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A computation of K-theory cochains and its applications

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Abstract

Let KU denote the $\mathbf{E}_\infty$-ring spectrum of complex K-theory. Let p be an arbitrary prime number, and let KU_p denote the p-completion of KU. Let X be a pointed space whose K-theory $KU_p^*(X)$ is an exterior algebra on a finite number of odd generators. (This includes odd spheres and many H-spaces.) We give a presentation for the KU_p-algebra spectrum $KU_p^{X_+}$ as the cofibre of a map between two ($K(1)$-localised) symmetric KU_p-algebra spectra. Combined with previous work of Bousfield, we use this result to show that if p is odd and X satisfies some additional conditions, then the algebra $\mathbf{S}_{K(1)}^{X_+}$ models the v_1-periodic homotopy type of X. More precisely, we show that the Behrens–Rezk comparison map from the Bousfield–Kuhn functor $\Phi_1 X$ to the $K(1)$-local TAQ-cohomology of $\mathbf{S}_{K(1)}^{X_+}$ is an equivalence.

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Conventions and notation

If p is a prime number, then $\mathbf{Z}/p$ denotes the ring of integers modulo p, and $\mathbf{Z}_p$ denotes the ring of p-adic integers.

The term *graded* means $\mathbf{Z}/2$-graded, unless explicitly stated otherwise. When a $\mathbf{Z}/2$-graded or $\mathbf{Z}$-graded ring is called *commutative*, it is always meant that it is *graded-commutative* (i.e., obeys the Koszul sign rule).

If R is a commutative ring and M an R-module, then $\mathrm{Sym}_R(M)$ and $\Lambda_R(M)$ denote the symmetric and exterior R-algebra on M, respectively.

We work with ∞-categories as our formalism for homotopy theory. Limits and colimits should always be understood as *homotopy* limits and colimits, unless explicitly stated otherwise. The ∞-category of spaces (respectively pointed spaces) is denoted by $\mathscr{S}$ (respectively $\mathscr{S}_*$), and the ∞-category of spectra is denoted by $\mathbf{Sp}$. If $\mathcal{C}$ is an ∞-category, its homotopy category is denoted by $\mathrm{h}\,\mathcal{C}$.

By abuse of notation, we think of every 1-category $\mathcal{C}$ as being an ∞-category by identifying $\mathcal{C}$ with its nerve $N(\mathcal{C})$.

The wedge sum and smash product of spectra are denoted by $\oplus$ and $\otimes$, respectively. The sphere spectrum is denoted by $\mathbf{S}$. The Spanier–Whitehead dual of a spectrum E is denoted by $E^\vee$.

The spectrum of complex K-theory is denoted by KU. If p is a prime number, then $KU_{(p)}$ and KU_p denote the p-localisation and p-completion of KU, respectively. Unless explicitly stated otherwise, the K-theory of a space or spectrum is viewed as a $\mathbf{Z}/2$-graded abelian group by recording only the groups in degrees 0 and 1.

By a *commutative ring spectrum*, we mean an $\mathbf{E}_\infty$*-ring spectrum* (and likewise for algebra spectra). All ring spectra appearing in this work are commutative.

If R is a commutative ring spectrum and M an R-module spectrum, then $\mathrm{Sym}_R(M)$ denotes the symmetric R-algebra spectrum on M.

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Introduction

Algebraic topology studies spaces by assigning algebraic invariants to them. Since the inception of this idea, many invariants have been defined and studied. They all seem to obey the same law: there is a trade-off between how computable an invariant is, and how powerful it is. The homotopy groups are the most important invariant of spaces. They are also among the hardest invariants to compute: even now we only know a small range of the homotopy groups of spheres, and rare are the spaces of which all homotopy groups are known.

One can try to ease the situation by studying only parts of the homotopy groups. Studying the easiest part, the free part, is tantamount to studying the *rational homotopy groups* of a pointed space X: the groups $\pi_*(X) \otimes \mathbf{Q}$. (This tensor product is well-defined if X is, e.g., simply-connected.) Unlike their torsion-infested originals, the rational homotopy groups are very amenable to computations. Using what is now known as the *Serre spectral sequence*, Serre [Ser51] first computed these groups for many spaces, including all spheres. Out of the seminal works of Quillen [Qui69] and Sullivan [Sul77], the field of *rational homotopy theory* was born, which gave an even more powerful framework for computing these groups. In the past decade, torsion-sensitive generalisations of rational homotopy theory have been developed. This thesis is essentially a computation using one of these generalisations. Before we can discuss those results, we need a little familiarity with rational homotopy theory.

Rational homotopy theory

If X is a space, then the rational cohomology groups $H^*(X;\mathbf{Q})$ carry a ring structure given by the cup product of cochains, making $H^*(X;\mathbf{Q})$ a graded-commutative ring. The cup product already lives on the cochain complex $C^\bullet(X;\mathbf{Q})$, but there it is not graded-commutative. However, this is only a minor defect of $C^\bullet(X;\mathbf{Q})$, for we do not care about $C^\bullet(X;\mathbf{Q})$ up to isomorphism, but only up to quasi-isomorphism (i.e., up to maps that induce isomorphisms on cohomology). Up to quasi-isomorphism, one can choose a model for $C^\bullet(X;\mathbf{Q})$ where the cup product does become graded-commutative (this can even be chosen functorially). In this way, the rational cochains have the structure of a *commutative differential graded algebra*, or *cdga* for short.

The rational cochains are very computable, at least if we work up to quasi-isomorphism. As a simple example, consider the k-sphere S^k, for $k \geqslant 1$. The cohomology of S^k is concentrated in degrees 0 and k. Let $A_k^\bullet$ denote the cochain complex with $\mathbf{Q}$ in degrees 0 and k, having the zero differential, and with the zero multiplication on elements of degree k. Then $C^\bullet(S^k;\mathbf{Q})$ is quasi-isomorphic to $A_k^\bullet$. In general, if the rational cohomology groups of a space are known, one can usually extract a good enough model (where 'good enough' depends on the context) for the rational cochains from this information.

The surprising feature of the rational cochains is that they also hold a lot of information. If X is a finite space (i.e., it is equivalent to a finite CW complex), then the cdga $C^\bullet(X;\mathbf{Q})$ captures essentially all homotopy-theoretic behaviour of X up to torsion phenomena. To make this precise, we need some terminology. For simplicity, let us work with simply-connected pointed spaces only. A map $f\colon X \to Y$ of simply-connected pointed spaces is called a *rational equivalence* if it induces an isomorphism on rational homotopy groups. An *augmentation* of a rational cochain complex $C^\bullet$ is a map $C^\bullet \to \mathbf{Q}$ to the cochain complex with only $\mathbf{Q}$ in degree 0. A choice of basepoint on X induces an augmentation of $C^\bullet(X;\mathbf{Q})$. The rational cochains set up an equivalence of categories

$$\mathbf{Top}_*^{\geqslant 2,\text{ fin}} \Big/ {\scriptstyle\substack{\text{rational}\\ \text{equivalence}}} \xrightarrow{\ \simeq\ } \{\,\text{augmented cdga's over } \mathbf{Q}\,\}^{\geqslant 2,\text{ fin}} \Big/ {\scriptstyle\substack{\text{quasi-}\\ \text{isomorphism.}}}$$

On both sides, the superscript '$\geqslant 2$' indicates simply-connected, and 'fin' indicates finite type. (A better formulation would be to say that we have an equivalence of homotopy theories.) The surprising feature of this equivalence is that the right-hand side is completely algebraic, even though one usually thinks of algebraic invariants as capturing much less information than do topological phenomena. The concreteness of this algebraic model gives it great computational power.

If X is a finite simply-connected pointed space, then the rational homotopy groups of X can be extracted from the rational cochains. Roughly speaking, there is an isomorphism between the $\mathbf{Q}$-linear dual of $\pi_k(X)\otimes\mathbf{Q}$ and the module of *derived indecomposables* of degree k of $C^\bullet(X;\mathbf{Q})$. The derived indecomposables take some care to define, because this notion has to be invariant under quasi-isomorphism. But once this has been done, computing them is again a very manageable and concrete matter.

The torsion-sensitive generalisation we are after uses topological K-theory in place of rational cohomology. This is the cohomology theory KU that sends a (compact) space X to the Grothendieck group of complex vector bundles over X (i.e., the group obtained by adding formal additive inverses for vector bundles). One can extend the definition of K-theory to have groups KU^n in degrees indexed by the integers $n \in \mathbf{Z}$. Note that this definition does not make use of cochain complexes, so it is unclear how we should proceed. It turns out that there is an object that plays the same role for K-theory as does $C^\bullet(-;\mathbf{Q})$ for rational cohomology, but it is not an algebraic object. Rather, it is an object from *higher algebra*.

Spectra and higher algebra

Higher algebra is the study of the type of algebraic structures that arise when using higher categories instead of ordinary categories. The role that abelian groups play in ordinary algebra is now played by *spectra*. For the purposes of this introduction, a spectrum is essentially a representing object for a cohomology theory. Consider rational cohomology $H^n(-;\mathbf{Q})$, which is represented by the Eilenberg–MacLane space $K(\mathbf{Q}, n)$: for X a pointed space,

$$\widetilde{H}^n(X;\mathbf{Q}) \cong [X,\ K(\mathbf{Q}, n)]_*.$$

The collection $\{\, K(\mathbf{Q}, n) \mid n \geqslant 0 \,\}$ of all representing spaces constitutes the spectrum underlying rational cohomology. Roughly speaking, one can think of this spectrum as a space with an abelian group structure up to homotopy, since the set of homotopy classes of maps from X to $K(\mathbf{Q}, n)$ has a natural abelian group structure. It turns out that any cohomology theory (i.e., a functor satisfying axioms similar to singular cohomology) has such a spectrum representing it. For example, the functor of complex K-theory KU is represented by $\mathbf{Z} \times BU$, with BU denoting the classifying space for the infinite unitary group. The functors KU^n for $n \in \mathbf{Z}$ are also representable.

Rational cohomology and K-theory are cohomology theories with a special property: they both have a graded-commutative ring structure. For rational cohomology this is given by the cup product; for K-theory this is given by the tensor product of vector bundles. One might wonder whether this transfers into a type of ring structure on the spectrum representing it. For these theories, it turns out that it does, and in fact the multiplication on the spectrum has a very rich structure: it is commutative up to *coherent homotopy*. This means that not only is it commutative up to homotopy, but moreover that the homotopies witnessing the commutativity are themselves unique up to specified homotopy, and those homotopies are unique up to specified homotopy, etc., ad infinitum. Spectra with such a ring structure are often called $\mathbf{E}_\infty$-*ring spectra*, though we will resort to calling them *commutative ring spectra*.

Compared to ordinary algebra, higher algebraic objects are built entirely out of topological spaces, and the theory hardly seems algebraic at all. It is a remarkable fact that there is a theory of ring spectra that closely parallels the theory of ordinary rings. For instance, one can speak of modules over a ring spectrum (which will be spectra with an 'action' of the ring spectrum), and study exact sequences of module spectra. Operations like the direct sum and the tensor product exist in this world, and satisfy many similar-looking properties. One can also localise or complete commutative ring spectra at a prime number p. There is even an analogue of the abelian group $\mathbf{Z}$: the *sphere spectrum* $\mathbf{S}$. It plays many similar roles: for example, a module over $\mathbf{S}$ is the same as a spectrum. A monumental treatise on higher algebra is given by Lurie [HA].

Using higher algebra, one can (as promised) construct an analogue of the cochain complex $C^\bullet(-;\mathbf{Q})$ for K-theory (or even for any commutative ring spectrum). Given a space X, one can define the *K-theory cochains* KU^{X_+}, which is a commutative algebra over the commutative ring spectrum KU. This algebra has the property that its homotopy groups are the K-theory of X:

$$\pi_* KU^{X_+} \cong KU^*(X).$$

This is a K-theoretic analogue of the rational cohomology groups $H^*(X;\mathbf{Q})$ being the cohomology groups of the cochain complex $C^\bullet(X;\mathbf{Q})$.

As with any invariant, there are two questions we should ask: how computable are these cochains, and how much information about the space do they retain? The first result of this thesis is geared to answering the first question: we compute this algebra (or rather, its p-completion) for many X. Before discussing this in more detail, let us discuss the answer to the second question.

Chromatic homotopy theory

The rational homotopy groups are the first (or rather, zeroth) in a long list of approximations to the homotopy groups. If p is a prime number, and $n \geqslant 0$ is a natural number called the *height*, chromatic homotopy theory defines the *v_n-periodic homotopy groups* of a space pointed X, denoted by $v_n^{-1}\pi_*(X)$. (The prime p is left implicit in the notation.) Taking $n = 0$ yields the rational homotopy groups. If $n > 0$, then these groups are periodic with a period depending on n and p. (One can think of these periods as different 'wavelengths', whence the name *chromatic*.) These groups detect torsion in the homotopy groups. It turns out that the v_n-periodic homotopy groups of a space X can be viewed as the homotopy groups of a spectrum $\Phi_n X$. Studying this spectrum gives a more rigid approach to studying the v_n-periodic homotopy groups.

Sadly, the complexity of computing v_n-periodic homotopy groups increases extremely quickly as n increases. The case $n = 1$ is the most amenable to computation; much less is known in the case $n = 2$, and very little for higher n. Among others, Bousfield has studied v_1-periodic homotopy theory. In [Bou99], working over an odd prime, he computed $v_1^{-1}\pi_*(X)$ when X is an odd-dimensional sphere (from this calculation, the case of even spheres quickly follows), and when X is a finite H-space subject to a few conditions. This computation applies in particular to all simply-connected compact Lie groups.

In more recent years, the approach of rational homotopy theory has been generalised to these v_n-periodic homotopy groups, but only to a certain extent. Behrens and Rezk [BR20b] constructed, for X a pointed space, a *comparison map* from $\Phi_n X$ to the 'indecomposables' of a cochain algebra on X. Specifically, they consider cochains $\mathbf{S}_{K(n)}^{X_+}$ with $\mathbf{S}_{K(n)}$ the "$K(n)$-local sphere spectrum". At height $n = 1$ this is very closely related to K-theory: the algebras KU^{X_+} and $\mathbf{S}_{K(1)}^{X_+}$ give more or less the same information (see the section *Results* below for a more precise statement). The precise name for the indecomposables of a commutative ring spectrum is its *topological André–Quillen cohomology* (or *TAQ-cohomology* for short).

The crucial difference with rational homotopy theory is that this map is *not* an equivalence for all X. As such, it does not always provide a way to compute $\Phi_n X$ from the 'indecomposables' of $\mathbf{S}_{K(n)}^{X_+}$. Behrens and Rezk identify a technical condition (termed *Φ_n-goodness*), which is equivalent to this map's being an equivalence (if X satisfies a finiteness condition). If the comparison map is an equivalence, then we think of the cochains $\mathbf{S}_{K(n)}^{X_+}$ as being a 'good model' for the v_n-periodic homotopy of X. It in particular means that the v_n-periodic homotopy groups of X can be extracted from the cochains on X.

Work has been done to determine when this comparison map is an equivalence. Behrens and Rezk [BR20b] prove that (for arbitrary n) it is an equivalence if X is a sphere. Kjaer [Kja19] works at height 1 and uses the aforementioned computation done by Bousfield to conclude that it is an equivalence on the spaces studied by Bousfield. However, the existing literature does not compute the indecomposables of the cochains on X by first understanding the cochains on X; rather they use technices to get the indecomposables directly from purely algebraic data. Yet these cochain algebras are interesting algebra spectra, and having a better understanding of them would be a worthwhile higher-algebraic result in and of itself.

In conclusion, the cochain algebras of higher algebra can detect certain torsion in the homotopy groups of a space, but this works best if the space satisfies a technical condition. The literature does not tend to compute these cochain algebras in this context. Doing so can shed light on how much information about v_n-periodic homotopy theory they carry.

Remark. There is a second side to rational homotopy theory that we have ignored so far, one that uses *differential graded Lie algebras* in the place of cdga's. This approach does generalise to an equivalence between v_n-periodic homotopy theory and a higher-algebraic analogue of Lie algebras, namely *spectral Lie algebras*. This was recently proved by Heuts [Heu21]. While spectral Lie algebras, unlike algebra spectra, thus give the 'correct model' for v_n-periodic homotopy theory, working with them is a challenge of its own and deserves a separate thesis. As our aim is to understand cochain algebra spectra, we do not discuss spectral Lie algebras in this work.

Results

This thesis contains two main results. Fixing an arbitrary prime p, let KU_p denote the p-completion of the commutative ring spectrum KU. Roughly speaking, let X be a space such that the ring $KU_p^*(X)$ is an exterior algebra on a finite number of odd generators. (This includes many H-spaces, among which all simply-connected compact Lie groups, and in fact includes all spaces studied by Bousfield in [Bou99].) We then compute the KU_p-cochain algebra $KU_p^{X_+}$ by giving a presentation of this algebra as the quotient of a 'free' KU_p-algebra.

This computation has applications to chromatic homotopy theory. Specifically, let p be an odd prime. Combining Bousfield's computation of $v_1^{-1}\pi_*(X)$ with the computation of $KU_p^{X_+}$, we can derive that the Behrens–Rezk comparison map is an equivalence on such X. We do this by computing the indecomposables of $KU_p^{X_+}$. By a method called *descent*, the algebra $\mathbf{S}_{K(1)}^{X_+}$ can be recovered from $KU_p^{X_+}$ (if p is odd and X satisfies a finiteness condition). We then show that the comparison map is an equivalence.

Outline

This thesis is divided into two parts. The first part is devoted to the computation of the KU_p-cochains, and the second to the implications of this computation to chromatic homotopy theory. Chapters on background information (Chapters 1, 2, and 4, and Appendix A) contain few proofs, often resorting to references to the literature. Chapters 3 and 5 contain original work. We work in an ∞-categorical setting, but this is mostly preferential, and our results do not make heavy use of this formulation of homotopy theory.

Chapter 1 discusses higher algebra in an ∞-categorical formulation, in order to make the foundations for the rest of this work explicit. This includes a definition of spectra, commutative ring spectra and their modules, and the theory of p-localisation and p-completion in higher algebra. This is by far the longest chapter, but may be treated as a reference work for the later chapters, instead of being read linearly from beginning to end. Appendix A concerns a variant of p-completion that arises naturally in higher algebra. This is briefly explained in the main text, but the reader may glance at the appendix if more background is desired. Chapter 2 focuses on K-theory and its cohomology operations known as *Adams operations*. The structure of a ring 'with Adams operations' is a called a *θ-algebra*. Understanding K-theory as a θ-algebra proves to be crucial for our main computation. Chapter 3 gives the promised computation of KU_p-cochains $KU_p^{X_+}$ on certain X, and afterwards discusses the question of how general the assumptions on X are. This discussion is a summary of results of Bousfield on the classification of bialgebras 'with Adams operations'. We end Chapter 3 with a short list of questions that naturally flow from this discussion.

Chapter 4 gives a brief introduction to v_1-periodic homotopy theory. Notably, this chapter discusses v_1-periodic homotopy groups, the functor Φ_1, the descent method, and the comparison map. Chapter 5 begins with a review of the height 1 computations done by Bousfield [Bou99], and then combines this with the result from Chapter 3 to show that the comparison map is an equivalence for the relevant spaces.

Prerequisites

This thesis is aimed at readers with a solid understanding of basic algebraic topology. We assume the reader is familiar with the language of ∞-categories (meaning *quasi-categories*) in the sense of Lurie [HTT]. Familiarity with K-theory is heavily recommended, although very brief reminders about the definition of K-theory and its cohomology operations are given. Previous familiarity with spectra is also recommended. Spectral sequences are lightly used in several places. Lastly, in order to properly grasp higher algebra, the reader will find a knowledge of commutative algebra (in the ordinary non-topological sense) to be very beneficial. In particular, we assume that the reader is comfortable with p-localisation and p-completion of commutative rings and modules.

1 Higher algebra

The goal of this chapter is to flesh out the sketch of commutative ring spectra that we gave in the Introduction. Developing this from the ground up would be way beyond the scope of this thesis, so instead we record the main definitions and theorems. As such, this chapter is (for a large part) a collection of results to be used in later chapters. We draw heavily from the monumental work of Lurie [HA], sometimes following the more condensed survey by Gepner [Gep19]. Readers new to the subject may also benefit from a recent course given by Nardin [Nar21].

We begin in §1.1 by recalling some basic terminology in ∞-categories that will play a special role. In §1.2 we define spectra and focus on two main features: first, that the ∞-category of spectra is a *stable* ∞-category, and second, that spectra represent (co)homology theories. In order to discuss commutative ring spectra, we need a formalism to keep track of the higher coherences. This is the formalism of *symmetric monoidal ∞-categories*; we discuss this in §1.3, and list a number of general results about them. In §1.4 we specialise to commutative ring spectra. Finally in §1.5 we discuss Bousfield localisation, with which we define p-localisation and p-completion of spectra and spaces. Along the way, we meet two commutative ring spectra: Eilenberg–MacLane spectra (Examples 1.26 and 1.67) and K-theory (Examples 1.28 and 1.69).

1.1 ∞-categories

Definition 1.1 The ∞-category of **spaces** $\mathscr{S}$ is the homotopy coherent nerve of the simplicially enriched category of Kan complexes. The ∞-category of **pointed spaces** $\mathscr{S}_*$ is the slice ∞-category $\mathscr{S}_{*/}$ under a point.

If $\mathcal{C}$ is an ∞-category and $X, Y \in \mathcal{C}$, we will often use the notation $[X, Y]$ to mean $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y)$. For pointed spaces $\mathscr{S}_*$ we reserve the notation $[X, Y]_*$ for this.

Occasionally we will need the technical notion of a *presentable ∞-category*.

Definition 1.2 An ∞-category $\mathcal{C}$ is called **presentable** if there exists a regular cardinal κ and a set $\mathcal{C}_0$ of objects satisfying the following conditions.

(i) The ∞-category $\mathcal{C}$ has all small colimits.

(ii) For any object $X \in \mathcal{C}_0$, the functor $\operatorname{Map}_{\mathcal{C}}(X, -)$ commutes with κ-filtered colimits.

(iii) Every object of $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{C}_0$.

Example 1.3 ([HTT], Ex. 5.5.1.8) The ∞-categories $\mathscr{S}$ and $\mathscr{S}_*$ are presentable. ▲

In an ∞-category $\mathcal{C}$, a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ will often be depicted as

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & Z. \end{tikzcd}$$

Definition 1.4 Let $\mathcal{C}$ be an ∞-category with a zero object. A **triangle** in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z, \end{tikzcd}$$

where 0 is a zero object of $\mathcal{C}$. We call this triangle a **fibre sequence**, or a **fibre** of g, if it is a pullback square. We call it a **cofibre sequence**, or a **cofibre** of f, if it is a pushout square.

