator overgaat, dan gebeurt er op de rand (de plek van overgang) iets vreemds. We zouden daarom kunnen zeggen dat deze vreemde isolator zich in een andere 'fase' bevindt, om dezelfde reden waarom we zeggen dat gas een andere fase is dan vloeibaar: om van de ene naar de andere over te gaan, moet er iets geks gebeuren. In plaats van het woord 'fase' te gebruiken voor deze isolator, zeggen we dat de isolator zich in een andere *topologische fase* bevindt.

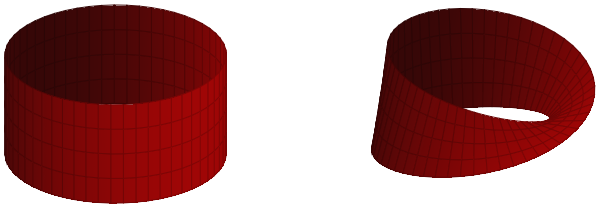
Het woord 'topologie' komt hier niet zomaar uit de lucht vallen. De topologie is een vakgebied in de wiskunde, wat de term 'topologische isolatoren' apart doet klinken voor wiskundigen. In de topologie is men geïnteresseerd of objecten naar elkaar omgevormd kunnen worden zonder iets te scheuren of te plakken. Bijvoorbeeld, kan je een lijn omvormen tot een cirkel, enkel door te buigen of uit te rekken, zonder te plakken?



Waarschijnlijk denk je "nee," en dat is ook het juiste antwoord. Echter, dit antwoord is

enkel gebaseerd op gevoel, wat een wiskundige niet een voldoende antwoord vindt. Hij wil een wiskundig bewijs zien. In één klap maakt dit de vraag veel, veel lastiger. Men heeft vele manieren bedacht om deze variant van de vraag te beantwoorden.

Een van deze manieren staat nu bekend als *K-theorie*. Deze theorie beantwoordt een iets specifiekere vraag dan de algemene vraag uit topologie. Namelijk, K-theorie is het stuk gereedschap dat je nodig hebt om een wiskundige te kunnen overtuigen dat een cilinder niet hetzelfde is als een *Möbius band*:



Dit laat zien dat topologie, en zo ook K-theorie, een behoorlijk abstract vakgebied is. Desalniettemin is K-theorie (een deelgebied van de topologie) de reden voor het woord 'topologie' in de naam 'topologische fasen'. Om een lang verhaal kort te maken: het blijkt dat K-theorie gebruikt kan worden om uit te rekenen *hoeveel* topologische fasen er zijn. Dit blijkt namelijk gecompliceerder te zijn dan de fasen van water. Water heeft drie fasen, onafhankelijk van de situatie waarin je je bevindt; voor topologische fasen werkt dit anders. Bijvoorbeeld, als je een sterke magneet aanzet, dan zal het aantal topologische fasen drastisch veranderen. Een magneetveld is niet het enige dat het aantal fasen doet veranderen, maar de andere situaties zijn lastiger uit te leggen. Het blijkt dat er 10 verschillende situaties zijn die het aantal fasen kunnen veranderen. Verder hangt het aantal ook af van de dimensie van het materiaal, oftewel: is het een plat vlak (twee dimensies), is het ruimtelijk (drie dimensies), of is het een lijn (één dimensie)?

Deze verschillende mogelijkheden kunnen we in een tabel zetten, welke men heeft ingevuld door K-theorie te gebruiken. De resulterende tabel werd bekend als het *periodieke systeem van topologische fasen*.

	0	1	2	3	4	5	6	7
A	∞	1	∞	1	∞	1	∞	1
AIII	1	∞	1	∞	1	∞	1	∞
AI	∞	1	1	1	∞	1	2	2
BDI	2	∞	1	1	1	∞	1	2
D	2	2	∞	1	1	1	∞	1
DIII	1	2	2	∞	1	1	1	∞
AII	∞	1	2	2	∞	1	1	1
CII	1	∞	1	2	2	∞	1	1
C	1	1	∞	1	2	2	∞	1
CI	1	1	1	∞	1	2	2	∞

Een vakje geeft het aantal topologische fasen aan in een bepaalde dimensie (gegeven door de kolom), en in een van de 10 gevallen die we eerder noemden (gegeven door de rij). In de gevallen waar een 1 staat, kunnen we het topologische gedrag dus niet zien: er is maar één fase, en dus kunnen er geen fase-overgangen plaatsvinden, dus zal er ook geen geleiding ontstaan! Opvallend is dat veel vakjes *oneindig veel* fasen hebben: dit laat weer zien dat topologische fasen drastisch anders zijn dan fasen uit het gewone leven.

Helaas is het periodieke systeem niet het volledige verhaal. Recentelijk hebben natuurkundigen ontdekt dat er vakjes missen, en wel ongelofelijk veel. Wat nog erger is, is dat de getallen die in deze vakjes moeten staan veel lastiger uit te rekenen zijn dan de vakjes in het periodieke systeem. Er zijn honderden, nee, duizenden vakjes die missen, en voor bijna al deze vakjes is de K-theorie nog te moeilijk om het ook echt uit te rekenen. De tijd zal leren of we ooit beter worden in K-theorie om dit probleem op te lossen.

In deze scriptie bekijk ik waarom zoveel vakjes hier missen, en belangrijker nog, welke van deze vakjes we *wel* kunnen uitrekenen, en hoe! Mijn uiteindelijke doel is dan om een aantal vakjes die nog leeg zijn uit te rekenen. Zo blijkt de kolom voor één dimensie niet uit 10 vakjes te bestaan, maar eigenlijk uit 30. Er missen dus 20 vakjes, die (naar het schijnt) nog niet berekend waren. Ietwat versimpeld komt hier de volgende tabel uit; merk op dat de rij die met "$R^2 = 0$" is aangegeven hetzelfde is als de kolom voor één dimensie uit het periodieke systeem.

	A	$AIII$	AI	BDI	D	$DIII$	AII	CII	C	CI
$R^2 = \ \ 0$	1	∞	1	∞	2	2	1	∞	1	1
$R^2 = +1$	1	1	1	1	1	1	1	1	1	1
$R^2 = -1$	1	1	1	1	1	2	2	1	1	1

Verder bekijk ik één specifiek geval in drie dimensies, een vakje dat niet terug te vinden is in het periodieke systeem. Als je dit uitrekent, wat in totaal zo'n 12 pagina's kost, vind je dat er 2 topologische fasen zijn. Dit klinkt niet heel spannend, maar voor de experts is dit verbazend: lange tijd dacht met dat er een 1 moest staan, maar blijkbaar niet!

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