We may abuse notation about fibre and cofibre sequences in several ways. First of all, if a triangle is a fibre or cofibre sequence, we may simply refer to it by the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

If this is a fibre sequence, we may write $\operatorname{fib} g$ for X, leaving the datum of the triangle implicit. Likewise, if it is a cofibre sequence, we may write $\operatorname{cofib} f$ for Z.

Definition 1.5 Let $\mathcal{C}$ be an ∞-category which has a zero object, and which admits fibres and cofibres. Let $X \in \mathcal{C}$ be an object. The **loop object** of X is the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd}$$

The **suspension** of X is the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & \Sigma X. \end{tikzcd}$$

Both Ω and Σ can be assembled into functors $\mathcal{C} \to \mathcal{C}$; see [HA, pp. 23, 24]. The functor Σ is left adjoint to Ω.

1.2 Spectra

Definition 1.6 The ∞-category of **spectra**, denoted $\mathbf{Sp}$, is the limit of the diagram

$$\cdots \longrightarrow \mathscr{S}_* \xrightarrow{\Omega} \mathscr{S}_* \xrightarrow{\Omega} \mathscr{S}_*$$

in the (large) ∞-category of ∞-categories.

Remark 1.7. The homotopy category $\mathrm{h}\,\mathbf{Sp}$ is known as the *stable homotopy category*.

Concretely, an object E of $\mathbf{Sp}$ is a sequence $(E_0, E_1, E_2, \dots)$ of pointed spaces, together with equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$ for every $n \geqslant 0$. Thus a spectrum is an *infinite loop space* together with a choice of an n-fold delooping for every $n \geqslant 0$. This is indeed a choice: in general such deloopings are not unique. Note that (up to equivalence) the space E_n recovers E_k for $k < n$.

If E and F are two spectra, then a morphism $f \colon E \to F$ consists of morphisms $f_n \colon E_n \to F_n$ of pointed spaces, together with diagrams

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d, "\simeq"'] & F_n \arrow[d, "\simeq"] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega F_{n+1}. \end{tikzcd}$$

In other words, a morphism is an *infinite loop map* between infinite loop spaces together with a choice of an n-fold delooping for every $n \geqslant 0$.

A homotopy H between two morphisms $f, g \colon E \to F$ consists of pointed homotopies $H_n \colon E_n \wedge [0,1]_+ \to F_n$ (with $\wedge$ denoting the smash product of pointed spaces, and $[0,1]_+$ the unit interval with an added disjoint basepoint), together with diagrams

$$\begin{tikzcd} E_n \wedge [0,1]_+ \arrow[r, "H_n"] \arrow[d] & F_n \arrow[d] \\ \Omega(E_n \wedge [0,1]_+) \arrow[r, "\Omega H_{n+1}"'] & \Omega F_{n+1}. \end{tikzcd}$$

Remark 1.8. What we call a spectrum, would by another convention be called an Ω*-spectrum*. What this other convention would call a spectrum, we would call a *prespectrum*. See [Gep19, Def. 3.1.13] for more information.

Example 1.9 The **zero spectrum** 0 is the spectrum where every space is a point, with the obvious delooping maps. This is a zero object in $\mathbf{Sp}$: for every spectrum E, the mapping spaces $\mathrm{Map}_{\mathbf{Sp}}(0, E)$ and $\mathrm{Map}_{\mathbf{Sp}}(E, 0)$ are contractible. A spectrum is called **contractible** if it is equivalent to the zero spectrum. ▲

If E is a spectrum, then the loop space structure on the zeroth space $E_0 \simeq \Omega E_1$ gives it an addition law

$$+ \colon E_0 \times E_0 \longrightarrow E_0$$

given by concatenation of loops. This is associative up to coherent homotopy. Because $E_0 \simeq \Omega^2 E_2$ is also a two-fold loop space, it is also commutative up to homotopy. The higher deloopings of E_0 make this operation commutative up to coherent homotopy. Maps of spectra respect this structure because they are infinite loop maps. In this sense spectra are a homotopical variant of abelian groups.

This analogy goes further, and in the next section we discuss how the ∞-category **Sp** can be said to behave in an algebraic way.

1.2.1 Categorical properties

Definition 1.10 An ∞-category $\mathcal{C}$ is called **stable** if it satisfies the following conditions.

(i) The ∞-category $\mathcal{C}$ has a zero object.

(ii) Every morphism in $\mathcal{C}$ admits a fibre and a cofibre.

(iii) A triangle in $\mathcal{C}$ is a pullback square if and only if it is a pushout square.

The notion of a stable ∞-category can be thought of as an abelian category 'up to homotopy'. We think of the (co)fibre of a morphism as a homotopical variant of the (co)kernel of a homomorphism. Condition (iii) above is an analogue of the condition on abelian categories that the image of a morphism should be isomorphic to its coimage. In this analogy with abelian categories, **Sp** plays the analogous role of **Ab**.

Note that the opposite of a stable ∞-category is again stable.

By [HA, Rmk. 1.1.3.5], in a stable ∞-category, finite products agree with finite coproducts. For this reason, in a stable ∞-category we will denote a coproduct by $\oplus$ and refer to a coproduct as a **direct sum**.

The homotopy category of a stable ∞-category $\mathcal{C}$ is naturally a *triangulated category*. See [HA, §1.1.2] for a discussion and proof.

The natural notion of a functor between stable ∞-categories is that of an *exact functor*.

Definition 1.11 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. The functor F is called **exact** if it sends zero objects to zero objects, and fibre sequences to fibre sequences.

Unlike in ordinary algebra, there is no need to distinguish between left and right exact functors between stable ∞-categories.

Proposition 1.12 ([HA], Prop. 1.1.4.1) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. The following are equivalent:*

(i) F preserves finite limits;

(ii) F preserves finite colimits;

(iii) F is exact.

The main example of a stable ∞-category is that of spectra.

Theorem 1.13 *The ∞ category* **Sp** *is presentable, stable, and admits all limits and colimits.*

Proof. See [HA], §1.4.3 and §1.4.4. ∎

In a stable ∞-category, the functors Σ and Ω are inverse equivalences. In the case of **Sp**, the loop functor Ω is (up to equivalence) given by 'shifting up' (and appending ΩE_0 at the bottom), and the suspension functor Σ is given by 'shifting down'. If $n \geqslant 0$, we will use the notation Σ^{-n} to mean Ω^n, and the notation Ω^{-n} to mean Σ^n.

Let $\Omega^\infty \colon \mathbf{Sp} \to \mathscr{S}_*$ denote the **zeroth space** functor, viz. the functor projecting onto the final copy of $\mathscr{S}_*$ in the diagram of Definition 1.6. The functor $\Omega^{\infty-n} := \Omega^\infty\Omega^{-n}$ maps a spectrum E to its n-th space. Henceforth we will write $\Omega^{\infty-n}E$ for the n-th space of a spectrum E, instead of E_n as we did above. This is to avoid confusion later on.

Proposition 1.14 ([HA], Prop. 1.4.4.4) *The functor $\Omega^\infty \colon \mathbf{Sp} \to \mathscr{S}_*$ admits a left adjoint. We denote this functor by Σ^∞, and call $\Sigma^\infty X$ of a pointed space X the* ***suspension spectrum*** *of X.*

Note that Ω^∞ commutes with Ω but not with Σ. Similarly, Σ^∞ commutes with Σ but not with Ω.

Example 1.15 The suspension spectrum of the zero-sphere S^0 is called the **sphere spectrum**, and is denoted by $\mathbf{S}$. If n is an integer, then we denote the n-fold suspension $\Sigma^n\mathbf{S}$ by $\mathbf{S}^n$. If n is nonnegative, then this is the suspension spectrum of S^n:

$$\Sigma^\infty S^n = \Sigma^\infty\Sigma^n S^0 = \Sigma^n\Sigma^\infty S^0 = \Sigma^n\mathbf{S}.$$

However, the negative suspensions $\mathbf{S}^{-n}$ are not in the essential image of Σ^∞. ▲

Remark 1.16. In pointed spaces $\mathscr{S}_*$, the coproduct is given by the wedge sum of spaces. As Σ^∞ is a left adjoint, we have a canonical equivalence

$$\Sigma^\infty \bigvee_i X_i \simeq \bigoplus_i \Sigma^\infty X_i.$$

For this reason the direct sum in **Sp** is also called the *wedge sum* of spectra, and denoted by $\vee$ by many authors. We use the notation $\oplus$ to emphasise the analogy with abelian groups.

Lemma 1.17 *Let E and F be spectra. Then the set $[E, F] = \pi_0 \operatorname{Map}_{\mathbf{Sp}}(E, F)$ naturally has the structure of an abelian group. Maps of spectra $E' \to E$ and $F \to F'$ induce group homomorphisms $[E, F] \to [E', F']$.*

Proof. This follows from the infinite loop space structure on the zeroth space of a spectrum. Maps of spectra are infinite loop maps, so they respect this structure. ∎

Definition 1.18 Let E be a spectrum, and n an integer. The ***n*-th homotopy group** of E is the group of homotopy classes of maps

$$\pi_n E := [\mathbf{S}^n, E].$$

The homotopy groups assemble to a functor

$$\pi_* \colon \mathbf{Sp} \longrightarrow \mathbf{Ab}^*$$

from spectra to $\mathbf{Z}$-graded abelian groups. Equivalently this is a functor $\mathrm{h}\,\mathbf{Sp} \to \mathbf{Ab}^*$, which is a functor between triangulated categories. This turns out to be an exact functor of triangulated categories. This means that to a (co)fibre sequence

$$X \longrightarrow Y \longrightarrow Z$$

in $\mathbf{Sp}$, there is an associated long exact sequence

$$\cdots \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \pi_n Z \longrightarrow \pi_{n-1} X \longrightarrow \cdots .$$

There is also a Whitehead theorem for spectra.

Theorem 1.19 ([HA], Rmk. 1.4.3.8) *A morphism $f \colon E \to F$ of spectra is an equivalence if and only if it induces an isomorphism $f_* \colon \pi_n E \to \pi_n F$ for every n.*

1.2.2 Homology and cohomology

A (generalised) **cohomology theory** is a sequence of contravariant functors $\{\,\tilde{h}^n\,\}_{n\in\mathbf{Z}}$ from pointed spaces to abelian groups, together with suspension isomorphisms of the form $\gamma^n \colon \tilde{h}^n(-) \cong \tilde{h}^{n+1}(\Sigma -)$, satisfying certain axioms. For a precise definition, see, e.g., [Gep19, Def. 3.1.7]. The *Brown Representability Theorem* says that every such cohomology theory is representable by a spectrum E, unique up to equivalence. This means that for every n we have a natural isomorphism

$$\tilde{h}^n(X) \cong [X,\, \Omega^{\infty-n}E]_*$$

and that the suspension isomorphism γ^n is induced by the delooping map

$$\Omega^{\infty-n}E \longrightarrow \Omega\,\Omega^{\infty-(n+1)}E.$$

Even though we formulated this using spectra, the representability of $\tilde{h}^n$ is a statement about homotopy classes of maps between (pointed) spaces. To obtain a similar representability result for homology theories, we have to work in the category of spectra. This has additional benefits: a spectrum gives rise to both a homology and a cohomology theory not only on spaces, but also on spectra themselves. To phrase this, we need two operations on spectra: the *smash product* and *mapping spectrum*.

Definition 1.20 Let E and F be spectra. The **smash product** $E \otimes F$ is the colimit of the diagram

$$\begin{array}{ccccc}
\Sigma^\infty(\Omega^\infty E \wedge \Omega^\infty F) & \longrightarrow & \Omega\Sigma^\infty(\Omega^{\infty-1}E \wedge \Omega^\infty F) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\Omega\Sigma^\infty(\Omega^\infty E \wedge \Omega^{\infty-1}F) & \longrightarrow & \Omega^2\Sigma^\infty(\Omega^{\infty-1}E \wedge \Omega^{\infty-1}F) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}$$

More briefly,

$$E \otimes F = \operatorname*{colim}_{n,m} \Omega^{n+m} \Sigma^\infty (\Omega^{\infty-n} E \wedge \Omega^{\infty-m} F).$$

Definition 1.21 Let E and F be spectra. The **mapping spectrum** $\mathrm{map}(E, F)$ is the spectrum with n-th space

$$\mathrm{Map}_{\mathbf{Sp}}(E,\, \Sigma^n F),$$

and with delooping maps

$$\mathrm{Map}_{\mathbf{Sp}}(E,\, \Sigma^n F) \simeq \mathrm{Map}_{\mathbf{Sp}}(E,\, \Omega\Sigma^{n+1} F) \simeq \Omega\, \mathrm{Map}_{\mathbf{Sp}}(E,\, \Sigma^{n+1} F).$$

Both of these assemble to bifunctors on $\mathbf{Sp}$. For a fixed spectrum X, the functor $X \otimes - \colon \mathbf{Sp} \to \mathbf{Sp}$ turns out to be left adjoint to $\mathrm{map}(X, -)$. We will revisit the smash product and mapping spectrum in more detail in §1.4.

The following property of the mapping spectrum will be useful.

Proposition 1.22 *Let X be a spectrum. The mapping spectrum functors* $\mathrm{map}(X, -) \colon \mathbf{Sp} \to \mathbf{Sp}$ *and* $\mathrm{map}(-, X) \colon \mathbf{Sp}^{\mathrm{op}} \to \mathbf{Sp}$ *are exact.*

Proof. The mapping space functor $\mathrm{Map}_{\mathbf{Sp}}(X, -) \colon \mathbf{Sp} \to \mathscr{S}_*$ preserves all limits. Limits in $\mathbf{Sp}$ are computed levelwise (because Ω^∞ commutes with limits), so the mapping spectrum $\mathrm{map}(X, -)$ also preserves all limits. It follows from Proposition 1.12 that $\mathrm{map}(X, -)$ is exact. Similarly one finds that $\mathrm{map}(-, X)$ is exact. ∎

Remark 1.23. Many others denote the smash product of spectra by $E \wedge F$, emphasising its relation with the smash product of spaces. We use the latter to emphasise the analogy with abelian groups.

Remark 1.24. The definition of the mapping spectrum makes sense in any stable ∞-category. This parallels the 1-categorical case of abelian categories, where the Hom-sets are abelian groups.

Definition 1.25 Let E and X be spectra, and n an integer.

(a) The ***n*-th *E*-homology group** of X is

$$E_n(X) := \pi_n(X \otimes E).$$

(b) The ***n*-th *E*-cohomology group** of X is

$$E^n(X) := \pi_{-n}\, \mathrm{map}(X, E) = [X,\, \Sigma^n E].$$

(c) The ***n*-th coefficient group** of E is

$$E_n := E_n(\mathbf{S}).$$

The **coefficients** of E is the $\mathbf{Z}$-graded abelian group E_*.

Note that $E_n = E^{-n}(\mathbf{S})$ for every n:

$$E_n = \pi_n(\mathbf{S} \otimes E) = [\mathbf{S}^n, E] = E^{-n}(\mathbf{S}).$$

Also note that the long exact sequence associated to a (co)fibre sequence implies that we get long exact sequences for E-homology and cohomology. Namely, if $X \to Y \to Z$ is a cofibre sequence, then $E \otimes X \to E \otimes Y \to E \otimes Z$ is also a cofibre sequence, because $E \otimes -$ is a left adjoint and therefore preserves colimits. Taking homotopy groups yields a long exact sequence

$$\cdots \longrightarrow E_n X \longrightarrow E_n Y \longrightarrow E_n Z \longrightarrow E_{n-1} X \longrightarrow \cdots.$$

Using that $\operatorname{map}(-, E)$ is exact (Proposition 1.22), we have a long exact sequence for cohomology as well:

$$\cdots \longrightarrow E^n X \longrightarrow E^n Y \longrightarrow E^n Z \longrightarrow E^{n+1} X \longrightarrow \cdots.$$

If $\widetilde{h}_* \colon \mathscr{S}_* \to \mathbf{Ab}^*$ is a homology theory on pointed spaces, then up to natural equivalence it factors as

$$\mathscr{S}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp} \xrightarrow{E_*(-)} \mathbf{Ab}^*$$

for a spectrum E that is unique up to equivalence. A similar statement holds for cohomology theories, which retrieves the Brown representability theorem for cohomology theories on spaces. If one wishes to work with unreduced (co)homology theories on $\mathscr{S}$ instead, then one should replace Σ^∞ by Σ^∞_+ (i.e., first adding a disjoint basepoint and then taking the suspension spectrum).

Example 1.26 If A is an abelian group, then singular cohomology with coefficients in A defines a generalised cohomology theory on pointed spaces. This is known to be represented by *Eilenberg–MacLane spaces*: for a pointed space X,

$$\widetilde{H}^n(X; A) \cong [X,\ K(A, n)]_*.$$

It is immediate that $K(A, n+1)$ is a delooping of $K(A, n)$. After choosing an equivalence $K(A, n) \simeq \Omega K(A, n+1)$ for every $n \geqslant 0$, we obtain a spectrum HA called the **Eilenberg–MacLane spectrum** of A. Its homotopy groups are

$$\pi_n HA \cong \begin{cases} A & n = 0, \\ 0 & n \neq 0. \end{cases}$$

If X is a spectrum, then $(HA)_n(X)$ is called the ***n*-th homology group** of X with coefficients in A. More commonly it is denoted by $H_n(X; A)$, or by $H_n(X)$ in the case $A = \mathbf{Z}$. Similarly $(HA)^n(X)$ is called the ***n*-th cohomology group** of X, and is denoted by $H^n(X; A)$. In the case that $X = \Sigma^\infty Y$ is the suspension spectrum of a pointed space Y, then these recover the (reduced) singular homology and cohomology of Y:

$$H_n(\Sigma^\infty Y; A) \cong \widetilde{H}_n(Y; A) \qquad \text{and} \qquad H^n(\Sigma^\infty Y; A) \cong \widetilde{H}^n(Y; A). \qquad \blacktriangle$$

Remark 1.27. The assignment $A \mapsto HA$ can be turned into a fully faithful functor $\mathbf{Ab} \to \mathbf{Sp}$ with essential image the *discrete spectra*: spectra E for which $\pi_n E = 0$ if $n \neq 0$. See [HA, Prop. 1.4.3.6]. An inverse equivalence $\mathbf{Sp}^{\mathrm{disc}} \to \mathbf{Ab}$ is given by π_0. In this way we can think of an abelian group as a spectrum.

Example 1.28 Consider *complex topological K-theory*. Recall that two vector bundles E and F over a space X are called *stably equivalent* if there are $n, m \geqslant 0$ such that $E \oplus \varepsilon^n \cong F \oplus \varepsilon^m$, with ε^n and ε^m denoting the trivial vector bundles of rank n and m, respectively. If X is a finite pointed space, then we define its **reduced K-group** $\widetilde{KU}(X)$ as the group of complex vector bundles over X up to stable equivalence, with group operation the direct sum.[1]

On finite spaces, the functor $\widetilde{KU}$ is represented by $\mathbf{Z} \times BU$, where BU denotes the classifying space of the infinite unitary group $U = \mathrm{colim}_n U(n)$: if X is finite,

$$\widetilde{KU}(X) \cong [X, \mathbf{Z} \times BU]_*.$$

See, e.g., [Ati, Prop. 2.1.10] for a proof. If X is an arbitrary pointed space, then we *define* its reduced K-group $\widetilde{KU}(X)$ to be this set of homotopy classes. We extend K-theory to negative degrees by defining $\widetilde{KU}^{-n}(X) := \widetilde{KU}(\Sigma^n X)$, or in other words, by defining $\widetilde{KU}^{-n}$ to be represented by $\Omega^n(\mathbf{Z} \times BU)$. Bott periodicity [Bot57] gives an equivalence between $\mathbf{Z} \times BU$ and its two-fold loop space $\Omega^2(\mathbf{Z} \times BU)$. This extends the grading in K-theory to all integers, and also gives us a spectrum KU, the **complex K-theory spectrum**. Its underlying spaces are

$$\Omega^{\infty-n} KU = \begin{cases} \mathbf{Z} \times BU & n \text{ even}, \\ U & n \text{ odd}, \end{cases}$$

and its homotopy groups are

$$\pi_n KU \cong \begin{cases} \mathbf{Z} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

This in particular shows that KU is neither a suspension spectrum nor an Eilenberg–MacLane spectrum, because it has nonvanishing negative homotopy groups.

If X is a spectrum, then we call $KU^*(X)$ the **K-theory** of X. In the case that $X = \Sigma^\infty Y$ is a suspension spectrum of a pointed space Y, it recovers the (reduced) K-theory of Y:

$$KU^*(\Sigma^\infty Y) \cong \widetilde{KU}^*(Y).$$ ▲

Remark 1.29. We advocate for denoting complex topological K-theory by KU rather than by K in order to stress the parallel with real topological K-theory KO, and to distinguish it from different kinds of K-theory (most notably, from algebraic K-theory).

[1] We do not need to take the Grothendieck group completion in the reduced case, because vector bundles over a finite space X up to stable equivalences already form a group under direct sum.

1.3 Symmetric monoidal ∞-categories

Roughly speaking, a *symmetric monoidal category* is a category $\mathcal{C}$ together with a product functor $\otimes\colon \mathcal{C} \to \mathcal{C} \to \mathcal{C}$ satisfying a number of coherence conditions, including associativity and symmetry. More precisely, these coherences are natural transformations that are themselves part of the data of a symmetric monoidal category. One can then speak of commutative algebra objects in a symmetric monoidal category, and modules over such an algebra object. This categorical language specialises to standard notions in algebra: commutative rings are commutative algebra objects in the symmetric monoidal category **Ab** of abelian groups, and modules over a commutative ring R are module objects over R in **Ab**.

It turns out that there is an analogous theory of *symmetric monoidal ∞-categories*. In this case there are more coherences, and they play a more important role, because the product functor is supposed to be associative and symmetric up to coherent homotopy. For this reason it turns out to be easier to give a different formulation of the definition, one using *cocartesian fibrations*. For a full review of cocartesian fibrations, see [HTT, §2.4]. Although it looks different at first sight, the definition is truly a generalisation of the 1-categorical case. In §1.3.1 we discuss commutative algebra objects in a symmetric monoidal ∞-category, and in §1.3.2 we discuss module objects over commutative algebra objects. In the end, using symmetric monoidal ∞-categories we can define commutative ring spectra and their module spectra in §1.4.

Lurie [HA] often works in the more general setting of ∞*-operads*; all results below have been specialised to the case of ∞-categories. Alternative sources include the survey by Gepner [Gep19] and the treatment by Groth [Gro15, §4]. These three sources also include additional motivation and intuition.

Remark 1.30. There is also a theory of *monoidal ∞-categories*, with which one could define *ring spectra* as opposed to commutative ring spectra. We do not discuss these in this work. Suffice it to say that a symmetric monoidal ∞-category has an underlying monoidal ∞-category, and a commutative ring spectrum has an underlying ring spectrum. (But unlike the algebraic case, commutativity is not a property of ring spectra, but additional data.)

The coherences that the product functor should satisfy can be modelled by the following 1-category.

Definition 1.31 For a natural number $n \geqslant 0$, let $\langle n \rangle$ denote the pointed set $\{0, 1, \dots, n\}$, pointed at the element 0. Let $\mathbf{Fin}_*$ denote the category with objects $\langle n \rangle$ for every $n \geqslant 0$, and morphisms the pointed maps. A morphism $\alpha\colon \langle n \rangle \to \langle m \rangle$ in $\mathbf{Fin}_*$ is called **inert** if for every nonzero $i \in \langle m \rangle$, the fibre $\alpha^{-1}\{i\}$ is a singleton.

Suppose that $p\colon \mathcal{D} \to \mathbf{Fin}_*$ is a functor of ∞-categories, i.e., a map $p\colon \mathcal{D} \to N(\mathbf{Fin}_*)$ of simplicial sets. We denote by $\mathcal{D}_{\langle n \rangle}$ the simplicial set that is the fibre of p over $\langle n \rangle \in N(\mathbf{Fin}_*)_0$. If the map $\mathcal{D} \to N(\mathbf{Fin}_*)$ is an inner fibration, then the simplicial set $\mathcal{D}_{\langle n \rangle}$ is also an ∞-category.

For $n \geqslant 0$ and $1 \leqslant i \leqslant n$, let ρ_i denote the morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathbf{Fin}_*$ defined by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

These are all the inert morphisms $\langle n \rangle \to \langle 1 \rangle$.

Definition 1.32 A **symmetric monoidal ∞-category** is an ∞-category $\mathcal{C}^\otimes$ together with a cocartesian fibration $p\colon \mathcal{C}^\otimes \to \mathbf{Fin}_*$ satisfying the **Segal condition**: for every $n \geqslant 0$, the map

$$\mathcal{C}^\otimes_{\langle n \rangle} \longrightarrow \left(\mathcal{C}^\otimes_{\langle 1 \rangle}\right)^{\times n}$$

induced by the morphisms ρ_i for $i = 1, \ldots, n$, is a categorical equivalence.

If $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ is a symmetric monoidal ∞-category, we write $\mathcal{C}$ for $\mathcal{C}^\otimes_{\langle 1 \rangle}$ and call it the *underlying ∞-category*. We say that $\mathcal{C}^\otimes$ *gives $\mathcal{C}$ the structure of a symmetric monoidal ∞-category*. We may abuse notation and simply call $\mathcal{C}$ the symmetric monoidal ∞-category, leaving the map $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ implicit.

The cocartesian fibration $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ gives rise to a product functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as follows. The morphism

$$\langle 2 \rangle \longrightarrow \langle 1 \rangle, \qquad \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 1 \end{cases}$$

in $\mathbf{Fin}_*$ lifts to a functor $\mathcal{C}^\otimes_{\langle 2 \rangle} \to \mathcal{C}$, well-defined up to contractible choice. Choosing an inverse of the equivalence $\mathcal{C}^\otimes_{\langle 2 \rangle} \simeq \mathcal{C} \times \mathcal{C}$ (guaranteed by the Segal condition) and precomposing with this gives a product functor. The structure of $\mathbf{Fin}_*$ implies in particular that this product is associative and commutative up to homotopy. The additional structure in the cocartesian fibration $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ records the higher coherence data.

In a similar way we can obtain a unit object of $\mathcal{C}$. Namely, by the Segal condition, the ∞-category $\mathcal{C}^\otimes_{\langle 0 \rangle}$ is equivalent to Δ^0. The unique morphism $\langle 0 \rangle \to \langle 1 \rangle$ in $\mathbf{Fin}_*$ lifts to a functor

$$\Delta^0 \longrightarrow \mathcal{C}$$

well-defined up to contractible choice. We call an object in the image of such a functor a **unit object** of $\mathcal{C}$, and denote it by $\mathbf{1}$. A unit object is in particular a two-sided unit object up to homotopy for the product functor $\otimes$.

Definition 1.33 A symmetric monoidal ∞-category $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ is called **closed** if for every object $X \in \mathcal{C}$, the functor $X \otimes -\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint.

In a closed symmetric monoidal ∞-category $\mathcal{C}$, we will often denote the right adjoint to $X \otimes -$ by $\mathrm{map}_{\mathcal{C}}(X, -)$. If X is an object in a closed symmetric monoidal ∞-category $\mathcal{C}$, we denote by $X^\vee$ its **dual object** $\mathrm{map}_{\mathcal{C}}(X, \mathbf{1})$.

Definition 1.34 Let $\mathcal{C}^\otimes \to \mathbf{Fin}_*$ be a closed symmetric monoidal ∞-category. An object $X \in \mathcal{C}$ is called **dualisable** if for every $Y \in \mathcal{C}$, the natural map

$$X^\vee \otimes Y \longrightarrow \mathrm{map}_{\mathcal{C}}(X, Y)$$

is an equivalence.

If X is dualisable, then so is its dual $X^\vee$.

Definition 1.35 Let $p\colon \mathcal{C}^\otimes \to \mathbf{Fin}_*$ and $q\colon \mathcal{D}^\otimes \to \mathbf{Fin}_*$ be two symmetric monoidal ∞-categories.

(a) A **lax monoidal functor** F from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$ is a functor $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ over $\mathbf{Fin}_*$ that sends p-cocartesian lifts of inert morphisms to q-cocartesian morphisms.

(b) A **symmetric monoidal functor** F from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$ is a functor $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ over $\mathbf{Fin}_*$ that sends p-cocartesian morphisms to q-cocartesian morphisms.

A lax (or symmetric) monoidal functor $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ has an underlying functor $F\colon \mathcal{C} \to \mathcal{D}$. We may abuse notation and call refer to the underlying functor $\mathcal{C} \to \mathcal{D}$ as the lax (or symmetric) monoidal functor, leaving the functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ implicit.

If $\mathcal{C}$ is a 1-category, then a symmetric monoidal structure on $N(\mathcal{C})$ regarded as an ∞-category turns out to be equivalent to a symmetric monoidal structure on $\mathcal{C}$ in the classical sense. See the introduction to Chapter 2 of [HA] for a discussion. In this way we regard a symmetric monoidal 1-category $\mathcal{C}$ (in the classical sense) as a symmetric monoidal ∞-category $N(\mathcal{C})^\otimes \to N(\mathcal{C})$. The notion of lax and symmetric monoidal functors also coincide.

1.3.1 Commutative algebra objects

Definition 1.36 Let $p\colon \mathcal{C}^\otimes \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. A **commutative algebra object** in $\mathcal{C}$ is a section $A\langle - \rangle\colon \mathbf{Fin}_* \to \mathcal{C}^\otimes$ of p that sends inert morphisms to p-cocartesian morphisms. Write $\mathbf{CAlg}(\mathcal{C})$ for the full subcategory of the ∞-category $\mathbf{Fun}_{\mathbf{Fin}_*}(\mathbf{Fin}_*, \mathcal{C}^\otimes)$ of functors over $\mathbf{Fin}_*$ on the commutative algebra objects.

If $A\langle - \rangle\colon \mathbf{Fin}_* \to \mathcal{C}^\otimes$ is a commutative algebra object, then $A := A\langle 1 \rangle$ lands in $\mathcal{C}^\otimes_{\langle 1 \rangle} = \mathcal{C}$. We call A the *underlying object*. We will often abuse notation by referring to A as the commutative algebra object. Precomposition with the functor $\langle 1 \rangle \to \mathbf{Fin}_*$ yields a forgetful functor $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ that sends a commutative algebra object to its underlying object.

The condition that $A\langle - \rangle$ should send inert morphisms to cocartesian morphisms implies, together with the Segal condition, that for $n \geqslant 1$, we have an equivalence $A\langle n \rangle \simeq A \otimes \cdots \otimes A$.

Example 1.37 Let $\mathcal{C}^\otimes \to \mathcal{C}$ be a symmetric monoidal ∞-category. There is an essentially unique commutative algebra object in $\mathbf{CAlg}(\mathcal{C})$ whose underlying object is the unit object $\mathbf{1}$; see [HA, Prop. 3.2.1.8]. We will refer to this object as the **unit algebra**, and also denote it by $\mathbf{1}$. It is an initial object in $\mathbf{CAlg}(\mathcal{C})$. ▲

The following is immediate from the definitions.

Proposition 1.38 *Let $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax monoidal functor between two symmetric monoidal ∞-categories. Then postcomposition with F induces a functor*

$$\mathbf{CAlg}(\mathcal{C}) \longrightarrow \mathbf{CAlg}(\mathcal{D})$$

which on underlying objects agrees with $F\colon \mathcal{C} \to \mathcal{D}$.

Definition 1.39 Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. An **augmented commutative algebra object** in $\mathcal{C}$ is a commutative algebra object A in $\mathcal{C}$ together with a morphism $A \to \mathbf{1}$ to the unit algebra of $\mathcal{C}$. Write $\mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C})$ for the slice category $\mathbf{CAlg}(\mathcal{C})_{/\mathbf{1}}$ over the unit algebra.

From Example 1.37 it follows that $\mathbf{1}$ is a zero object of $\mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C})$.

Remark 1.40. There is also a notion of a nonunital commutative algebra object; see [GL, Var. 3.1.3.8]. There is an equivalence between $\mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C})$ and $\mathbf{CAlg}^{\mathrm{nu}}(\mathcal{C})$: see [HA, Prop. 5.4.4.10]. Informally, this equivalence sends an augmented algebra $A \to \mathbf{1}$ to the fibre of the augmentation; the inverse equivalence sends a nonunital algebra $\widetilde{A}$ to $\mathbf{1} \oplus \widetilde{A}$.

We now turn to limits and colimits of commutative algebra objects. Limits are the most straightforward.

Theorem 1.41 ([HA], Cor. 3.2.2.5) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and K a simplicial set. Suppose that $\mathcal{C}$ admits K-indexed limits. Then the ∞-category $\mathbf{CAlg}(\mathcal{C})$ admits K-indexed limits, and the forgetful functor $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ preserves and reflects these.*

Corollary 1.42 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. The forgetful functor $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ is conservative.*

Proof. By [Ker, Tag 02JR], a morphism $X \to Y$ in an ∞-category $\mathcal{D}$ exhibits X as a limit of the diagram $\{Y\} \to \mathcal{D}$ if and only if $X \to Y$ is an equivalence. Now apply the above theorem. ∎

In general, colimits in $\mathbf{CAlg}(\mathcal{C})$ are much more complicated. We only need a specific case, namely that of coproducts, from which we will later deduce the case of pushouts. (Since limits behave so differently from colimits, it follows that even if $\mathcal{C}$ is stable, the ∞-category $\mathbf{CAlg}(\mathcal{C})$ need not be stable.)

In [HA], Construction 3.2.4.1, Lurie constructs a symmetric monoidal structure on $\mathbf{CAlg}(\mathcal{C})$, such that the forgetful functor $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ is a symmetric monoidal functor. We shall denote the product in $\mathbf{CAlg}(\mathcal{C})$ by $\otimes$ also.

Proposition 1.43 ([HA], Prop. 3.2.4.7) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. Then the coproduct of $A, B \in \mathbf{CAlg}(\mathcal{C})$ is given by $A \otimes B$.*

We conclude this section with the notion of 'free' commutative algebra objects. In good settings such objects always exist.

Theorem 1.44 ([HA], Prop. 3.1.3.13) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. Assume that the underlying category $\mathcal{C}$ has all countable colimits, and that the product functor* *preserves these in each variable separately. Then the forgetful functor* $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ *admits a left adjoint.*

We will denote such a left adjoint by $\mathrm{Sym}\colon \mathcal{C} \to \mathbf{CAlg}(\mathcal{C})$. For $X \in \mathcal{C}$, we call $\mathrm{Sym}(X)$ the **symmetric algebra** on X. We think of the counit map $X \to \mathrm{Sym}(X)$ as the inclusion of X in the symmetric algebra on X.

If $\mathcal{C}$ has a zero object 0, then $\mathrm{Sym}(0)$ is an initial object of $\mathbf{CAlg}(\mathcal{C})$, i.e., it is the unit algebra $\mathbf{1}$. For every $X \in \mathcal{C}$, a morphism $X \to 0$ induces an augmentation $\mathrm{Sym}(X) \to \mathbf{1}$ of $\mathrm{Sym}(X)$. Thus in this case we may also think of Sym as a functor $\mathcal{C} \to \mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C})$ to augmented commutative algebra objects.

Remark 1.45. We use the notation Sym in analogy with the symmetric algebra $\mathrm{Sym}_R(M)$ (where R is a commutative ring and M an R-module).

1.3.2 Module objects

Spelling out the complete definition of module objects in a symmetric monoidal ∞-category would take up a considerable amount of space, so we give a summary of a construction. An alternative overview is given by Gepner [Gep19, §3.4].

Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. In Definition 4.2.1.13 of [HA], Lurie defines an isofibration $\mathbf{LMod}(\mathcal{C}) \to \mathbf{CAlg}(\mathcal{C})$. Intuitively speaking, the ∞-category $\mathbf{LMod}(\mathcal{C})$ consists of 'pairs' (A, M) with A a commutative algebra object of $\mathcal{C}$, and M an object of $\mathcal{C}$ with a left multiplication by A. The map $\mathbf{LMod}(\mathcal{C}) \to \mathbf{CAlg}(\mathcal{C})$ maps such a pair (A, M) to A. If $A \in \mathbf{CAlg}(\mathcal{C})$ is a commutative algebra object, we define the ∞-category of **left A-module objects** in $\mathcal{C}$ to be the pullback

$$\mathbf{LMod}_A(\mathcal{C}) := \mathbf{LMod}(\mathcal{C}) \times_{\{A\}} \mathbf{CAlg}(A).$$

There is a forgetful functor $\mathbf{LMod}_A(\mathcal{C}) \to \mathcal{C}$ (see [HA, Ex. 4.2.1.18]).

Alternatively, one could define an ∞-category of right A-modules, or (A, A)-bimodules, or 'commutative' A-modules in the sense of [HA, §3.3]. Since we work with commutative algebra objects, all these resulting ∞-categories are equivalent: see [HA, p. 592] for a summary. For this reason we will simply write $\mathbf{Mod}_A(\mathcal{C})$ for $\mathbf{LMod}_A(\mathcal{C})$ in the remainder of this text.

Let $A \in \mathbf{CAlg}(\mathcal{C})$. In §4.4 of [HA], Lurie defines the *relative tensor product* of A-module objects $M, N \in \mathbf{Mod}_A(\mathcal{C})$, denoted by $M \otimes_A N$.

Theorem 1.46 ([HA], Thm. 4.5.2.1) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$. Suppose $\mathcal{C}$ admits all geometric realisations and $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves these in each variable separately. Then the ∞-category $\mathbf{Mod}_A(\mathcal{C})$ has the structure of a symmetric monoidal ∞-category given by the relative tensor product.*

We write $\mathbf{Mod}_A^{\otimes}(\mathcal{C}) \to \mathbf{Fin}_*$ for the symmetric monoidal structure of $\mathbf{Mod}_A(\mathcal{C})$.

Example 1.47 Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category. By Proposition 3.4.2.1 of [HA], the forgetful functor $\mathbf{Mod}_{\mathbf{1}}^{\otimes}(\mathcal{C}) \to \mathcal{C}^{\otimes}$ is an equivalence. In other words, every object of $\mathcal{C}$ is a module over $\mathbf{1}$ in an essentially unique way. ▲

Let $A \to B$ be a morphism of commutative algebra objects in $\mathcal{C}$. In §3.4.3 of [HA], a forgetful functor $\mathbf{Mod}_B^{\otimes}(\mathcal{C}) \to \mathbf{Mod}_A^{\otimes}(\mathcal{C})$ over $\mathbf{Fin}_*$ is constructed. In good settings this functor has a left adjoint, resulting in an 'extension of scalars' adjunction.

Theorem 1.48 (Extension of scalars; [HA], Thm. 4.5.3.1, Thm. 4.6.2.17) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \to B$ be a morphism in $\mathbf{CAlg}(\mathcal{C})$. Suppose $\mathcal{C}$ admits all geometric realisations and $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves these in each variable separately. Then the forgetful functor $\mathbf{Mod}_B(\mathcal{C}) \to \mathbf{Mod}_A(\mathcal{C})$ has a left adjoint given by the relative tensor product $M \mapsto B \otimes_A M$. Moreover, the forgetful functor $\mathbf{Mod}_B^{\otimes}(\mathcal{C}) \to \mathbf{Mod}_A^{\otimes}(\mathcal{C})$ is lax monoidal, and the relative tensor product $\mathbf{Mod}_A^{\otimes}(\mathcal{C}) \to \mathbf{Mod}_B^{\otimes}(\mathcal{C})$ is symmetric monoidal.*

Limits and colimits in $\mathbf{Mod}_A(\mathcal{C})$ are very closely related to limits and colimits in $\mathcal{C}$.

Theorem 1.49 ([HA], Cor. 4.2.3.3, Cor. 4.2.3.5) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$. Let K be a simplicial set.*

(a) *Suppose $\mathcal{C}$ admits K-indexed limits. Then $\mathbf{Mod}_A(\mathcal{C})$ admits K-indexed limits, and the forgetful functor $\mathbf{Mod}_A(\mathcal{C}) \to \mathcal{C}$ preserves and reflects these.*

(b) *Suppose $\mathcal{C}$ admits K-indexed colimits and that the tensor product functor $M \mapsto A \otimes M$ preserves these. Then $\mathbf{Mod}_A(\mathcal{C})$ admits K-indexed colimits, and the forgetful functor $\mathbf{Mod}_A(\mathcal{C}) \to \mathcal{C}$ preserves and reflects these.*

By the same reasoning as in Corollary 1.42, this implies the following.

Corollary 1.50 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$. The forgetful functor $\mathbf{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative.*

Because colimits in $\mathbf{Mod}_A(\mathcal{C})$ are so closely related to colimits in $\mathcal{C}$ (which was not the case for commutative algebra objects), we also find the following.

Corollary 1.51 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category such that $\mathcal{C}$ is stable. Let $A \in \mathbf{CAlg}(\mathcal{C})$ be a commutative algebra object. Suppose that the tensor product functor $M \mapsto A \otimes M$ commutes with pushouts. Then $\mathbf{Mod}_A(\mathcal{C})$ is a stable ∞-category.*

Theorem 1.52 ([HA], Cor. 4.4.2.15) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$. Let K be a simplicial set. Assume that the tensor product functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realisations and K-indexed colimits in each variable separately. Then the relative tensor product functor $\otimes_A\colon \mathbf{Mod}_A(\mathcal{C}) \times \mathbf{Mod}_A(\mathcal{C}) \to \mathbf{Mod}_A(\mathcal{C})$ preserves K-indexed colimits in each variable separately.*

Corollary 1.53 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a closed symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$ be a commutative algebra object. Then the symmetric monoidal ∞-category $\mathbf{Mod}_A(\mathcal{C})$ is also closed.*

Definition 1.54 Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let $A \in \mathbf{CAlg}(\mathcal{C})$ be a commutative algebra object. A **commutative A-algebra object** in $\mathcal{C}$

is a commutative algebra object in $\mathbf{Mod}_A(\mathcal{C})$. Write $\mathbf{CAlg}_A(\mathcal{C})$ for the ∞-category $\mathbf{CAlg}(\mathbf{Mod}_A(\mathcal{C}))$.

Theorem 1.55 ([HA], Cor. 3.4.1.7) *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and A a commutative algebra object of $\mathcal{C}$. Then we have a categorical equivalence*

$$\mathbf{CAlg}(\mathbf{Mod}_A(\mathcal{C})) \simeq \mathbf{CAlg}(\mathcal{C})_{A/}.$$

In particular, we see that A is a unit algebra for $\mathbf{CAlg}_A(\mathcal{C})$.

Corollary 1.56 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category, and let*

$$B \longleftarrow A \longrightarrow C$$

be a diagram in $\mathbf{CAlg}(\mathcal{C})$. Then the pushout of this diagram is given by $B \otimes_A C$.

Proof. Write f for the given diagram. This determines a diagram $g\colon \Delta^0 \sqcup \Delta^0 \to \mathbf{CAlg}(\mathcal{C})_{A/}$, and a colimit of g in $\mathbf{CAlg}(\mathcal{C})_{A/}$ is the same as a colimit of f in $\mathbf{CAlg}(\mathcal{C})$. By Proposition 1.43, the colimit of g in $\mathbf{CAlg}(\mathcal{C})_{A/} \simeq \mathbf{CAlg}(\mathbf{Mod}_A(\mathcal{C}))$ is given by $B \otimes_A C$. ■

Note that the extension of scalars result for module objects (Theorem 1.48) also gives an extension of scalars result for commutative A-algebra objects. (This is because both functors are at least lax monoidal: see Proposition 1.38.) We record a useful result about the compatibility of symmetric algebra objects with extension of scalars.

Proposition 1.57 *Let $\mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a symmetric monoidal ∞-category satisfying the conditions of Theorem 1.48. Let $A \in \mathbf{CAlg}(\mathcal{C})$ and let $X \in \mathcal{C}$. Then we have a natural equivalence*

$$\mathrm{Sym}_{\mathbf{Mod}_A(\mathcal{C})}(A \otimes X) \simeq A \otimes \mathrm{Sym}_{\mathcal{C}}(X)$$

in $\mathbf{CAlg}_A(\mathcal{C})$.

Proof. Using extension of scalars (Theorem 1.48), we see that for any $B \in \mathbf{CAlg}_A(\mathcal{C})$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathbf{CAlg}_A(\mathcal{C})}(\mathrm{Sym}_{\mathbf{Mod}_A(\mathcal{C})}(A \otimes X),\ B) &\simeq \mathrm{Map}_{\mathbf{Mod}_A(\mathcal{C})}(A \otimes X,\ B) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(X, B) \\
&\simeq \mathrm{Map}_{\mathbf{CAlg}(\mathcal{C})}(\mathrm{Sym}_{\mathcal{C}}(X),\ B) \\
&\simeq \mathrm{Map}_{\mathbf{CAlg}_A(\mathcal{C})}(A \otimes \mathrm{Sym}_{\mathcal{C}}(X),\ B).
\end{aligned}$$

Thus the Yoneda lemma implies the result. ■

1.4 Commutative ring spectra

Previously in Definition 1.20 we defined the smash product of spectra objectwise. It has a lot more structure than this: it turns $\mathbf{Sp}$ into a symmetric monoidal ∞-category.

Theorem 1.58 **([HA], Cor. 4.8.2.19)** *There is a closed symmetric monoidal structure on* $\mathbf{Sp}$*, unique up to contractible choice, such that*

(*i*) *the product functor* $\otimes\colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ *preserves colimits in each variable separately;*

(*ii*) *the sphere spectrum* $\mathbf{S}$ *is a two-sided unit.*

We fix a choice of such a structure and call the associated product the *smash product*. The right adjoint to the functor $X \otimes -$ is the mapping spectrum $\mathrm{map}(X, -)$ from Definition 1.21.

The concepts from the previous section now specialise to give the theory of *commutative ring spectra*.

Definition 1.59

(a) A **commutative ring spectrum** is a commutative algebra object in the symmetric monoidal ∞-category $\mathbf{Sp}$. We denote the ∞-category of commutative ring spectra by $\mathbf{CAlg}(\mathbf{Sp})$.

(b) If R is a commutative ring spectrum, then an ***R*-module spectrum** (or *R-module* for short) is an R-module object in $\mathbf{Sp}$. We denote the ∞-category of R-module spectra by $\mathbf{Mod}_R$.

(c) If R is a commutative ring spectrum, then a **commutative *R*-algebra spectrum** (or *commutative R-algebra* for short) is a commutative R-algebra object in $\mathbf{Sp}$. We denote the ∞-category of commutative R-algebra spectra by $\mathbf{CAlg}_R$.

By Theorem 1.55, a commutative R-algebra spectrum A is the same as a commutative ring spectrum together with a map $R \to A$ of commutative ring spectra. The ∞-category $\mathbf{Mod}_R$ is stable, but $\mathbf{CAlg}_R$ and $\mathbf{CAlg}(\mathbf{Sp})$ are not.

Remark 1.60. Often the term $\mathbf{E}_\infty$-*ring spectrum* is used for what we call a commutative ring spectrum. This is because there is an infinite hierarchy of commutativity conditions in the world of higher algebra, which are denoted by $\mathbf{E}_1$, $\mathbf{E}_2$, etc. Of these, $\mathbf{E}_\infty$ is the most complex: it describes commutativity up to coherent homotopy. (See [HA, §7.2] for more information.) This is the only type of commutativity we need in this text, so we use the word 'commutative' to mean $\mathbf{E}_\infty$ throughout.

Example 1.61 The sphere spectrum $\mathbf{S}$ is naturally a commutative ring spectrum, because it is the unit of the symmetric monoidal structure on $\mathbf{Sp}$ (see Example 1.37). A module spectrum over $\mathbf{S}$ is the same as a spectrum, $\mathbf{Mod}_\mathbf{S} \simeq \mathbf{Sp}$ (see Example 1.47). It is also the initial commutative ring spectrum. In this sense $\mathbf{S}$ plays the role that $\mathbf{Z}$ does in ordinary algebra. The $\mathbf{S}$-linear dual $X^\vee = \mathrm{map}(X, \mathbf{S})$ of a spectrum X is more commonly referred to as the **Spanier–Whitehead dual** of X. ▲

If R is a commutative ring spectrum, then the relative smash product $\otimes_R$ on $\mathbf{Mod}_R$ preserves colimits in each variable separately, because the smash product on $\mathbf{Sp}$ does so (Theorem 1.52). Therefore all results from the previous section apply to ring spectra and module spectra. Let us draw attention to a few particular cases.

Proposition 1.62 *A map of commutative ring spectra is an equivalence if and only if it induces an isomorphism on homotopy groups. If R is a commutative ring spectrum, then a map of R-module spectra (or commutative R-algebra spectra) is an equivalence if and only if it induces an isomorphism on homotopy groups.*

Proof. Combine Corollaries 1.42 and 1.50 with Theorem 1.19. ∎

Example 1.63 Let R be a commutative ring spectrum. The ∞-category $\mathbf{Mod}_R$ is a closed symmetric monoidal ∞-category by Corollary 1.53. We write $\mathrm{map}_R(M, N)$ for the R-module spectrum of maps from M to N. If M is an R-module spectrum, then its ***R*-linear dual** is the R-module spectrum $M^\vee := \mathrm{map}_R(M, R)$. The R-linear dual functor $(-)^\vee \colon \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ preserves limits, and so by Proposition 1.12 it is an exact functor. Since $\mathbf{Mod}_R$ is a stable ∞-category, in practise this means that $(-)^\vee$ preserves cofibre sequences of R-module spectra. ▲

Example 1.64 Let R be a commutative ring spectrum. The symmetric objects from Theorem 1.44 exist in the symmetric monoidal ∞-category $\mathbf{Mod}_R$. If M is an R-module spectrum, then we call $\mathrm{Sym}_R(M)$ the **symmetric *R*-algebra spectrum** on M. In the absolute case where $R = \mathbf{S}$, we omit the subscript and simply write $\mathrm{Sym}(X)$ for the *symmetric ring spectrum* on a spectrum X. ▲

In the remainder of this section we discuss results which are specific to ring spectra.

Proposition 1.65 *The functor $\pi_* \colon \mathbf{Sp} \to \mathbf{Ab}^*$ is lax monoidal.*

Proof sketch. It suffices to give $\mathrm{h}\,\mathbf{Sp} \to \mathbf{Ab}^*$ a lax monoidal structure. If E and F are spectra, and $\mathbf{S}^n \to E$ and $\mathbf{S}^m \to F$ are elements of $\pi_n E$ and $\pi_m F$, respectively, then taking their smash product gives a map

$$\mathbf{S}^n \otimes \mathbf{S}^m \simeq \mathbf{S}^{n+m} \longrightarrow E \otimes F,$$

i.e., an element of $\pi_{n+m} E \otimes F$. This defines a bilinear pairing between $\pi_n E$ and $\pi_m F$, giving rise to a natural transformation $\pi_*(-) \otimes \pi_*(-) \to \pi_*(- \otimes -)$, making π_* lax monoidal. ∎

Corollary 1.66 *If R is a commutative ring spectrum, then $\pi_* R$ is a graded-commutative $\mathbf{Z}$-graded ring. If M is an R-module spectrum, then $\pi_* M$ is a $\mathbf{Z}$-graded $\pi_* R$-module. If A is a commutative R-algebra spectrum, then $\pi_* A$ is a graded-commutative $\mathbf{Z}$-graded $\pi_* R$-algebra.*

Example 1.67 If R is an ordinary commutative ring, then its Eilenberg–MacLane spectrum HR is naturally a commutative ring spectrum. We can even make this functorial, as follows. Recall the Eilenberg–MacLane spectrum functor $\mathbf{Ab} \to \mathbf{Sp}$ from Remark 1.27. This functor lands in connective spectra (i.e., spectra whose homotopy vanishes in negative degrees), and as a functor $\mathbf{Ab} \to \mathbf{Sp}^{\mathrm{cn}}$ it has a left adjoint, given by the zeroth homotopy group (see [HA, Ex. 2.2.1.10]). The functor $\pi_0 \colon \mathbf{Sp}^{\mathrm{cn}} \to \mathbf{Ab}$ is symmetric monoidal, because $\pi_0 \colon \mathrm{h}\,\mathbf{Sp}^{\mathrm{cn}} \to \mathbf{Ab}$ is so: the natural transformation

$$\pi_0(X) \otimes \pi_0(X) \longrightarrow \pi_0(X \otimes Y)$$

is an isomorphism if X and Y are connective. Thus its right adjoint, the Eilenberg–MacLane spectrum functor, is lax monoidal by [HA, Cor. 7.3.2.7]. It therefore induces a functor $\mathbf{CAlg}(\mathbf{Ab}) \to \mathbf{CAlg}(\mathbf{Sp})$ on commutative algebra objects, i.e., a functor $\mathbf{CRing} \to \mathbf{CAlg}(\mathbf{Sp})$ from ordinary commutative rings to commutative ring spectra. In this way, the Eilenberg–MacLane spectrum on a ring has the structure of a commutative ring spectrum. ▲

Remark 1.68. The functor $\mathbf{CRing} \to \mathbf{CAlg}(\mathbf{Sp})$ from the previous example is fully faithful, with essential image the discrete commutative ring spectra.

Example 1.69 The complex K-theory spectrum KU from Example 1.28 can be upgraded to a commutative ring spectrum. There are several ways of doing this; we provide a sketch of one approach. We start with the suspension spectrum $\Sigma^\infty_+ \mathbf{CP}^\infty$. The space $\mathbf{CP}^\infty$ is an Eilenberg–MacLane space, and its multiplication induces a commutative ring spectrum structure on its suspension spectrum. One can invert the Bott element $\beta \in \pi_2 \Sigma^\infty_+ \mathbf{CP}^\infty$, resulting in a commutative ring spectrum $\Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}]$. This commutative ring spectrum has the property that there is a graded ring isomorphism

$$\pi_*(\Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}]) \cong \pi_*(\Sigma^\infty_+ \mathbf{CP}^\infty)[\beta^{-1}].$$

By a theorem of Snaith [Sna81], the underlying spectrum of $\Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}]$ is equivalent to KU. ▲

Theorem 1.70 (Tor spectral sequence, [HA], Prop 7.2.1.19) *Let R be a commutative ring spectrum, and let M and N be two A-module spectra. There is a spectral sequence with E_2-page*

$$E_2^{p,q} = \mathrm{Tor}^{\pi_* R}_{p,q}(\pi_* M,\ \pi_* N)$$

converging to $\pi_*(M \otimes_R N)$.

1.4.1 Cochain algebra spectra

Let R be a commutative ring spectrum, and Y a spectrum. An alternative notation for the R-module $\mathrm{map}(Y, R)$ is R^Y. In the case where $Y = \Sigma^\infty_+ X$ is the pointed suspension spectrum of a space X, this R-module can be given the structure of a commutative R-algebra. Recall from Theorem 1.41 that $\mathbf{CAlg}_R$ has all limits, and that the forgetful functor $\mathbf{CAlg}_R \to \mathbf{Mod}_R$ preserves these. Note that if X is a space, then X is also an ∞-groupoid, so we can talk about X-shaped diagrams in an ∞-category.

Definition 1.71 Let X be a pointed space and R a commutative ring spectrum. The ***R*-cochains** of X is the commutative R-algebra spectrum

$$R^{X_+} := \lim_X R,$$

where the limit is taken in $\mathbf{CAlg}_R$.

This definition gives a functor $\mathscr{S}^{\mathrm{op}} \to \mathbf{CAlg}_R$.

Remark 1.72. One can also define R^X if X is a pointed space, but this will be a nonunital commutative R-algebra.

Note that if $X = *$ is a point, then $R^{*_+} = R$. If X is a pointed space, then the inclusion of the basepoint induces an augmentation $R^{X_+} \to R$. In fact, taking slices under a point, the functor $\mathscr{S}^{\mathrm{op}} \to \mathbf{CAlg}_R$ induces a functor $\mathscr{S}_*^{\mathrm{op}} \to \mathbf{CAlg}_R^{\mathrm{aug}}$ to augmented R-algebras.

Proposition 1.73 *Let R be a commutative ring spectrum. The composite functor*

$$\mathscr{S} \xrightarrow{R^{(-)_+}} \mathbf{CAlg}_R \longrightarrow \mathbf{Mod}_R$$

is equivalent to the functor $\operatorname{map}(\Sigma_+^\infty -, R)$*.*

Proof. The functor $\mathscr{S}^{\mathrm{op}} \to \mathbf{CAlg}_R$ is the unique functor (up to equivalence) that preserves limits and that sends a point to R. As the forgetful functor $\mathbf{CAlg}_R \to \mathbf{Mod}_R$ preserves limits, we find that the composite $\mathscr{S}^{\mathrm{op}} \to \mathbf{Mod}_R$ is also the unique functor preserving limits and sending a point to R. The functor $\operatorname{map}(\Sigma_+^\infty -, R)$ satisfies the same property. ■

Later in this work we may use the notation R^{X_+} to refer to both the R-algebra and the underlying R-module; the context should make clear what we mean by the notation.

Corollary 1.74 *If X is a space and R a commutative ring spectrum, then the cohomology $R^*(X)$ is naturally a graded-commutative $\mathbf{Z}$-graded ring.*

As in the case of the singular cochains from singular cohomology, one should expect the functor $R^{(-)_+}$ to behave best on finite spaces. In the following result, note that the maps $R \to R^{X_+}$ and the natural map $\mathbf{S}^{X_+} \to R^{X_+}$ together induce (using Proposition 1.43) a natural map

$$R \otimes \mathbf{S}^{X_+} \longrightarrow R^{X_+}.$$

See Definition 1.34 for the notion of dualisability of an object.

Proposition 1.75 *Let R be a commutative ring spectrum and X a space. Suppose that $\Sigma_+^\infty X$ is dualisable in the ∞-category* $\mathbf{Sp}$*. Then the natural map*

$$R \otimes \mathbf{S}^{X_+} \xrightarrow{\simeq} R^{X_+}$$

is an equivalence.

Proof. It suffices to check that the map is an equivalence on underlying spectra. On underlying spectra, this is the natural map

$$R \otimes \operatorname{map}(\Sigma_+^\infty X, \mathbf{S}) \longrightarrow \operatorname{map}(\Sigma_+^\infty X, R).$$

If $\Sigma_+^\infty X$ is dualisable in $\mathbf{Sp}$, then this is an equivalence. ■

Remark 1.76. The ∞-category of *finite spectra* $\mathbf{Sp}^{\mathrm{fin}}$ is the full subcategory of $\mathbf{Sp}$ that contains $\mathbf{S}$ and is closed under finite colimits. A spectrum is dualisable if and only if it is finite. We phrased the above result using dualisability to make the proof more transparent. It is also how we can later generalise this result: see Proposition 4.21.

1.5 Bousfield localisation

It turns out that algebraic operations such as localisation and completion at a prime p can also be done with spaces, spectra, commutative ring spectra and module spectra. These operations are special instances of an ∞-categorical construction called *localisation*. Informally, a localisation of an ∞-category at a collection of morphisms W is another ∞-category obtained by inverting all morphisms in W.

Definition 1.77 Let $\mathcal{C}$ be an ∞-category and let W be a class of morphisms in $\mathcal{C}$. A **localisation of $\mathcal{C}$ at W** is an ∞-category $\mathcal{D}$ together with a functor $L\colon \mathcal{C} \to \mathcal{D}$ that sends W to equivalences, satisfying the following universal property. For every ∞-category $\mathcal{E}$, precomposition with L yields a categorical equivalence

$$L^*\colon \mathbf{Fun}(\mathcal{D},\mathcal{E}) \longrightarrow \mathbf{Fun}_W(\mathcal{C},\mathcal{E}),$$

where $\mathbf{Fun}_W(\mathcal{C},\mathcal{E})$ denotes the full subcategory of $\mathbf{Fun}(\mathcal{C},\mathcal{E})$ on the functors that send W to equivalences. A localisation is called **reflective** if L admits a fully faithful right adjoint.

Remark 1.78. This terminology is different from that of [HTT], where the term localisation is meant to refer to a *reflective* localisation. See Definition 5.2.7.2 and Warning 5.2.7.3 of op. cit.

In certain cases, we can construct a localisation of $\mathcal{C}$ at W as a subcategory of $\mathcal{C}$ of a particular form. These types of localisations are called *Bousfield localisations*. It was originally developed by Bousfield in the setting of spaces [Bou75] and spectra [Bou79], but the ideas can be used in a much more general context. We summarise the ∞-categorical formulation of Bousfield localisation, and then specialise to the case of spectra, ring and module spectra, and spaces. This section is heavily based on the exposition by Lawson [Law20], with parts taken from Heuts [Heu19] and Barthel and Bousfield [BB17].

We start with some terminology.

Definition 1.79 Let $\mathcal{C}$ be an ∞-category and let S be a class of morphisms in $\mathcal{C}$.

(a) An object $A \in \mathcal{C}$ is **S-local** if for every morphism $f\colon X \to Y$ in S, the map

$$f^*\colon \mathrm{Map}_{\mathcal{C}}(Y,A) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X,A)$$

is an equivalence. Write $L^S\mathcal{C}$ for the full subcategory on the S-local objects.

(b) A morphism $f\colon X \to Y$ in $\mathcal{C}$ is an **S-equivalence** if for every S-local object A, the map

$$f^*\colon \mathrm{Map}_{\mathcal{C}}(Y,A) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X,A)$$

is an equivalence.

(c) An **S-localisation** of an object $A \in \mathcal{C}$ is an S-equivalence $A \to A'$ with A' an S-local object.

Proposition 1.80 **([Law20], Prop. 3.9)** *Let $\mathcal{C}$ be an ∞-category and let S be a class of morphisms in $\mathcal{C}$. Then $L^S\mathcal{C}$ is closed under limits taken in $\mathcal{C}$.*

Proposition 1.81 *Let $\mathcal{C}$ be a stable ∞-category and let S be a class of morphisms in $\mathcal{C}$. Then the ∞-category $L^S\mathcal{C}$ is stable.*

Proof. By [HA, Prop. 1.4.2.11], it suffices to show $L^S\mathcal{C}$ is closed under suspensions in $\mathcal{C}$. This follows from the observation that $\mathrm{Map}_{\mathcal{C}}(X, -)$ commutes with suspensions, since

$$\mathrm{Map}_{\mathcal{C}}(X, \Sigma Y) \simeq \Sigma\Omega\,\mathrm{Map}_{\mathcal{C}}(X, \Sigma Y) \simeq \Sigma\,\mathrm{Map}_{\mathcal{C}}(X,\ \Omega\Sigma Y) \simeq \Sigma\,\mathrm{Map}_{\mathcal{C}}(X, Y)$$

because $\mathcal{C}$ is stable. ∎

Proposition 1.82 *Let $\mathcal{C}$ be a closed symmetric monoidal ∞-category, and let S be a class of morphisms in $\mathcal{C}$. Let $A \in \mathcal{C}$ be an S-local object. Then for every object $X \in \mathcal{C}$, the object $\mathrm{map}_{\mathcal{C}}(X, A)$ is S-local.*

Proof. Let $Y \to Z$ be an S-equivalence. Since the functor $\mathrm{map}(X, -)$ is left adjoint to $- \otimes X$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(Z,\ \mathrm{map}_{\mathcal{C}}(X, A)) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Map}_{\mathcal{C}}(Y,\ \mathrm{map}_{\mathcal{C}}(X, A)) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathcal{C}}(Z \otimes X,\ A) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(Y \otimes X,\ A).
\end{tikzcd}$$

Because A is S-local, the bottom horizontal map is an equivalence. We find that the top horizontal map is an equivalence, proving the claim. ∎

We now turn to the question of the existence of localisations of this form, i.e., when there is a localisation functor $\mathcal{C} \to L^S\mathcal{C}$. First, some additional terminology.

Definition 1.83 Let $\mathcal{C}$ be an ∞-category and let W be a class of morphisms in $\mathcal{C}$. The class W is called **strongly saturated** if it satisfies the following conditions.

(i) For every pushout diagram in $\mathcal{C}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Y \arrow[d] \\
X' \arrow[r, "f'"'] & Y',
\end{tikzcd}$$

if f belongs to W, then so does f'.

(ii) The class W is closed under colimits.

(iii) The class W is closed under equivalence, and its image in the homotopy category satisfies the 2-out-of-3 axiom.

Lemma 1.84 **([HTT], Rmk. 5.5.4.7)** *Let $\mathcal{C}$ be an ∞-category. Every collection S of morphisms in $\mathcal{C}$ generates a smallest strongly saturated class $\overline{S}$.*

In the case of presentable ∞-categories, given a collection S of morphisms, we can always localise at $\overline{S}$, and this localisation is equivalent to the full subcategory on the S-local objects.

Theorem 1.85 **([HTT], Prop. 5.5.4.15)** *Let $\mathcal{C}$ be a presentable ∞-category and let S be a (small) collection of morphisms of $\mathcal{C}$. Let $\overline{S}$ denote the strongly saturated class of morphisms generated by S.*

(*a*) *For each object $A \in \mathcal{C}$, there exists a morphism $s \colon A \to A'$ such that A' is S-local and s belongs to $\overline{S}$.*

(*b*) *The ∞-category $L^S\mathcal{C}$ is presentable.*

(*c*) *The inclusion $L^S\mathcal{C} \to \mathcal{C}$ has a left adjoint L.*

(*d*) *For every morphism f of $\mathcal{C}$, the following are equivalent:*

  (*i*) *f is an S-equivalence.*

  (*ii*) *f belongs to $\overline{S}$.*

  (*iii*) *Lf is an equivalence.*

Since left adjoints are unique up to contractible choice, we can pick any left adjoint to the inclusion functor $L^S\mathcal{C} \to \mathcal{C}$ and call it a *localisation functor*. The counit gives a natural transformation $\mathrm{id}_{\mathcal{C}} \to L$. Note that combining (d) with (a), we see that every object of $\mathcal{C}$ has an S-localisation in the sense of Definition 1.79(c).

The localisation $L \colon \mathcal{C} \to L^S\mathcal{C}$ is called **Bousfield localisation of $\mathcal{C}$ at S**. Note that in the terminology of Definition 1.77, this is the localisation of $\mathcal{C}$ at $\overline{S}$. Bousfield localisations are always reflective.

Corollary 1.86 *Let $\mathcal{C}$ be an ∞-category and let S be a class of morphisms in $\mathcal{C}$. Suppose $\mathcal{C}$ has all limits and colimits.*

(*a*) *The limit of a diagram in $L^S\mathcal{C}$ is computed by taking the limit in $\mathcal{C}$.*

(*b*) *The colimit of a diagram in $L^S\mathcal{C}$ is computed by applying L to the colimit in $\mathcal{C}$.*

In particular, the ∞-category $L^S\mathcal{C}$ has all limits and colimits.

Proof. The first assertion because the limit in $\mathcal{C}$ of S-local objects is S-local (Proposition 1.80). The second assertion is formal: L is a left adjoint and left inverse to the inclusion $L^S\mathcal{C} \to \mathcal{C}$, so it creates colimits. ■

In good cases, we can 'upgrade' a localisation functor on a symmetric monoidal ∞-category to a symmetric monoidal functor.

Theorem 1.87 **([HA], Prop. 2.2.1.9)** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and S a class of morphisms in $\mathcal{C}$. Let $L \colon \mathcal{C} \to L^S\mathcal{C}$ be a localisation functor. Suppose that for every $X, Y \in \mathcal{C}$, the map*

$$L(X \otimes Y) \longrightarrow L(LX \otimes LY)$$

is an equivalence. Then the subcategory $L^S\mathcal{C}$ of local objects has the structure of a symmetric monoidal ∞-category, and L is a symmetric monoidal functor.

Lemma 1.88 **([Law20], Prop. 12.3)** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and S a class of morphisms in $\mathcal{C}$. Suppose that for every S-equivalence f and every object $X \in \mathcal{C}$, the morphism $f \otimes \mathrm{id}_X$ is an S-equivalence. Then the localisation functor $L\colon \mathcal{C} \to L^S\mathcal{C}$ satisfies the condition of the previous theorem.*

1.5.1 Localisation of spectra

Knowing the general theory, we specialise to the case of spectra.

Definition 1.89 Let E be a spectrum. A map $f\colon X \to Y$ of spectra is called an ***E*-equivalence** if the induced map $E_*(X) \to E_*(Y)$ is an isomorphism.

Equivalently, $f\colon X \to Y$ is an E-equivalence if the cofibre $\operatorname{cofib} f$ has zero E-homology, i.e., if and only if $E \otimes \operatorname{cofib} f$ is contractible, i.e., if and only if $E \otimes f$ is an equivalence. The class of E-equivalences is a strongly saturated class.

We write $\mathbf{Sp}_E$ for the full subcategory on E-local spectra, and write $L_E\colon \mathbf{Sp} \to \mathbf{Sp}_E$ for the localisation functor. We write $\mathbf{S}_E$ for the E-localisation of the sphere spectrum. By Proposition 1.81, the ∞-category $\mathbf{Sp}_E$ is stable. Because E-localisations satisfy the condition of Lemma 1.88, the ∞-category $\mathbf{Sp}_E$ becomes a symmetric monoidal ∞-category, and L_E is a symmetric monoidal functor. Concretely, the product of $X, Y \in \mathbf{Sp}_E$ is given by the *E-local smash product*

$$L_E(X \otimes Y).$$

By Proposition 1.82, it is even a closed symmetric monoidal ∞-category, with the E-local mapping spectrum being the same as the normal mapping spectrum.

If X is an E-local spectrum, then its E-local Spanier–Whitehead dual is $X^\vee = \operatorname{map}(X, \mathbf{S}_E)$. An E-local spectrum X is called ***E*-locally dualisable** if it is dualisable in the ∞-category $\mathbf{Sp}_E$, i.e., if for all E-local spectra Y, the natural map

$$L_E(X^\vee \otimes Y) \longrightarrow \operatorname{map}(X, Y)$$

is an equivalence.

Some localisations are of a special form: they are given by smashing with the localisation $\mathbf{S} \to \mathbf{S}_E$ of the sphere spectrum.

Definition 1.90 Let E be a spectrum. We call E-localisation a **smashing localisation** if for all spectra X, the map

$$X \simeq X \otimes \mathbf{S} \longrightarrow X \otimes \mathbf{S}_E$$

is an E-localiation.

A commutative ring spectrum R is called E-local when the underlying spectrum is so, and similarly for module spectra. The ∞-category of E-local commutative ring spectra is $\mathbf{CAlg}(\mathbf{Sp}_E)$. If R is an E-local commutative ring spectrum, then we write $\mathbf{Mod}_{R,E}$ for $\mathbf{Mod}_R(\mathbf{Sp}_E)$. Likewise we write $\mathbf{CAlg}_{R,E}$ for $\mathbf{CAlg}(\mathbf{Mod}_{R,E})$. Note that the ∞-category $\mathbf{Mod}_{R,E}$ is stable because $\mathbf{Mod}_R$ is. Also note that $\mathbf{Mod}_{R,E}$ is equivalent to the subcategory of $\mathbf{Mod}_R$ on the E-local modules. Indeed, both satisfy the universal property for localisation at the E-equivalences. Similarly, $\mathbf{CAlg}_{R,E}$ is equivalent to the subcategory on the E-local algebras.

The localisation $L_E\colon \mathbf{Sp} \to \mathbf{Sp}_E$ of spectra induces a localisation $L_E\colon \mathbf{Mod}_R \to \mathbf{Mod}_{R,E}$ of modules. This functor is symmetric monoidal, so it also induces a localisation $L_E\colon \mathbf{CAlg}_R \to \mathbf{CAlg}_{R,E}$ of commutative algebras.

For particular spectra E, the operation of E-localisation is so important that it deserves a special name.

Example 1.91 Localisation with respect to the Eilenberg–MacLane spectrum $H\mathbf{Q}$ is **rationalisation**. We denote the rationalisation of a spectrum X by $X_{\mathbf{Q}}$. A spectrum X is $H\mathbf{Q}$-local if and only if its homotopy groups $\pi_* X$ are rational vector spaces. A map $X \to Y$ of spectra is an $H\mathbf{Q}$-equivalence if and only if it induces an isomorphism on rational homotopy groups $\pi_*(X) \otimes \mathbf{Q} \to \pi_*(Y) \otimes \mathbf{Q}$. This is a smashing localisation: the functor $X \mapsto X \otimes H\mathbf{Q}$ is equivalent to rationalisation (see [Bou79, Prop. 2.4]). ▲

Example 1.92 Let p be a prime number. Let $\mathbf{SZ}_{(p)}$ denote the $\mathbf{Z}_{(p)}$-Moore spectrum, i.e., the spectrum with

$$\begin{aligned} \pi_n(\mathbf{SZ}_{(p)}) &= 0 \quad \text{for } n < 0, \\ \pi_0(\mathbf{SZ}_{(p)}) &= H_0(\mathbf{SZ}_{(p)}) = \mathbf{Z}_{(p)}, \\ H_n(\mathbf{SZ}_{(p)}) &= 0 \quad \text{for } n > 0. \end{aligned}$$

Localisation with respect to $\mathbf{SZ}_{(p)}$ is called ***p*-localisation** of spectra. We denote the p-localisation of a spectrum X by $X_{(p)}$. A spectrum X is $\mathbf{SZ}_{(p)}$-local if and only if its homotopy groups $\pi_* X$ are p-local abelian groups. By [Bou79, Prop. 2.4], this is a smashing localisation. ▲

Example 1.93 Let p be a prime number. Let $\mathbf{S}/p$ denote the *mod p Moore spectrum*: the cofibre of the degree p map $\mathbf{S} \to \mathbf{S}$. Localisation with respect to $\mathbf{S}/p$ is called ***p*-completion** of spectra. We denote the p-completion of X by $X_p^\wedge$. It turns out that this is equivalent to the limit

$$X_p^\wedge \simeq \lim_n X/p^n.$$

This is not a smashing localisation.

As similar as it looks, the relevant algebraic shadow is not p-completion in the classical sense, but *derived p-completion*. An abelian group A is derived p-complete if and only if the natural map

$$A \longrightarrow \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/p^\infty,\, A)$$

is an isomorphism, where $\mathbf{Z}/p^\infty$ denotes $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$. If A is a finitely generated or a flat abelian group, then this coincides with classical p-completion. See Appendix A for an extended discussion of derived p-completion. We write

$$L_0 A := \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/p^\infty,\ A) \qquad \text{and} \qquad L_1 A := \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/p^\infty,\ A)$$

If A has bounded p-torsion, then $L_1 A = 0$.

A spectrum X is p-complete if and only if its homotopy groups $\pi_* X$ are derived p-complete. By [Bou79, Prop. 2.5], there is a split short exact sequence

$$0 \longrightarrow L_0\, \pi_n X \longrightarrow \pi_n(X^\wedge_p) \longrightarrow L_1\, \pi_{n-1} X \longrightarrow 0.$$ ▲

In some cases, we use special notation for the p-completion of a spectrum. We write $\mathbf{S}_p$ for the p-completion of the sphere spectrum $\mathbf{S}$, and KU_p for the p-completion of the complex K-theory spectrum KU. This is by analogy with the notation $\mathbf{Z}_p$ for the p-adic integers (rather than writing $\mathbf{Z}^\wedge_p$), as $\mathbf{S}_p$ and KU_p as the base rings over which most of the work in this thesis is done.

1.5.2 Localisation of spaces

Definition 1.94 Let E be a spectrum. A map $f\colon X \to Y$ of (pointed) spaces is called an ***E*-equivalence** if the induced map $E_*(X) \to E_*(Y)$ is an isomorphism.

We will make light use of the localisation of spaces. Most of the time we will consider pointed spaces only, and for this reason we write $\mathscr{S}_E$ for the full subcategory of $\mathscr{S}_*$ on the E-local pointed spaces.

The definitions of rationalisation, p-localisation and p-completion of spaces are the same as in the case for spectra. These localisations work best on simply-connected spaces, or more generally on nilpotent spaces. For example, a nilpotent pointed space is $H\mathbf{Q}$-local if and only if its homotopy groups are rational vector spaces; for nilpotent pointed spaces, there is an analogous short exact sequence for the homotopy groups of its p-completion. For more information, the reader may consult [Law20, §7, 9.1], and [BB17] for a specific discussion of p-completion of spaces and spectra (and how these compare).

K-theory and Adams operations 2

If R is a commutative ring spectrum and X a space, we saw in Corollary 1.74 that the R-cohomology $R^*(X)$ naturally has the structure of a graded-commutative ring. The commutative structure on R is much richer than this, and in fact it endows $R^*(X)$ with more structure: $R^*(X)$ carries cohomology operations. These cohomology operations depend on the ring spectrum R. For mod 2 cohomology $H\mathbf{F}_2$, these are the Steenrod squares. For p-completed K-theory KU_p, these are the *Adams operations*, which are additive and multiplicative homomorphisms ψ^k indexed by p-adic integers $k \in \mathbf{Z}_p$.

In this chapter we discuss the Adams operation on p-completed K-theory. Specifically, we look at the KU_p-homology of spectra, and the KU_p-cohomology of spaces. The K-homology of spectra has Adams operations indexed by p-adic units $\mathbf{Z}_p^\times$. The K-homology of commutative ring spectra, as well as the K-cohomology of spaces, forms a *θ-algebra*, which is a refinement of a ring with Adams operations indexed by all p-adic integers $\mathbf{Z}_p$. The structure of a θ-algebra captures essentially all the algebraic information in these groups: by a result of McClure, the K-homology of a symmetric ring spectrum is a *free* θ-algebra.

More precisely, we work with a completed variant of KU_p-homology, which has proved to be a more natural object than the non-completed variant. We introduce this in §2.1. In §2.2 we give a short introduction to Adams operations, both on the K-theory of spaces and of spectra. In §2.3 we collect the precise results about the structure that the Adams operations give to K-theory. In §2.4 we discuss free θ-algebras, which then allows us to discuss the K-theory of symmetric ring spectra in §2.5.

Throughout this chapter, p denotes a fixed prime number. As in Appendix A, we use an asterisk in the notation $\mathbf{Mod}^*_{\mathbf{Z}_p}$ to signify $\mathbf{Z}/2$-graded modules, and similarly for $\mathbf{CAlg}^*_{\mathbf{Z}_p}$.

Convention 2.1 The coefficients of (p-adic) K-theory are 2-periodic:

$$(KU)_* = \mathbf{Z}[u^{\pm}] \qquad \text{and} \qquad (KU_p)_* = \mathbf{Z}_p[u^{\pm}],$$

with u in degree 2 denoting the Bott element. As such, the homotopy of a module (or commutative algebra) over KU (or KU_p) can either be viewed as a 2-periodic $\mathbf{Z}$-graded module, or as a $\mathbf{Z}/2$-graded module. In this chapter, and even this whole thesis, we shall view these as $\mathbf{Z}/2$-graded unless explicitly stated otherwise. That is, $KU_*(X)$ denotes the $\mathbf{Z}/2$-graded abelian group $KU_0(X) \oplus KU_1(X)$, and for cohomology we use the negative degree version $KU^0(X) \oplus KU^{-1}(X)$. The ring structure on cohomology is defined via

$$KU^{-1}(X) \otimes KU^{-1}(X) \longrightarrow KU^{-2}(X) \xrightarrow{\cong} KU^{0}(X),$$

where the second map is division by u. Throughout this work, the term 'graded' will mean '$\mathbf{Z}/2$-graded' unless explicitly stated otherwise.

2.1 Morava K-theory

Definition 2.2 The **first Morava K-theory** $K(1)$ is the spectrum KU/p, i.e., the cofibre in $\mathbf{Sp}$ of the degree p map $KU \to KU$.

The prime p is left implicit in the notation, which is commonplace. (This is common in the field of chromatic homotopy theory: see Chapter 4.) We will only use the notation $K(1)$ when working over a fixed prime, so this should not cause confusion.

Remark 2.3. Even though KU is a commutative ring spectrum, $K(1)$ does not admit the structure of a commutative ring spectrum.

Remark 2.4. The above definition is slightly unusual. Most commonly $K(1)$ refers to a summand of KU/p called the *Adams summand*. However, localisation with respect to this summand yields an equivalent functor to localisation with respect to KU/p. We are only interested in localisation with respect to $K(1)$, so the above definition should not cause issues. We use the notation $K(1)$ instead of KU/p to reflect related literature more closely.

Proposition 2.5 *The functor $L_{K(1)}\colon \mathbf{Sp} \to \mathbf{Sp}$ is equivalent to the composite $L_{\mathbf{S}/p}L_{KU}$.*

Proof. See, e.g., [Hov93]. ■

Lemma 2.6 ([Law20], Prop. 9.17) *Let E be a commutative ring spectrum. Then E-module spectra are E-local.*

Corollary 2.7 *On KU-module spectra, $K(1)$-localisation is the same as p-completion.*

Proposition 2.8 ([Bou79], §4) *The localisation L_{KU} is a smashing localisation* (*see Definition* 1.90).

Corollary 2.9 *Let X be a spectrum, and let Y be a KU-local spectrum. Then the $K(1)$-localisation of $X \otimes Y$ is the same as its p-completion.*

Proof. The KU-localisation of $X \otimes Y$ is given by $X \otimes Y \to X \otimes Y \otimes \mathbf{S}_{KU}$. The map $Y \to Y \otimes \mathbf{S}_{KU}$ is an equivalence because Y is KU-local. The result now follows from Proposition 2.5. ■

In the style of §1.5.1, we write $\mathbf{Mod}_{KU_p,\, K(1)}$ for the ∞-category of $K(1)$-local KU_p-module spectra, and $\mathbf{CAlg}_{KU_p,\, K(1)}$ for $K(1)$-local commutative KU_p-algebra spectra.

As discussed in Example 1.93, the algebraic shadow of p-completion of spectra is *derived p-completion*. Appendix A discusses the precise type of derived p-completion we need here, viz. the setting of $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-modules. We use the same notation as we do there, writing $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ for the category of derived p-complete $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-modules. We write $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ for the category of derived p-complete graded-commutative $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebras.

Proposition 2.10 ([BF15], Cor. 3.14) *Let M be a KU_p-module spectrum. Then M is $K(1)$-local if and only if the $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module $\pi_* M$ is derived p-complete.*

Thus the homotopy groups give rise to functors

$$\pi_*\colon \mathbf{Mod}_{KU_p,\, K(1)} \longrightarrow \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p} \qquad \text{and} \qquad \pi_*\colon \mathbf{CAlg}_{KU_p,\, K(1)} \longrightarrow \widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}.$$

It turns out that the KU_p-homology of a space or spectrum X, as defined in Definition 1.25, is not a very natural invariant to consider. This is because $(KU_p)_*(X)$ is defined as $\pi_*(X \otimes KU_p)$, but the smash product $X \otimes KU_p$ need not be $K(1)$-local even if X is. This makes it more difficult to study: even for reasonable spectra, $(KU_p)_*(X)$ can be intractable. It is better to consider the completed alternative.

Definition 2.11 Let X be a spectrum. The **completed KU_p-homology** of X is the $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module

$$(KU_p)^\wedge_*(X) := \pi_*(L_{K(1)}(KU_p \otimes X)).$$

By the above, completed KU_p-homology is derived p-complete. Strictly speaking this is not a homology theory, because it does not preserve coproducts (since $L_{K(1)}$ does not preserve coproducts as a functor $\mathbf{Sp} \to \mathbf{Sp}$).

Remark 2.12. There is no need for a completed variant of KU_p-cohomology: the mapping spectrum $\operatorname{map}(X, KU_p)$ is already $K(1)$-local for every spectrum X (see Proposition 1.82). So for any spectrum X, the cohomology $KU_p^*(X)$ is derived p-complete.

A detailed treatment of $K(1)$-localisation and completed KU_p-homology is given by Hovey and Strickland [HS99]. (Note that they work with so-called *Morava E-theory* at height n; taking $n = 1$ retrieves the present setting.) The following result will be particularly useful.

Theorem 2.13 ([HS99], Thm. 8.6) *Let X be a $K(1)$-local spectrum. The following are equivalent:*

(i) *the spectrum X is $K(1)$-locally dualisable;*

(ii) *the graded $\mathbf{Z}_p$-module $(KU_p)^\wedge_*(X)$ is finitely generated;*

(iii) *the graded $\mathbf{Z}_p$-module $KU_p^*(X)$ is finitely generated.*

2.2 Adams operations

Theorem 2.14 (Adams [Ada62]) *There exist cohomology operations $\psi^k\colon KU^0(-) \to KU^0(-)$ on the complex K-theory of spaces, with $k \in \mathbf{Z}$, satisfying the following conditions, where X is a space, where $x, y \in KU^0(X)$, and where $k, \ell \geqslant 0$.*

(i) *We have* $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$.

(ii) *We have* $\psi^k(xy) = \psi^k(x) \cdot \psi^k(y)$.

(iii) *If* x *is the class of a line bundle, then* $\psi^k(x) = x^k$.

(iv) *We have* $\psi^k(\psi^\ell(x)) = \psi^{k\ell}(x)$.

(v) *If* p *is a prime number, then* $\psi^p(x) \equiv x^p \bmod p$.

(vi) *If* $n \geqslant 0$ *and* $x \in \widetilde{KU}^0(S^{2n})$*, then* $\psi^k(x) = k^n \cdot x$.

These conditions uniquely determine the cohomology operations ψ^k for $k \in \mathbf{Z}$. We call these (unstable) **Adams operations**. The operation ψ^0 sends a vector bundle to its rank, and ψ^{-1} sends a vector bundle to its linear dual bundle.

If $KU^0(X)$ has no p-torsion, then Condition (v) implies that there is a unique operation $\theta^p \colon KU^0(X) \to KU^0(X)$ such that

$$\psi^p(x) = x^p + p \cdot \theta^p(x).$$

If $KU^0(X)$ has p-torsion, then there would be multiple choices for such a θ^p. By a refinement of the construction of the Adams operations, there is a canonical choice for a map θ^p which is natural in the space X (see, e.g., [BF15], §5.3). In the presence of p-torsion it is best to record θ^p instead of ψ^p, because it holds more information.

In order to move from the K-theory of spaces to the K-theory of spectra, we need to stabilise these Adams operations. Let us temporarily consider K-theory as $\mathbf{Z}$-graded rather than $\mathbf{Z}/2$-graded. So far we have only looked at KU^0. To move to negative degrees KU^{-n}, one could consider the Adams operation ψ^k on $\widetilde{KU}(\Sigma^n_+ X) = KU^{-n}(X)$. But this choice does not turn the Adams operations into stable cohomology operations. Indeed, if $n \geqslant 1$, then the square

$$\begin{array}{ccc} KU^0(X) & \xrightarrow{\psi^k} & KU^0(X) \\ {\scriptstyle u^n}\downarrow & & \downarrow{\scriptstyle u^n} \\ KU^{-2n}(X) & \xrightarrow{\psi^k} & KU^{-2n}(X), \end{array}$$

with u denoting the Bott element, does *not* commute: by Condition (vi) above, we have (where $x \in KU(X)$)

$$\psi^k(u^n \cdot x) = \psi^k(u^n) \cdot \psi^k(x) = k^n \cdot u^n \cdot \psi^k(x).$$

If $k \in \mathbf{Z}$ is a unit in $KU^0(X)$, then we can remedy this: instead of considering ψ^k on $\widetilde{KU}(\Sigma^{2n}_+ X)$, we take $1/k^n \cdot \psi^k$ on $\widetilde{KU}(\Sigma^{2n}_+ X)$ as operation. Then trivially for every $n \geqslant 0$ the square

$$\begin{array}{ccc} KU^0(X) & \xrightarrow{\psi^k} & KU^0(X) \\ {\scriptstyle u^n}\downarrow & & \downarrow{\scriptstyle u^n} \\ KU^{-2n}(X) & \xrightarrow{1/k^n \cdot \psi^k} & KU^{-2n}(X), \end{array}$$

commutes.

When working with p-completed K-theory KU_p instead of ordinary K-theory, then for every X, the integers coprime to p are invertible in $KU_p(X)$. Thus $KU_p(X)$ has Adams operations ψ^k for k coprime to p. This even extends to ψ^k for $k \in \mathbf{Z}_p^\times$. In the end, for every $k \in \mathbf{Z}_p^\times$ we get a map of spectra $\psi^k \colon KU_p \to KU_p$.

2.3 θ-algebras

In this section we record results about the structure that the Adams operations give to the p-adic K-theory of spectra. The K-theory of a spectrum forms a *Morava module*. The K-theory of a commutative ring spectrum has additional structure: it is a *θ-algebra*. We closely follows Goerss and Hopkins [GH, §5.2].

Definition 2.15 A ***p*-adic Morava module** is a derived p-complete topological $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module M with a continuous action of $\mathbf{Z}_p^\times$ by degree-preserving maps, such that the quotient M/p is a discrete $\mathbf{Z}_p^\times$-module. A **morphism** of Morava modules is a continuous morphism of graded $\mathbf{Z}_p$-modules that intertwines the action $\mathbf{Z}_p^\times$. Write $\mathbf{MorMod}_p$ for the category of Morava modules.

Often we will write the action of $k \in \mathbf{Z}_p^\times$ on a Morava module M as ψ^k (or by ψ^k_M if we wish to emphasize the module M) and call these the *Adams operations* of M. If the prime p is obvious from the context, we will often omit it and simply refer to p-adic Morava modules as *Morava modules*.

Remark 2.16. In [Bou99], Bousfield uses the name *stable Adams module* (see Definition 2.7 of op. cit.) for a closely related concept. Roughly speaking, he uses the name *Adams module* to mean a stable Adams module which also has a compatible ψ^p operation.

Proposition 2.17 ([GH], Prop. 2.2.2) *Let X be a spectrum. The completed KU_p-homology $(KU_p)^\wedge_*(X)$ is naturally a p-adic Morava module. If $f \colon X \to Y$ is a morphism of spectra, then the induced map $f_* \colon (KU_p)^\wedge_*(X) \to (KU_p)^\wedge_*(Y)$ is a morphism of p-adic Morava modules.*

We call a $\mathbf{Z}/2$-graded algebra A **strictly commutative** if it is graded-commutative and $x^2 = 0$ for all $x \in A$ of odd degree.

Definition 2.18 A ***p*-adic θ-algebra** is a strictly commutative $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebra A which is also a Morava module, together with continuous maps

$$\theta^p \colon A_0 \longrightarrow A_0 \qquad \text{and} \qquad \theta^p \colon A_1 \longrightarrow A_1,$$

satisfying the following conditions.

(i) For all $k \in \mathbf{Z}_p^\times$, the operation ψ^k on A is $\mathbf{Z}_p$-linear, and for $x, y \in A$ homogeneous,

$$\psi^k(xy) = \begin{cases} \psi^k(x) \cdot \psi^k(y) & |x| = 0 \quad \text{or} \quad |y| = 0, \\ \frac{1}{k} \cdot \psi^k(x) \cdot \psi^k(y) & |x| = |y| = 1. \end{cases}$$

(ii) We have $\theta^p(1) = 0$.

(iii) For all $k \in \mathbf{Z}_p^\times$, we have $\theta^p \circ \psi^k = \psi^k \circ \theta^p$.

(iv) For $x, y \in A$ homogeneous,

$$\theta^p(x+y) = \begin{cases} \theta^p(x) + \theta^p(y) - \frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i} \cdot x^i y^{p-i} & |x| = |y| = 0, \\ \theta^p(x) + \theta^p(y) & |x| = |y| = 1. \end{cases}$$

(v) For $x, y \in A$ homogeneous,

$$\theta^p(xy) = \begin{cases} \theta(x) \cdot y^p + x \cdot \theta^p(y) + p \cdot \theta^p(x) \cdot \theta^p(y) & |x| = 0 \quad \text{or} \quad |y| = 0, \\ \theta^p(x) \cdot \theta^p(y) & |x| = |y| = 1. \end{cases}$$

A **morphism** of p-adic θ-algebras is a morphism of graded $\mathbf{Z}_p$-algebras that is a morphism of Morava modules and that intertwines the action of θ^p. Write $\mathbf{Alg}_{\theta,p}$ for the category of p-adic θ-algebras.

As with Morava modules, if the prime p is clear from the context, we will simply refer to these algebras as *θ-algebras*. In that case we may also write θ for θ^p.

Some comments are in place. By definition a p-adic θ-algebra is derived p-complete. We extend θ^p to be a map $A \to A$ by putting $\theta^p(x+y) = \theta^p(x) + \theta^p(y)$ for $x \in A_0$ and $y \in A_1$. We define an Adams operation ψ^p by

$$\psi^p \colon A \longrightarrow A, \quad x \longmapsto x^p + p \cdot \theta^p(x).$$

This map commutes with θ^p and with ψ^k for all $k \in \mathbf{Z}_p^\times$. Note that $\psi^p(x) = p \cdot \theta^p(x)$ if x is of odd degree. The conditions on θ^p imply that ψ^p is an additive operation satisfying a multiplicativity condition similar to the one on ψ^k above: for $x, y \in A$ homogeneous,

$$\psi^p(x) \cdot \psi^p(y) = \begin{cases} \psi^p(xy) & |x| = 0 \quad \text{or} \quad |y| = 0, \\ p \cdot \psi^p(xy) & |x| = |y| = 1. \end{cases}$$

If A has no p-torsion, then this operation ψ^p recovers θ^p via the formula

$$\theta^p(x) = \frac{\psi^p(x) - x^p}{p}.$$

Moreover, part of the multiplicativity condition on θ^p can be phrased as, for $x \in A_0$ and $y \in A_1$,

$$\theta^p(xy) = \psi^p(x) \cdot \theta^p(y). \tag{2.19}$$

Remark 2.20. In the literature, the term *θ-algebra* is often used to refer to a non-graded ring A with an operation $\theta \colon A \to A$ satisfying the conditions on θ^p above in the case of x, y even. This is also called a *δ-ring* by others. We use the adjective "p-adic" to indicate the additional structure.

Remark 2.21. Bousfield (in [Bou96a; Bou96b; Bou99]) uses different notation and terminology than the above. He defines a $\mathbf{Z}/2$-graded θ-algebra as a ring A with operations $\theta\colon A_0 \to A_0$ and $\psi\colon A_1 \to A_1$ satisfying the conditions we put on $\theta^p\colon A_0 \to A_0$ and $\theta^p\colon A_1 \to A_1$ above, respectively. The Adams operations ψ^k are not part of the datum of a θ-algebra; he uses the separate term *θ-algebras with Adams operations* when including the Adams operations. Compared to the notation introduced above, the difference is particularly unfortunate: the operation ψ^p is *not* what Bousfield means when writing ψ.

Remark 2.22. In addition to a difference in notation, Bousfield uses the term "p-adic θ-algebra" to refer to a different concept than the above. See [Bou96a, §6] or [Bou96b, §1] for precise definitions. Roughly speaking, his definition requires the algebra to be complete in a stronger sense than (derived) p-completeness. Every Bousfield p-adic θ-algebra determines a p-adic θ-algebra in the sense of Definition 2.18. The difference can be seen particularly clearly in the case of free θ-algebras below in §2.4: see Remark 2.32.

Example 2.23 The ring $\mathbf{Z}_p$ is a p-adic θ-algebra under the operations

$$\psi^k = \mathrm{id} \quad (\text{for } k \in \mathbf{Z}_p^\times) \qquad \text{and} \qquad \theta^p(x) = (x - x^p)/p.$$

The operation ψ^p is thus also the identity. In this work, when we view $\mathbf{Z}_p$ as a θ-algebra, it is always with these Adams operations. ▲

Theorem 2.24 *Let A be a commutative ring spectrum. The completed KU_p-homology $(KU_p)^\wedge_*(A)$ is naturally a p-adic θ-algebra. If $f\colon A \to B$ is a morphism of commutative ring spectra, then the induced map $f_*\colon (KU_p)^\wedge_*(A) \to (KU_p)^\wedge_*(B)$ is a morphism of p-adic θ-algebras.*

Proof. This follows from the work of McClure [McC], as explained by Barthel and Frankland [BF15, §6]. (Note though that they use the notation that Bousfield does, writing ψ instead of θ^p on degree 1; see Remark 2.21.) ■

Remark 2.25. McClure's work actually shows a stronger statement, namely that for every $\mathbf{H}_\infty$-ring spectrum A, the completed homology $(KU_p)^\wedge_*(Y)$ is a p-adic θ-algebra.

Proposition 2.26 *Let X be a space. Then $KU_p^*(X)$ is naturally a p-adic θ-algebra. If $f\colon X \to Y$ is a map of spaces, then the induced map $f_*\colon KU_p^*(X) \to KU_p^*(Y)$ is a morphism of p-adic θ-algebras.*

Proof. See, e.g., [Bou96a, Thm. 1.11], bearing in mind Remark 2.22. ■

Concretely, the Adams operations on this θ-algebra are the ones described in §2.2.

The K-theory of a point is $\mathbf{Z}_p$ with the θ-algebra structure from Example 2.23. Thus for any space X, we have a splitting of p-adic θ-algebras

$$KU_p^*(X) = \mathbf{Z}_p \oplus \widetilde{KU}_p(X).$$

On $\mathbf{Z}_p$, all Adams operations ψ^k for $k \in \mathbf{Z}_p$ are the identity. Therefore, for any space X, the multiplicativity condition (2.19) on θ^p shows that θ^p is $\mathbf{Z}_p$-linear on elements of odd degree in K-theory.

Example 2.27 Let n be an odd natural number. The K-theory of S^n is generated by a class x in odd degree:

$$KU_p^*(S^n) = \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot x,$$

with $x^2 = 0$ (which is enforced by graded-commutativity). The properties of the Adams operations (see Theorem 2.14) imply that for $k \in \mathbf{Z}$, we have

$$\psi^k(x) = k^{(n+1)/2} \cdot x.$$

By continuity, the operation ψ^k for $k \in \mathbf{Z}_p$ is given by the same formula. We recover θ^p from ψ^p because the K-theory is torsion-free:

$$\theta^p(x) = \frac{\psi^p(x) - x^p}{p} = \frac{p^{(n+1)/2} \cdot x}{p} = p^{(n-1)/2} \cdot x.$$

Using that θ^p is $\mathbf{Z}_p$-linear on elements of odd degree, we see that θ^p on $\widetilde{KU}_p^*(S^n)$ is given by multiplication by $p^{(n-1)/2}$. ▲

Example 2.28 Let n be an even natural number. The K-theory of S^n is generated by a class y in even degree:

$$KU_p^*(X) = \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot y,$$

with $y^2 = 0$. Using the same reasoning as in the previous example, we find that for $k \in \mathbf{Z}_p$, we have

$$\psi^k(x) = k^{n/2} \cdot y.$$

In this case θ^p becomes

$$\theta^p(y) = \frac{\psi^p(y) - y^p}{p} = \frac{p^{n/2} \cdot y}{p} = p^{n/2-2} \cdot y. \qquad ▲$$

2.4 Free θ-algebras

There is a θ-algebra analogue of the free algebra functor on a module M over a ring. Given a module M with prescribed Adams operations (more precisely, given a Morava module M), we can freely add an operation θ^p to it and freely turn it into a ring. The result is the *free θ-algebra* on M.

Theorem 2.29 *Let M be a p-adic Morava module. Then there exists a p-adic θ-algebra $\mathrm{Free}_{\theta,p}(M)$ with a morphism $M \to \mathrm{Free}_{\theta,p}(M)$ of Morava modules, unique up to unique isomorphism, satisfying the following universal property. For every θ-algebra A and morphism $M \to A$ of Morava modules, there is a unique morphism $\mathrm{Free}_{\theta,p}(M) \to A$ of θ-algebras making the diagram*

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & A \\ & \mathrm{Free}_{\theta,p}(M) \arrow[u, dashed] \end{tikzcd}$$

commute.

We call $\mathrm{Free}_{\theta,p}(M)$ the **free θ-algebra** on M. We get a functor $\mathrm{Free}_{\theta,p}\colon \mathbf{MorMod}_p \to \mathbf{Alg}_{\theta,p}$ that is left adjoint to the forgetful functor $\mathbf{Alg}_{\theta,p} \to \mathbf{MorMod}_p$. If the prime p is clear from the context, we may write Free_θ for $\mathrm{Free}_{\theta,p}$. If $f\colon M \to A$ is a map of Morava modules to a θ-algebra A, we may be sloppy and also write f for the map $\mathrm{Free}_{\theta,p}(M) \to A$.

We do not need the above result in full generality; for us the following cases will suffice.

Example 2.30 Let us write θ for θ^p in this example; this means we write θ^t for the t-th iterate of θ. If M is a Morava module of the form $\mathbf{Z}_p \cdot x$ with x in even degree, then we write the free θ-algebra on M as $\mathrm{Free}_\theta[x]$. As an algebra this is

$$\mathrm{Free}_\theta[x] = \mathbf{Z}_p[x, \theta x, \theta^2 x, \ldots]^\wedge_p.$$

Note that in this case classical p-completion coincides with derived p-completion, because the polynomial ring is free as $\mathbf{Z}_p$-module (see Theorem A.4). The θ-action sends $\theta^t x$ to $\theta^{t+1}x$, and this uniquely determines θ on the entire algebra. For $k \in \mathbf{Z}_p^\times$, the Adams operation ψ^k is extended from the action of ψ^k on M.

Let A be a θ-algebra and $f\colon M \to A$ a morphism of Morava modules. Then the induced morphism $\mathrm{Free}_\theta(M) \to A$ is given by sending $\theta^t x$ to $\theta^t_A(f(x))$ for all $t \geqslant 0$ (where we understand $\theta^0 x$ to mean x). ▲

Example 2.31 Again we write θ for θ^p in this example. If M is the Morava module $\mathbf{Z}_p \cdot y$ with y in odd degree, then as an algebra, the free θ-algebra on M is

$$\mathrm{Free}_\theta[y] = \Lambda_{\mathbf{Z}_p}[y, \theta y, \theta^2 y, \ldots]^\wedge_p.$$

The maps θ and ψ^k are induced from the operations on M.

More generally, if M is a Morava module concentrated in odd degree, then we can form $\mathrm{Free}_\theta(M)$ as follows. Let FM denote the module $M \oplus M \oplus \cdots$, which inherits a topology and Adams operations from M. (Although strictly speaking it is not a Morava module, as it is not derived p-complete when M is nonzero.) Define a $\mathbf{Z}_p$-linear homomorphism $\theta^p\colon FM \to FM$ by shifting each copy of M one to the right. Now define as algebras

$$\mathrm{Free}_\theta(M) := L_0\, \Lambda_{\mathbf{Z}_p}(FM).$$

(In the language of §A.3, this is the free derived p-complete graded algebra on FM.) The Adams operations ψ^k are induced from the Adams operations on FM. The homomorphism θ^p on FM uniquely determines an operation θ^p on $\mathrm{Free}_\theta(M)$ that turns it into a p-adic θ-algebra.

Let A be a θ-algebra and $f\colon M \to A$ a morphism of Morava modules. Then the induced morphism $\mathrm{Free}_\theta(M) \to A$ can be described as follows. First define a map $Ff\colon FM \to A$ given by applying the map $\theta^t_A \circ f$ to the t-th copy of M. Note that Ff is a $\mathbf{Z}_p$-module homomorphism that intertwines the Adams operations. By the universal property of the free derived p-complete graded algebra, the map Ff induces a map $L_0\,\Lambda(FM) \to A$, which is the map $\mathrm{Free}_\theta(M) \to A$. ▲

The universal property of free θ-algebras shows that they are naturally augmented: take $\mathrm{Free}_\theta(M) \to \mathbf{Z}_p$ to be the map induced by the zero map $M \to \mathbf{Z}_p$.

Remark 2.32. Because Bousfield's use of the term "p-adic θ-algebra" is different from ours (see Remark 2.22), his definition of the term "free p-adic θ-algebra" is different also. For example, Bousfield's free p-adic θ-algebra on a generator x in even degree is a power series algebra $\mathbf{Z}_p[\![x, \theta x, \ldots]\!]$. The above notion is more relevant for us because it appears in McClure's Theorem described in the next section. In Chapter 5, Bousfield's free p-adic θ-algebras will make an appearance.

2.5 McClure's Theorem

McClure's Theorem computes the completed KU_p-homology of a symmetric ring spectrum $\mathrm{Sym}(X)$, for X a spectrum. (See Example 1.64 for the definition of symmetric ring spectra.) In other words, it computes the homotopy groups of

$$L_{K(1)}\, KU_p \otimes \mathrm{Sym}(X).$$

By Proposition 1.57 this is naturally equivalent to $L_{K(1)} \mathrm{Sym}_{KU_p}(KU_p \otimes X)$, so one can also think of this as computing the homotopy of certain $K(1)$-localised symmetric KU_p-algebra spectra. Note also that the functor $L_{K(1)} \mathrm{Sym}_{KU_p}$ is the functor $\mathrm{Sym}_{\mathcal{C}}$ for the symmetric monoidal ∞-category $\mathcal{C} = \mathbf{Mod}_{KU_p,\, K(1)}$ of $K(1)$-local KU_p-modules.

Theorem 2.33 (McClure) *Let X be a spectrum such that $(KU_p)^\wedge_*(X)$ is a flat $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module. Then we have a natural isomorphism of p-adic θ-algebras*

$$(KU_p)^\wedge_*(\mathrm{Sym}(X)) \cong \mathrm{Free}_{\theta,p}((KU_p)^\wedge_*(X)).$$

Moreover, this isomorphism is compatible with the free-forgetful adjunctions: if R is a commutative ring spectrum, then the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{CAlg}(\mathbf{Sp})}(\mathrm{Sym}(X),\ R) \arrow[r, "\simeq"] \arrow[d, "(KU_p)^\wedge_*"'] & \mathrm{Map}_{\mathbf{Sp}}(X, R) \arrow[d, "(KU_p)^\wedge_*"] \\
\mathrm{Hom}_\theta(\mathrm{Free}_{\theta,p}((KU_p)^\wedge_*(X)),\ (KU_p)^\wedge_*(R)) \arrow[r, "\cong"] & \mathrm{Hom}((KU_p)^\wedge_*(X),\ (KU_p)^\wedge_*(R))
\end{tikzcd}$$

commutes up to natural isomorphism.

Proof. This follows from the work by McClure [McC], as explained by Barthel and Frankland [BF15, §6]. (Note again that they use the notation of Bousfield; see Remark 2.21). ■

The augmentation on $\mathrm{Sym}(X)$ induces an augmentation on the free θ-algebra. As a special case of the above, this agrees with the natural augmentation of the free θ-algebra.

Computation of K-theory cochains 3

Let p be a prime number. Consider a pointed space X whose p-adic K-theory $KU_p^*(X)$ is an exterior algebra on a finite number of odd generators. More precisely, assume there is a p-adic Morava module M, together with a morphism of Morava modules $\theta_M^p\colon M \to M$, such that

(i) M is finitely generated and free as a $\mathbf{Z}_p$-module;

(ii) M is concentrated in degree 1;

(iii) there is an isomorphism
$$KU_p^*(X) \cong \Lambda_{\mathbf{Z}_p}(M)$$
of p-adic θ-algebras, where the θ^p-map on $\Lambda(M)$ is induced from θ_M^p.

This class of spaces includes odd spheres, as well as many finite H-spaces, including all simply-connected compact Lie groups; we discuss this in detail in §3.3. Note that Condition (i) is a type of finiteness condition on X. Henceforth, we shall refer to Condition (iii) by saying that the θ-algebra $KU_p^*(X)$ is *an exterior algebra on the pair* (M, θ_M^p).

Consider the KU_p-cochains on such an X, i.e., the commutative KU_p-algebra spectrum $KU_p^{X_+}$ from Definition 1.71. The homotopy groups of this algebra spectrum is the K-theory $KU_p^*(X)$. Since $KU_p^*(X)$ is free as a graded algebra, one might guess that $KU_p^{X_+}$ is itself free as a KU_p-algebra spectrum, i.e., that it is a symmetric KU_p-algebra in the sense of Example 1.64. This turns out to be naive, because $KU_p^*(X)$ is not free as a θ-algebra. Rather, we can give a θ-algebra presentation of $KU_p^*(X)$ of the form

$$\mathrm{Free}_{\theta,p}(M) \longrightarrow \mathrm{Free}_{\theta,p}(M) \longrightarrow KU_p^*(X).$$

This presentation does generalise: in this chapter we prove that there exists a cofibre sequence in $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p,\,K(1)}$ of the form

$$L_{K(1)}\, KU_p \otimes \mathrm{Sym}(E) \longrightarrow L_{K(1)}\, KU_p \otimes \mathrm{Sym}(E) \longrightarrow KU_p^{X_+}$$

where E is a spectrum. This result is motivated and inspired by closely related work done by Bousfield [Bou99] (we summarise his results in §5.1).

The spectrum E can in fact be taken to be a topological analogue of the module M. We introduce this spectrum in §3.1, and shall denote it by $\mathscr{M}(M)$ to stress this analogy.

In §3.2 we construct the advertised presentation, first constructing the maps in the sequence, and then proving these fit into a cofibre sequence (which, by Definition 1.4, is a particular kind of pushout square). The precise result is described in Theorem 3.12, which is the main result of this chapter. In §3.3 we discuss the spaces that this theorem applies to. This leads us to a short list of questions in §3.4.

Throughout this chapter, p denotes a fixed prime. In §3.2, X denotes a pointed space as above, M a p-adic Morava module as above, and $\theta^p_M \colon M \to M$ a morphism as above. We follow Convention 2.1, viewing K-theory as $\mathbf{Z}/2$-graded; in particular we make the identification $(KU_p)_* = \mathbf{Z}_p$.

3.1 K-theory Moore spectra

Let G be a p-adic Morava module, concentrated either in even or odd degree, and which is finitely generated as $\mathbf{Z}_p$-module. In [Bou85, Prop. 8.7], Bousfield constructs a KU-local spectrum $\mathscr{M}_{(p)}(G)$ along with an isomorphism

$$(KU_{(p)})_*(\mathscr{M}_{(p)}(G)) \cong G$$

respecting stable p-local Adams operations ψ^k for $k \in \mathbf{Z}_{(p)}^\times$.

Definition 3.1 Let G be a p-adic Morava module, concentrated either in even or odd degree, and which is finitely generated as $\mathbf{Z}_p$-module. We write $\mathscr{M}(G)$ for the spectrum $\mathscr{M}_{(p)}(G)^\wedge_p$.

Note that $\mathscr{M}(G)$ is a $K(1)$-local spectrum.

Lemma 3.2 *The isomorphism $(KU_{(p)})_*(\mathscr{M}_{(p)}(G)) \cong G$ induces an isomorphism*

$$(KU_p)^\wedge_*(\mathscr{M}(G)) \cong G$$

of p-adic Morava modules.

Proof. Since G is derived p-complete, we have

$$L_0\Big((KU_{(p)})_*(\mathscr{M}_{(p)}(G))\Big) = L_0 G \cong G.$$

Using Corollary 2.9, we find that

$$\begin{aligned}(KU_p)^\wedge_*(\mathscr{M}(G)) &= \pi_*\Big(L_{K(1)}(KU_p \otimes \mathscr{M}(G))\Big) \\ &= \pi_*\Big(\big(KU_{(p)} \otimes \mathscr{M}_{(p)}(G)\big)^\wedge_p\Big),\end{aligned}$$

and using the short exact sequence for the homotopy groups of the p-completion of a spectrum (see Example 1.93), this evaluates to

$$\begin{aligned}(KU_p)^\wedge_*(\mathscr{M}(G)) &\cong L_0\, \pi_*\Big(KU_{(p)} \otimes \mathscr{M}_{(p)}(G)\Big) \\ &= L_0\Big((KU_{(p)})_*(\mathscr{M}_{(p)}(G))\Big) \\ &\cong G.\end{aligned}$$

The Adams operations ψ^k for $k \in \mathbf{Z}_{(p)}^\times$ are respected by this isomorphism, hence by continuity so are all Adams operations ψ^k for $k \in \mathbf{Z}_p^\times$. ■

Remark 3.3. The notation $\mathscr{M}(G)$ is taken from Bousfield [Bou99], although his use there is slightly different from ours. We discuss the difference in §5.1 (specifically, see Remark 5.4).

Remark 3.4. Up to equivalence, the spectrum $\mathscr{M}(G)$ is the unique $K(1)$-local spectrum with the property of Lemma 3.2. This follows, e.g., by combining Proposition 3.5 and Proposition 4.25 below.

3.2 Construction of the presentation

The construction of the cofibre sequence will be a topological version of the resolution of the θ-algebra ΛM hinted at earlier in this chapter. In fact, taking homotopy groups of the resolution of $KU_p^{X_+}$ will yield the resolution of ΛM as a θ-algebra. To build the maps in the cofibre sequence, we require some results relating maps of spectra with homomorphisms on their K-theory.

Recall from Definition A.12 that a module is called *pro-free* when it is of the form $L_0 F$ with F a free module. If Y and Z are spectra, then we have a natural map

$$\pi_* \operatorname{map}(Y,\, L_{K(1)} KU_p \otimes Z) \longrightarrow \operatorname{Hom}_{(KU_p)_*}((KU_p)^\wedge_*(Y),\, (KU_p)^\wedge_*(Z))$$

evaluating on completed KU_p-homology.

Proposition 3.5 ([BH16], Prop. 1.14) *Let Y and Z be spectra. Suppose that $(KU_p)^\wedge_*(Y)$ is pro-free. Then the natural map*

$$\pi_* \operatorname{map}(Y,\, L_{K(1)} KU_p \otimes Z) \longrightarrow \operatorname{Hom}_{(KU_p)_*}((KU_p)^\wedge_*(Y),\, (KU_p)^\wedge_*(Z))$$

is an isomorphism.

Since M is finitely generated and free, it is immediate that ΛM is pro-free. In what follows, we use the construction of $\mathrm{Free}_\theta(M)$ given in Example 2.31: we write FM for $M \oplus M \oplus \cdots$, and let θ^p act on this by shifting every copy of M one to the right. The free θ-algebra is then $L_0\,\Lambda_{\mathbf{Z}_p}(FM)$. This in particular shows that $\mathrm{Free}_\theta(M)$ is pro-free also.

Corollary 3.6 *Let Y be a spectrum such that $(KU_p)^\wedge_*(Y)$ is pro-free. Then we have an isomorphism*

$$KU_p^*(Y) \cong \operatorname{Hom}_{(KU_p)_*}((KU_p)^\wedge_*(Y),\, (KU_p)_*).$$

Proof. Take $Z = \mathbf{S}$ in Proposition 3.5. ■

If Y and Z are spectra, we have a natural map

$$KU_p^*(Y) \otimes_{(KU_p)_*} KU_p^*(Z) \longrightarrow KU_p^*(Y \otimes Z).$$

Lemma 3.7 *Let Y and Z be spectra. Suppose that $KU_p^*(Y)$ is finitely generated and free. Then the natural map*

$$KU_p^*(Y) \otimes_{(KU_p)_*} KU_p^*(Z) \longrightarrow KU_p^*(Y \otimes Z)$$

is an isomorphism.

Proof. Because $KU_p^*(Y)$ is free, the Künneth spectral sequence for $KU_p^*(Y \otimes Z)$ (as in, e.g., [Hov13, Thm. 5.3]) collapses, yielding the claim. ∎

We can now construct the maps in the cofibre sequence.

(1) **The map**

$$\boldsymbol{L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \longrightarrow KU_p^{X_+}.}$$

The KU_p-algebra $KU_p^{X_+}$ is $K(1)$-local, so it is equivalent to give a map

$$KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \longrightarrow KU_p^{X_+}$$

of KU_p-algebras. This is equivalent to giving a map $\mathscr{M}(M) \to KU_p^{X_+}$ of spectra, which is the same as giving a map

$$\mathscr{M}(M) \otimes \Sigma_+^\infty X \longrightarrow KU_p$$

of spectra, i.e., a cohomology class in $KU_p^*(\mathscr{M}(M) \otimes \Sigma_+^\infty X)$. Note that Lemma 3.7 applies because $KU_p^*(\mathscr{M}(M)) \cong M$ is a free $\mathbf{Z}_p$-module, giving us an isomorphism

$$KU_p^*(\mathscr{M}(M) \otimes \Sigma_+^\infty X) \cong KU_p^*(\mathscr{M}(M)) \otimes_{(KU_p)_*} KU_p^*(X).$$

Using Corollary 3.6 and the fact that M is finitely generated and free, the right-hand side is

$$\mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \Lambda M \cong \mathrm{Hom}_{\mathbf{Z}_p}(M,\ \Lambda M).$$

In conclusion, giving the desired map is equivalent to giving a homomorphism $M \to \Lambda M$. This we take to be the natural inclusion.

(2) **The map**

$$\boldsymbol{L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \longrightarrow L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)).}$$

By the same reasoning, it is equivalent to give a map

$$\mathscr{M}(M) \longrightarrow L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M))$$

of spectra. By Proposition 3.5, such a map is determined up to equivalence by the homomorphism it induces on K-homology. McClure's Theorem 2.33 says that the K-homology of $\mathrm{Sym}(\mathscr{M}(M))$ is

$$\mathrm{Free}_\theta((KU_p)_*^\wedge(\mathscr{M}(M)) \cong \mathrm{Free}_\theta(M).$$

We take the homomorphism on K-homology to be

$$\theta^p - F\theta_M^p \colon M \longrightarrow \mathrm{Free}_\theta(M), \quad x \longmapsto (0, x, 0, 0, \dots) - (\theta_M^p(x), 0, 0, 0, \dots),$$

where θ^p denotes the free operation on the free θ-algebra, and $F\theta_M^p$ denotes the componentwise action of θ_M^p on $FM = M \oplus M \oplus \cdots$. By abuse of notation we will also denote the resulting map of KU_p-algebras by $\theta^p - F\theta_M^p$.

Note that in the first step, we could also have used a cohomology class in $KU_p^*(\Sigma^\infty_+ X \otimes L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)))$. From this point of view, the map is classified by the morphism of θ-algebras $\mathrm{Free}_\theta(M) \to \Lambda M$ induced by the morphism $M \to \Lambda M$ of Morava modules.

In a general ∞-category, recall that a map is called *nullhomotopic* if it factors over a zero object. In augmented KU_p-algebras, the algebra KU_p is a zero object.

Lemma 3.8 *The composite of the above two maps is a nullhomotopic map of augmented KU_p-algebra spectra.*

Proof. By the same reasoning as above, the composite map is classified by the corresponding map

$$\mathscr{M}(M) \longrightarrow KU_p^{X_+},$$

which is classified by the corresponding cohomology class in

$$KU_p^*(\mathscr{M}(M)) \otimes_{(KU_p)_*} KU_p^*(X) \cong \mathrm{Hom}_{\mathbf{Z}_p}(M, \Lambda M).$$

By definition, the map

$$\theta^p - F\theta^p_M \colon \mathscr{M}(M) \longrightarrow L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M))$$

induces on K-theory

$$\theta^p - F\theta^p_M \colon M \longrightarrow \mathrm{Free}_\theta(M).$$

Thus the map $M \to \Lambda M$ classifying the composite map of spectra is equal to

$$\begin{array}{ccccc} M & \xrightarrow{\theta^p - F\theta^p_M} & \mathrm{Free}_\theta(M) & \longrightarrow & \Lambda M, \\ x & \longmapsto & (-\theta^p_M(x),\ x,\ 0,\ 0,\ \dots) & \longmapsto & -\theta^p_M(x) + \theta^p_M(x). \end{array}$$

As this is the zero map, we are done. ∎

Henceforth we fix a choice of nullhomotopy; in the end the choice does not impact the cofibre sequence. This choice assembles to a triangle

$$\begin{tikzcd} L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \arrow[r, "\theta^p - F\theta^p_M"] \arrow[d] & L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \arrow[d] \\ KU_p \arrow[r] & KU_p^{X_+} \end{tikzcd} \tag{3.9}$$

in the category $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p}$ of augmented KU_p-algebras. (Note again that KU_p is a zero object in augmented KU_p-algebras.) Our main result is that this is a pushout square. To prove this, we need an algebraic computation.

Lemma 3.10 (Shifting generators trick) *Let $I' \subseteq I$ and $J' \subseteq J$ be two inclusions of sets. Let A be the free $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebra on generators $\{ x_i \}_{i \in I'}$ in even degree and $\{ y_j \}_{j \in J'}$ in odd degree. Let B be the analogous algebra on generators indexed by I and J. Then the map $L_0 A \to L_0 B$ induced by the inclusion is flat.*

Proof. Evidently the inclusion $A \to B$ is flat: the A-module B splits as a direct sum of copies of A. Forming direct sums in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is exact (Proposition A.11), so $L_0 A \to L_0 B$ is also flat. ∎

Lemma 3.11 *The map* $\mathrm{Free}_\theta(M) \to \mathrm{Free}_\theta(M)$ *induced by* $\theta^p - F\theta^p_M$ *is a flat map of* $\mathbf{Z}_p$-*modules.*

Proof. In this proof, we will write θ for the operation θ^p in the free θ-algebra. Let $x_1, \ldots, x_n$ be generators for M. The underlying algebra of $\mathrm{Free}_\theta(M)$ is of the form

$$L_0\, \Lambda_{\mathbf{Z}_p}[\, x_i,\ \theta x_i,\ \theta^2 x_i,\ \theta^3 x_i,\ \ldots\,]_{i=1,\ldots,n}.$$

For $i = 1, \ldots, n$, write $\theta^p_M(x_i) = \sum_{j=1}^n \lambda_{ij} x_j$ for $\lambda_{ij} \in \mathbf{Z}_p$. The map $\theta - F\theta^p_M \colon \mathrm{Free}_\theta(M) \to \mathrm{Free}_\theta(M)$ is obtained by applying L_0 to the map

$$\begin{aligned} f\colon \Lambda_{\mathbf{Z}_p}[\, x_i,\ \theta x_i,\ \theta^2 x_i,\ \ldots\,]_i &\longrightarrow \Lambda_{\mathbf{Z}_p}[\, x_i,\ \theta x_i,\ \theta^2 x_i,\ \ldots\,]_i, \\ \theta^t x_i &\longmapsto \theta^{t+1} x_i - \sum_{j=1}^n \lambda_{ij} \cdot \theta^t x_j. \end{aligned}$$

Here we use the convention that $\theta^0 x_i = x_i$. Denote the image of $\theta^t x_i$ by α_{it}. The elements α_{it} for $i = 1, \ldots, n$ and $t \geqslant 0$, together with the elements x_j for $j = 1, \ldots, n$, also serve as exterior algebra generators. Thus up to isomorphism f is of the form described in the previous lemma, and therefore $L_0 f = \theta - F\theta^p_M$ is flat. ∎

Theorem 3.12 *Let p be a prime number. Let X be a pointed space, let M be a p-adic Morava module, and $\theta^p_M \colon M \to M$ a morphism of Morava modules, such that*

(i) *M is finitely generated and free as a $\mathbf{Z}_p$-module;*

(ii) *M is concentrated in degree 1;*

(iii) *the θ-algebra $KU^*_p(X)$ is an exterior algebra on (M, θ^p_M), i.e., there is an isomorphism*

$$KU^*_p(X) \cong \Lambda_{\mathbf{Z}_p}(M)$$

of p-adic θ-algebras, where the θ^p-map on $\Lambda(M)$ is induced from θ^p_M.

Then the triangle

$$L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \xrightarrow{\theta^p - F\theta^p_M} L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \longrightarrow KU_p^{X_+}$$

constructed above is a cofibre sequence in both $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p}$ *and* $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p,\, K(1)}$.

Proof. As all terms in the triangle are $K(1)$-local, it suffices to prove that the diagram (3.9) is a pushout in $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p}$. By [HTT, Prop. 1.2.13.8], the pushout in $\mathbf{CAlg}^{\mathrm{aug}}_{KU_p}$ is the same as the pushout in $\mathbf{CAlg}_{KU_p}$. Let A denote a pushout of the diagram (3.9) in $\mathbf{CAlg}_{KU_p}$. The diagram induces a map $A \to KU_p^{X_+}$; we prove this is an equivalence. It suffices to show that it induces an isomorphism on homotopy groups (Proposition 1.62). By Corollary 1.56, A is the relative smash product

$$A = L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \otimes_{L_{K(1)}\, KU_p \otimes \mathrm{Sym}(\mathscr{M}(M))} KU_p.$$

The homotopy of a relative tensor product is computed by the Tor spectral sequence of Theorem 1.70. The map $\theta^p - F\theta^p_M \colon \operatorname{Free}_\theta(M) \to \operatorname{Free}_\theta(M)$ is flat (Lemma 3.11), so this spectral sequence collapses immediately. This means we have an isomorphism

$$\pi_* A \cong \operatorname{Free}_\theta(M) \otimes_{\operatorname{Free}_\theta(M)} \mathbf{Z}_p,$$

or in other words: after taking homotopy groups, the pushout square for A becomes a pushout square

$$\begin{tikzcd} \operatorname{Free}_\theta(M) \arrow[r, "\theta^p - F\theta^p_M"] \arrow[d] & \operatorname{Free}_\theta(M) \arrow[d] \\ \mathbf{Z}_p \arrow[r] & \pi_* A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathbf{CAlg}^*_{\mathbf{Z}_p}$. This pushout is a quotient of $\operatorname{Free}_\theta(M)$. A quotient of a derived p-complete module is again derived p-complete (Proposition A.9), so $\pi_* A$ is derived p-complete. Therefore the above square is also a pushout in $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$.

We now compute the pushout in $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ in a different way. Recall the notation $FM = M \oplus M \oplus \cdots$, and recall that $\operatorname{Free}_\theta(M) = L_0 \Lambda(FM)$. We have a pushout square

$$\begin{tikzcd} FM \arrow[r, "\theta^p - F\theta^p_M"] \arrow[d] & FM \arrow[d] \\ 0 \arrow[r] & M \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathbf{Mod}^*_{\mathbf{Z}_p}$. The free derived p-complete algebra functor from §A.3 is a left adjoint, and therefore preserves colimits. The three nonzero terms in this square are concentrated in odd degree, so on these objects this functor is given by $L_0 \Lambda$. As such, the square

$$\begin{tikzcd} \operatorname{Free}_\theta(M) \arrow[r, "\theta^p - F\theta^p_M"] \arrow[d] & \operatorname{Free}_\theta(M) \arrow[d] \\ \mathbf{Z}_p \arrow[r] & \Lambda M \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout square in $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$. We find that the natural map $\pi_* A \to \Lambda M$ is an isomorphism. Since the map $A \to KU_p^{X_+}$ induces this natural map $\pi_* A \to \Lambda M = KU_p^*(X)$ on homotopy groups, we find that the map $A \to KU_p^{X_+}$ is an equivalence. ■

3.3 Applicability

We discuss two types of spaces that satisfy the conditions of Theorem 3.12: odd spheres, and H-spaces.

3.3.1 Odd spheres

Let n be an odd natural number. Recall from Example 2.27 that the K-theory of S^n is given by

$$KU_p^*(S^n) = \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot x$$

with x in odd degree, and

$$\psi^k(x) = k^{(n+1)/2} \cdot x \qquad \text{and} \qquad \theta^p(x) = p^{(n-1)/2} \cdot x.$$

Write $M = \mathbf{Z}_p \cdot x$ for the submodule on x. As algebras, we evidently have an isomorphism

$$KU_p^*(S^n) \cong \Lambda_{\mathbf{Z}_p}(M).$$

We can restrict the Adams operations ψ^k and the operation θ^p to M by precomposing with the inclusion $M \to KU_p^*(S^n)$ and postcomposing with the quotient $KU_p^*(S^n) \to M$. Denote these restrictions by ψ_M^k and θ_M^p. Note that θ_M^p is linear because M is concentrated in odd degree. These restricted operations θ_M^p and ψ_M^k induce operations on $\Lambda(M)$, turning it into a θ-algebra. (Here on $\mathbf{Z}_p$ in $\Lambda(M)$, we take the usual θ-algebra structure on $\mathbf{Z}_p$ from Example 2.23.) The above isomorphism $KU_p^*(S^n) \cong \Lambda(M)$ is then even an isomorphism of θ-algebras, and therefore S^n satisfies the conditions of Theorem 3.12.

In this case, the spectrum $\mathscr{M}(M)$ has a very concrete description. Recall from Remark 3.4 that among the $K(1)$-local spectra, the spectrum $\mathscr{M}(M)$ is uniquely characterised by its completed KU_p-homology. Note that $M = \widetilde{KU}_p^*(S^n)$. Since $(\mathbf{S}^n)^\vee \simeq \mathbf{S}^{-n}$ and $\mathbf{S}^n$ is dualisable, we find an isomorphism of Morava modules

$$(KU_p)_*^\wedge(\mathbf{S}^{-n}) \cong KU_p^*(\mathbf{S}^n) = \widetilde{KU}_p^*(S^n) \cong M.$$

Thus the spectrum $\mathscr{M}(M)$ is $\mathbf{S}_{K(1)}^{-n}$.

Any space whose p-adic K-theory resembles that of an odd sphere closely enough also satisfies the conditions of Theorem 3.12. Bousfield studies such spaces in [Bou99, §5] under the name of *odd KU/p-homology spheres*.

3.3.2 H-spaces

This section is mostly a summary of results of Bousfield, who extended previous work on the classification of the K-theory of H-spaces. His approach is based on a θ-algebra variant of the Milnor–Moore Theorem. His methods are rather technical, so rather than spell out all the details, we present an overview of his results. It is our hope that this provides an easier introduction to the relevant papers. This discussion leads naturally to questions for further research; see §3.4.

To put things in their proper context, let us first review the more classical setting of the *Milnor–Moore Theorem*. Let k be a field. Roughly speaking, a **bialgebra** over k is a $\mathbf{Z}$-graded augmented k-algebra A, which is also equipped with a *comultiplication*

$$A \longrightarrow A \otimes_k A,$$

which is coassociative (i.e., it satisfies the dual versions of associativity) and where the augmentation serves as a counit (i.e., satisfying the dual axiom of a unit element). The multiplication and comultiplication and units are required to be compatible in a suitable sense. A **Hopf algebra** over k is a bialgebra over k together with an *antipode* map $A \to A$ satisfying a compatibility condition. See [MM65, Def. 4.1] for precise definitions.

If A is an augmented k-algebra, the **augmentation ideal** $\widetilde{A}$ is the kernel of the augmentation $A \to k$. If A is a Hopf algebra over k, then its space of **primitives** PA is the kernel of the comultiplication $\widetilde{A} \to \widetilde{A} \otimes_k \widetilde{A}$ restricted to the augmentation ideal. This has a natural structure of a $\mathbf{Z}$-graded Lie algebra over k, with Lie bracket induced by the (graded) commutator bracket of A. One can ask when these primitive elements serve as generators for the algebra A; phrased differently, when the natural map $U(PA) \to A$ from the universal enveloping algebra of PA to A is an isomorphism. If this is the case, we say A is *primitively generated*.

Theorem 3.13 (Milnor–Moore [MM65]) *Let k be a field of characteristic zero, and let A be a Hopf algebra over k. If the Hopf algebra A is*

(*i*) *connected* (*i.e.*, $A_0 = k$ *and* A *vanishes in negative degrees*);

(*ii*) *cocommutative* (*i.e., the comultiplication satisfies the dual axiom for commutativity*);

(*iii*) *degree-wise finite dimensional,*

then the natural map $U(PA) \to A$ *is an isomorphism.*

This has applications in, e.g., the computation of the rational cohomology of H-spaces. If X is an H-space, then the multiplication $X \times X \to X$ induces a comultiplication on the cohomology of X. In good cases this comultiplication is coassociative and cocommutative (e.g., when the multiplication on X is homotopy-associative and homotopy-commutative). In those cases, the Milnor–Moore Theorem gives a purely algebraic way to compute the rational cohomology of X.

Bousfield in [Bou96b] proved an analogue of this classical result for a θ-algebra version of bialgebras. This allows one to compute the p-adic K-theory of H-spaces using purely algebraic structure on K-theory. The precise discussion can be found in §3 and §4 of op. cit., culminating in Theorems 4.4 and 4.7. The original discussion is rather technical at certain points; to increase readability we focus on the applications of these results to the K-theory of H-spaces. We also discuss consequences of this result listed in [Bou99].

A technical detail to note first is that in [Bou96b], Bousfield only considers modules and rings with a ψ^p-operations, without considering Adams operations ψ^k for other k. (Most rings considered here will be torsion-free, in which case the map ψ^p gives the same data as would the operation θ^p.) If p is odd, then one can use the topological cyclicity of $\mathbf{Z}_p^\times$ to adapt the results from [Bou96b] to this case. This is done implicitly in [Bou99], working over an odd prime always. Henceforth we assume p is odd in order to add in the missing Adams operations in these results.

A ***p*-adic *θ*-bialgebra** is a $\mathbf{Z}/2$-graded p-adic θ-algebra A in the sense of Bousfield (which is not precisely the same as our Definition 2.18 — see Remark 2.22), together with a comultiplication

$$A \longrightarrow A \hat{\otimes} A,$$

with $\hat{\otimes}$ denoting the completed p-adic θ-algebra tensor product in the sense of Bousfield, and where this comultiplication is coassociative and counital. If X is an H-space such that $KU_p^*(X)$ is torsion-free, then the multiplication on X induces a comultiplication of this form.

In Definition 4.3 of [Bou96b], Bousfield defines an *augmented primitive element functor* $\overline{P}$, which is (roughly speaking) a functor from θ-bialgebras to augmented $\mathbf{Z}/2$-graded modules $M \to H$ that are both equipped with an Adams operation ψ^p. The target module H is what Bousfield calls a *linear* module (see Definition 2.3 of op. cit., or [Bou99, Def. 4.2]). On the K-theory $KU_p^*(X)$ of a space X, the augmented primitives turn out to be the map (using Example 1.8 of op. cit.)

$$P(KU_p^*(X)) \longrightarrow H^2(X;\mathbf{Z}_p) \oplus H^1(X;\mathbf{Z}_p), \tag{3.14}$$

where $H^2(X;\mathbf{Z}_p)$ is placed in degree 0, and $H^1(X;\mathbf{Z}_p)$ in degree 1, and where P denotes the primitives as defined above. Both the source and the target of this map inherit Adams operations ψ^k for $k \in \mathbf{Z}_p^\times$ and a map θ^p from $KU_p^*(X)$. For the primitives $P(KU_p^*(X))$ this is done (as we did in §3.3.1 above) by precomposing with the inclusion and postcomposing with the quotient.

Next, also in Definition 4.3 of op. cit., he defines a functor U, which is a left adjoint to $\overline{P}$. This is an analogue of the universal enveloping algebra on a Lie algebra, but only in certain cases. If an augmentation $M \to H$ has target $H = 0$ and if M is torsion-free, then U agrees with the free $\mathbf{Z}/2$-graded algebra. In the case of K-theory $KU_p^*(X)$, this happens if and only if $H^1(X;\mathbf{Z}_p)$ and $H^2(X;\mathbf{Z}_p)$ vanish and $KU_p^*(X)$ is torsion-free.

However, in the case of an augmentation $M \to H$ with $M = 0$, then U takes on a different character. In that case $U(0 \to H)$ is a type of group ring on H. Write H_0 and H_1 for the even and odd part of H, respectively. Let $\mathbf{Z}_p[\![H_0]\!]$ denote the *completed group ring*

$$\mathbf{Z}_p[\![H_0]\!] := \lim_{\alpha} \mathbf{Z}_p[(H_0)_\alpha],$$

where the $(H_0)_\alpha$ are the finite quotients of H_0, and where $\mathbf{Z}_p[(H_0)_\alpha]$ denotes the group ring on the group $(H_0)_\alpha$. In the case where $H = \mathbf{Z}_p \cdot x$ with x in even degree, this is the power series algebra $\mathbf{Z}_p[\![x]\!]$, explaining the notation. We give $\mathbf{Z}_p[(H_0)_\alpha]$ a map θ^p by requiring that $\theta^p(h) = 0$ for all $h \in (H_0)_\alpha$, and extending it uniquely so that it satisfies the usual requirements for θ^p. The Adams operations ψ^k we take to be $\psi^k(h) = h^k$ for $h \in (H_0)_\alpha$ and for integers k coprime to p, and extending this in the natural way. This also gives $\mathbf{Z}_p[\![H_0]\!]$ the structure of a θ-algebra. The θ-bialgebra $U(0 \to H)$ is then

$$U(0 \to H) = \mathbf{Z}_p[\![H_0]\!] \hat{\otimes} \Lambda_{\mathbf{Z}_p}(H_1).$$

(In Bousfield's terminology, in this case U agrees with the functor J from Definition 3.4 of op. cit.) An example of a space whose K-theory is of this form is given in Example 3.19.

In general the functor U is a combination of these two types of θ-bialgebras. With the necessary definitions in place, Bousfield then proves the following.

Theorem 3.15 ([Bou96b], Thm. 4.4) *If A is a torsion-free bicommutative $\mathbf{Z}/2$-graded p-adic θ-bialgebra in the sense of Bousfield, then the map $U\overline{P}A \to A$ is an isomorphism.*

The reader should be cautioned that this result does not imply that such an A is always primitively generated. This is due to the mixed nature of the functor U. In certain cases however, it does imply that A is primitively generated. This is the case for a wide range of H-spaces, as witnessed by the following result.

Theorem 3.16 ([Bou99], Thm. 6.2) *Let X be a simply-connected H-space such that*

(*i*) *the primitives $P((KU/p)_0(X))$ vanish;*

(*ii*) *$KU_p^*(X)$ is coassociative.*

Then $KU_p^*(X)$ is torsion-free, and we have an isomorphism of θ-algebras

$$KU_p^*(X) \cong \Lambda_{\mathbf{Z}_p}(P(KU_p^1(X)))$$

and the module $P(KU_p^1(X))$ is torsion-free.

Proof overview. Bousfield shows that under these assumptions, $KU_p^*(X)$ is torsion-free and also cocommutative, and so Theorem 3.15 applies to it. The module $P(KU_p^1(X))$ is an extension of what Bousfield calls a *strictly nonlinear* submodule and a *linear* quotient module. (Such modules are called *regular* by Bousfield — see [Bou99, Def. 4.4].) This implies that the functor U discussed above is the free algebra functor. As the module $P(KU_p^1(X))$ is concentrated in degree 1, the free algebra functor on $P(KU_p^1(X))$ is the exterior algebra as claimed. ■

Note that the result of this theorem even includes the claim that the Adams operations ψ^k for $k \in \mathbf{Z}_p$ (in particular ψ^p) are induced from those on the module $P(KU_p^1(X))$. Since $KU_p^*(X)$ is torsion-free, this means that θ^p is also induced from the corresponding θ_M^p on M. Therefore if X is a space satisfying the conditions of Theorem 3.16 such that $KU_p^*(X)$ is finitely generated, then the module $M = P(KU_p^1(X))$ witnesses that the conditions of Theorem 3.12 are fulfilled for such X.

The conditions of Theorem 3.16 may seem cryptic. The following result, which follows from previous results of Lin and Kane, shows that they are in fact very general.

Theorem 3.17 ([Bou99], Thm. 6.3) *Let X be a simply-connected H-space such that*

(*i*) *$H^*(X;\mathbf{Q})$ is associative;*

(*ii*) *$H^*(X;\mathbf{Z}_{(p)})$ is finitely generated over $\mathbf{Z}_{(p)}$ (in particular, it vanishes above some degree).*

Then X satisfies the conditions of the previous theorem.

Simply-connected compact Lie groups clearly satisfy the conditions of Theorem 3.17. Let us consider an example.

Example 3.18 Let $n \geqslant 2$ and consider $X = \mathrm{SU}(n)$. As Bousfield [Bou99, Ex. 9.3] notes, the primitives of $KU_p^*(\mathrm{SU}(n))$ are isomorphic (as a Morava module) to $\widetilde{KU}_p^*(\mathbf{CP}^{n-1})$. Write $M = \widetilde{KU}_p^*(\mathbf{CP}^{n-1}) \cong P(KU_p^*(\mathrm{SU}(n)))$. Again we can use the uniqueness of $\mathscr{M}(M)$ (see Remark 3.4) to give a concrete expression for $\mathscr{M}(M)$. The spectrum $L_{K(1)}\Sigma^\infty\mathbf{CP}^{n-1}$ is $K(1)$-locally dualisable by Theorem 2.13, so we have an isomorphism

$$(KU_p)_*^\wedge\left(L_{K(1)}\left(\Sigma^\infty\mathbf{CP}^{n-1}\right)^\vee\right) \cong KU_p^*(\Sigma^\infty\mathbf{CP}^{n-1}) = \widetilde{KU}_p^*(\mathbf{CP}^{n-1}).$$

Therefore we find

$$\mathscr{M}(M) \simeq L_{K(1)}\left(\Sigma^\infty\mathbf{CP}^{n-1}\right)^\vee.$$

We can also describe the Adams operations on M. The K-theory of $\mathbf{CP}^{n-1}$ is a truncated polynomial ring

$$KU_p^*(\mathbf{CP}^{n-1}) = \mathbf{Z}_p[x]/(x^n)$$

where x is the class $\gamma_{n-1} - 1$, with γ_{n-1} the tautological line bundle on $\mathbf{CP}^{n-1}$. For $k \in \mathbf{N}$, the operation ψ_M^k is the Adams operation ψ^k on $\mathbf{CP}^{n-1}$, which is given by

$$\psi^k(x) = (x+1)^k - 1 = \sum_{j=1}^{k}\binom{k}{j}x^j.$$

(This follows because $x+1$ is a line bundle and the properties of ψ^k from Theorem 2.14.) We find that $\theta_M^p\colon M \to M$ is given by

$$\theta_M^p(x) = \frac{1}{p}\sum_{j=1}^{p-1}\binom{p}{j}x^j. \qquad \blacktriangle$$

Not all H-spaces are of the form of Theorem 3.16. An illustrative counterexample is $\mathbf{CP}^\infty$.

Example 3.19 The K-theory of $\mathbf{CP}^\infty$ is a power series algebra

$$KU_p^*(\mathbf{CP}^\infty) \cong \mathbf{Z}_p[\![x]\!],$$

where x is the class $\gamma - 1$, with γ the tautological line bundle on $\mathbf{CP}^\infty$. Like in the case of $\mathbf{CP}^n$, the Adams operation ψ^k for $k \in \mathbf{N}$ is given by

$$\psi^k(x) = (x+1)^k - 1 = \sum_{j=1}^{k}\binom{k}{j}x^j.$$

In this case the augmented primitives of $KU_p^*(\mathbf{CP}^\infty)$ turns out to be the map

$$0 \longrightarrow H^2(\mathbf{CP}^\infty;\mathbf{Z}_p).$$

Thus Theorem 3.15 expresses $KU_p^*(\mathbf{CP}^\infty)$ as the completed group ring

$$KU_p^*(\mathbf{CP}^\infty) \cong \mathbf{Z}_p[\![H^2(\mathbf{CP}^\infty;\mathbf{Z}_p)]\!].$$

Here the natural map from the group ring to K-theory is induced by the natural map

$$H^2(-) \longrightarrow KU(-)$$

that (on finite spaces) sends a complex line bundle to its class in K-theory. (Recall that $H^2(-)$ is represented by $\mathbf{CP}^\infty$, and homotopy classes of maps $X \to \mathbf{CP}^\infty$ classify complex line bundles on X if X is finite.) Thus, in the end, Theorem 3.15 proves that the K-theory of $\mathbf{CP}^\infty$ is a power series algebra $\mathbf{Z}_p[\![\gamma]\!]$ on the tautological line bundle. This is a fundamental difference compared to the H-spaces discussed previously, because the Adams operations are *not* induced from an action on γ. Indeed, write $M = \mathbf{Z}_p \cdot \gamma$. Note that $\psi^k(\gamma) = \gamma^k$ for all $k \in \mathbf{N}$. If $k > 1$, then the induced map ψ^k_M on M is zero, because ψ^k_M is obtained by postcomposing with the quotient $KU^*_p(\mathbf{CP}^\infty) \to M$. In particular, the operations ψ^k_M do not recover the Adams operations of $KU^*_p(\mathbf{CP}^\infty)$, and in fact M is not even a Morava module (since the maps ψ^k_M for $k > 1$ are not invertible). ▲

3.4 Questions

Theorem 3.15 shows that the structure of bicommutative p-adic θ-bialgebras is very rigid. This has strong implications for the p-adic K-theory of H-spaces, showing in many cases that, as a θ-algebra, it is an exterior algebra (Theorem 3.16). This structure on K-theory is reflected in higher algebra, as witnessed by Theorem 3.12. But Theorem 3.15 also demonstrates that (completed) group rings naturally come up in the classification of bicommutative θ-bialgebras. A first example is the case of $KU^*_p(\mathbf{CP}^\infty)$ from Example 3.19.

(Q1) To what extent does the KU_p-algebra $KU_p^{\mathbf{CP}^\infty_+}$ form a higher algebra analogue of a group ring?

The rigidity of θ-bialgebras would suggest there to be a rigidity of some sort to a higher-algebraic version of Hopf algebras. In the higher algebra literature, there are various notions of Hopf algebra spectra.

(Q2) Is there a (similarly rigid) structure theorem for bicommutative (in an appropriately coherent sense) Hopf algebra spectra over KU_p?

4 Chromatic homotopy theory

To understand the homotopy groups of a pointed space X (or better, its homotopy type), it suffices to work p-locally one prime p at a time. If X is a p-local space (say pointed and simply-connected), then the easiest part of X is its rationalisation. On homotopy groups $\pi_k(X)$, this comes down to inverting the degree p map $S^k \to S^k$. A main result in rational homotopy theory is that the rational cochains give an equivalence

$$\mathscr{S}_{\mathbf{Q}}^{\geqslant 2,\text{ fin}} \simeq \mathbf{CAlg}(\mathbf{Ch}_{\mathbf{Q}})^{\geqslant 2,\text{ fin}}$$

between rational simply-connected finite pointed spaces, and simply-connected rational cdga's of finite type.

The remaining part in the homotopy groups is captured by the *mod p homotopy groups*, or more generally the *mod p^n homotopy groups*.

Definition 4.1 Let X be a pointed space, and let $n, k \geqslant 1$. Let S^k/p^n denote the cofibre of the degree p^n map $S^k \to S^k$. The k-th **mod p^n homotopy group** of X is

$$\pi_k(X; \mathbf{Z}/p^n) = [S^{k+1}/p^n,\ X]_*.$$

This is a group for $k \geqslant 1$, and an abelian group for $k \geqslant 2$. The difficulty of computing these groups is on par with that of the usual homotopy groups. Chromatic homotopy theory studies these groups by cutting them up further. The next easiest 'slice' of these groups are the *v_1-periodic homotopy groups*. In this chapter we give an overview of the necessary tools in the analysis of v_1-periodic homotopy groups. We are particularly interested in generalising the rational cochain approach from rational homotopy theory. The role of rational cochains turns out to be played by KU_p-cochain algebra spectra. However, unlike in rational homotopy theory, these cochains do not give an equivalence between v_1-periodic homotopy theory and higher algebra. There is a map that measures how well the cochain algebra spectrum models the v_1-periodic homotopy of a space: the *Behrens–Rezk comparison map*.

This chapter is mostly drawn from the expositions by Behrens and Rezk [BR20a] and Heuts [Heu19]. In §4.1 we define the v_1-periodic homotopy groups, culminating in the discussion of the *Bousfield–Kuhn functor*. The next goal is to study the Behrens–Rezk comparison map. This requires some preliminaries: in §4.2 we define *topological André–Quillen cohomology*, which appears in the comparison map. In §4.3 we study descent, where it becomes important that we work over an odd prime p. We end the chapter with an overview of the Behrens–Rezk comparison map, including some basic computational tools, in §4.4. We include some comments about more general v_n-periodic homotopy theory to give the reader some context, although we will only study v_1-periodic homotopy theory in the next chapter.

Throughout this chapter, p denotes a fixed prime, which is assumed to be odd in §4.3 and §4.4.

4.1 v_1-periodic homotopy groups

Definition 4.2 Let X be a p-local pointed space. A **v_1-self map** on X is a map $\Sigma^d X \to X$ (for some $d \geqslant 1$) that induces the zero map on rational homology, and an isomorphism on p-adic complex K-theory.

If $f\colon \Sigma^d X \to X$ is a v_1-self map, then for $i \geqslant 1$ we define its ***i*-th iterate** f^i to be

$$\Sigma^{di} X \xrightarrow{\Sigma^{d(i-1)} f} \Sigma^{d(i-1)} X \longrightarrow \cdots \longrightarrow \Sigma^{2d} X \xrightarrow{\Sigma^d f} \Sigma^d X \xrightarrow{f} X.$$

Because f induces an isomorphism on K-theory, none of the iterates f^i for $i \geqslant 1$ are nullhomotopic.

As a special case of the *periodicity theorem* of Hopkins and Smith, v_1-self maps exist after sufficient suspension, and are asymptotically unique.

Theorem 4.3 *Let X be the p-localisation of a finite pointed space.*

(*a*) *There exist $i \geqslant 0$ and $d \geqslant 1$ such that there is a v_1-self map $\Sigma^{d+i} X \to \Sigma^i X$ on $\Sigma^i X$.*

(*b*) *Suppose $f\colon \Sigma^d X \to X$ and $g\colon \Sigma^e X \to X$ are v_1-self maps on X. Then there are $i, j \geqslant 1$ such that the iterates f^i and g^j are homotopic.*

Proof. See, e.g., [Rav, Thm. 1.5.4]. ■

Fix a k and n such that S^k/p^n admits a v_1-self map. We pick such a map and denote it by $\alpha\colon \Sigma^d S^k/p^n \to S^k/p^n$. This map acts on the mod p^n homotopy groups of a space: if $\ell \geqslant k-1$, then

$$\pi_\ell(X;\mathbf{Z}/p^n) = [S^{\ell+1}/p,\ X]_* \xrightarrow{-\circ(\Sigma^{\ell+1-k}\alpha)} [\Sigma^d S^{k+1}/p,\ X]_* = \pi_{k+d}(X;\mathbf{Z}/p^n).$$

In short, this endows the $\mathbf{Z}$-graded abelian group $\pi_*(X;\mathbf{Z}/p^n)$, for $*$ large enough, with an action of $\mathbf{Z}[\alpha]$. Inverting this action yields the v_1-*periodic homotopy groups*.

Definition 4.4 Let X be a pointed space.

(a) Let $n \geqslant 0$ be a natural number. The **v_1-periodic mod p^n homotopy groups** of X are

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p^n) = \mathbf{Z}[\alpha^{\pm}] \otimes_{\mathbf{Z}[\alpha]} \pi_*(X;\mathbf{Z}/p^n).$$

(b) The **v_1-periodic homotopy groups** of X are

$$v_1^{-1}\pi_*(X) = \operatorname*{colim}_n v_1^{-1}\pi_*(X;\mathbf{Z}/p^n).$$

(c) The **completed v_1-periodic homotopy groups** of X are

$$v_1^{-1}\pi_*^{\wedge}(X) = \lim_n v_1^{-1}\pi_*(X; \mathbf{Z}/p^n).$$

Note that the prime p is left implicit in the notation. The above definitions do not depend on α because of the asymptotic uniqueness of v_1-self maps. The v_1-periodic groups are only well-defined for large enough index, but the resulting groups are periodic with period d, so the definition extends to all integers.

Remark 4.5. The terminology of 'completed' v_1-periodic homotopy groups is taken from Behrens and Rezk [BR20a; BR20b]. We distinguish between these two notions to make comparison with existing literature easier. Bousfield [Bou99] studies the uncompleted version.

Remark 4.6. As the notation hints at, there is also a notion of v_n-periodic homotopy groups, for any natural number $n \geqslant 1$. The natural number n is referred to as the *height*.

Definition 4.7 A map $f\colon X \to Y$ of pointed spaces is called a **v_1-periodic equivalence** if it induces an isomorphism on $v_1^{-1}\pi_*$.

By [Bou99, Cor. 7.6], a map is a v_1-periodic equivalence if and only if it induces an isomorphism on the v_1-periodic mod p homotopy groups.

Remark 4.8. One can localise (in the sense of Definition 1.77) the ∞-category $\mathcal{S}_*$ at the v_1-periodic equivalences, yielding the ∞-category $\mathcal{S}_{v_1}$ of *v_1-periodic spaces*. This can be proved using results of Bousfield; see [Heu21, §3.2] for an ∞-categorical formulation at a general height n, where this ∞-category is denoted by $\mathcal{M}_n^f$. However, this is *not* a Bousfield localisation in the sense of §1.5: it is not even a reflective localisation, because the localisation functor does not preserve all limits. We will not use the existence of this localisation in any serious way.

The periodicity of v_1-periodic homotopy groups implies that for every $k \geqslant 0$, the truncation map $X \to X\langle k\rangle$ is a v_1-periodic equivalence. In particular, spaces whose homotopy groups vanish above a certain degree (e.g., Eilenberg–MacLane spaces) are v_1-periodically contractible.

Studying these groups via the above definitions above is not the best approach. A much more robust approach uses the fact that v_1-periodic homotopy groups of a space X are actually the homotopy groups of a spectrum $\Phi_1 X$.

4.1.1 Monochromatic layers

Recall from Definition 2.2 that $K(1)$ denotes KU/p.

Definition 4.9 Let X be a spectrum. The **first monochromatic layer** M_1X of X is the fibre of the natural map

$$L_{H\mathbf{Q}\oplus K(1)}X \longrightarrow L_{H\mathbf{Q}}X.$$

Write $M_1\mathbf{Sp}$ for the full subcategory of $\mathbf{Sp}$ on the spectra of the form M_1X for $X \in \mathbf{Sp}$.

The operation M_1 can be made into a functor $M_1\colon \mathbf{Sp} \to M_1\mathbf{Sp}$. This is closely related to $K(1)$-localisation.

Theorem 4.10 ([BR20a], §5) *The functors*

$$M_1\mathbf{Sp} \underset{M_1}{\overset{L_{K(1)}}{\rightleftarrows}} \mathbf{Sp}_{K(1)}$$

are mutually inverse categorical equivalences.

As ∞-categories therefore, the distinction between $M_1\mathbf{Sp}$ and $\mathbf{Sp}_{K(1)}$ is not very important, but as subcategories of $\mathbf{Sp}$ the distinction does matter. For instance, if $X \in \mathbf{Sp}_{K(1)}$, then $\pi_* X$ need not be the same as $\pi_* M_1 X$.

Remark 4.11. For $n \geqslant 2$, there is also a height-n Morava K-theory $K(n)$, although the definition does not involve complex (or real) K-theory. One often writes $K(0)$ for $H\mathbf{Q}$ in this context. In general, for $n \geqslant 1$, the height-n monochromatic layer $M_n X$ is the fibre of the natural map

$$L_{K(0)\oplus\cdots\oplus K(n)} X \longrightarrow L_{K(0)\oplus\cdots\oplus K(n-1)} X.$$

There is a similar equivalence between $\mathbf{Sp}_{K(n)}$ and $M_n\mathbf{Sp}$; see [BR20a, §5] for an extended discussion.

4.1.2 The Bousfield–Kuhn functor

It turns out that the v_1-periodic homotopy groups of X are the homotopy groups of a spectrum $\Phi_1 X$. Specifically, the works of Bousfield and Kuhn (see [Kuh08] for an overview, or [Bou99, §7] for a more concise account) give a construction of a functor $\Phi_1\colon \mathscr{S}_* \to \mathbf{Sp}_{K(1)}$ satisfying the following properties.

(i) If k and n are such that S^k/p^n admits a v_1-self map, then we have a natural isomorphism
$$\pi_* \operatorname{map}(\mathbf{S}^k/p^n,\ \Phi_1 X) \cong v_1^{-1}\pi_*(X;\mathbf{Z}/p^n).$$

(ii) We have natural isomorphisms

$$\pi_*\, \Phi_1 X \cong v_1^{-1}\pi_*^{\wedge}(X),$$
$$\pi_*\, M_1\Phi_1 X \cong v_1^{-1}\pi_*(X).$$

(iii) If E is a spectrum, then we have a natural equivalence

$$\Phi_1(\Omega^\infty E) \simeq L_{K(1)}\, E.$$

(iv) The functor Φ preserves finite limits.

(v) A map $f\colon X \to Y$ of pointed spaces is a v_1-periodic equivalence if and only if the induced map $\Phi X \to \Phi Y$ is an equivalence.

The functor Φ_1 is called the **Bousfield–Kuhn functor** (at height 1). As before, the prime p is left implicit in the notation.

Remark 4.12. Condition (iii) is a very strong delooping result for $K(1)$-local spectra: it gives a functorial way to reconstruct (up to equivalence) a $K(1)$-local spectrum from its zeroth space. For general spectra there can be no such results, because a space can have inequivalent deloopings.

Remark 4.13. The construction of the Bousfield–Kuhn functor works at a general height n, yielding a functor Φ_n. In this work (specifically, Chapter 5) we may simply write Φ instead of Φ_1 if we work specifically at height 1.

Remark 4.14. In the literature, often the Bousfield–Kuhn functor Φ_n lands in so-called $T(n)$-local spectra, rather than $K(n)$-local spectra. A $K(n)$-local version can then be obtained by taking $L_{K(n)}\Phi_n$. See [BR20a, §8] for a further discussion. The *telescope conjecture* says that $T(n)$-localisation of spectra is the same as $K(n)$-localisation. This is a theorem in the case of $n = 1$, but is widely believed to be false for $n > 1$; see [Bar19] for an overview. In [BR20b], Behrens and Rezk work with the $K(n)$-variant of the Bousfield–Kuhn functor when doing computations. Because the two localisations agree at height 1, which is our main case of interest, we will ignore the issue from hereon out.

4.2 Topological André–Quillen cohomology

If R is a commutative ring and A an augmented commutative R-algebra, then its module of *indecomposables* is the R-module $\widetilde{A}/\widetilde{A}^2$, with $\widetilde{A}$ the kernel of the augmentation $A \to R$. If A is a symmetric R-algebra $R[x_1, \ldots, x_n]$, then its indecomposables is the module $\langle x_1, \ldots, x_n \rangle$. André–Quillen homology is essentially the left derived functor of the indecomposables functor $\mathbf{CAlg}_R^{\mathrm{aug}} \to \mathbf{Mod}_R$; for this reason it is also referred to as the *derived indecomposables*. This takes some care to make precise because $\mathbf{CAlg}_R^{\mathrm{aug}}$ is not an abelian category. In practise it can be computed by replacing the algebra A with a suitable free alternative. In the world of higher algebra, this has an analogue: *topological André–Quillen homology*.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. In [GL, §4.2], Gaitsgory and Lurie give a definition of the functor $\mathrm{TAQ}_{\mathcal{C}} \colon \mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ (although they call it the *cotangent fibre*). We are mostly interested in the version where $\mathcal{C} = \mathbf{Mod}_R$ is the ∞-category of modules over a commutative ring spectrum R, or where $\mathcal{C} = \mathbf{Mod}_{R,E}$ is the ∞-category of E-local R-modules for some spectrum E. The main property we need is the following.

Theorem 4.15 *Let $\mathcal{C}$ be a symmetric monoidal ∞-category which admits countable colimits, and such that the product functor preserves these. By Theorem* 1.44*, the forgetful functor* $\mathbf{CAlg}(\mathcal{C}) \to \mathcal{C}$ *has a left adjoint* $\mathrm{Sym}_{\mathcal{C}}$.

(*a*) *The functor* $\mathrm{TAQ}_{\mathcal{C}} \colon \mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathbf{Mod}(\mathcal{C})$ *admits a right adjoint.*

(*b*) *There is a natural equivalence, for $X \in \mathcal{C}$,*

$$\mathrm{TAQ}_{\mathcal{C}}(\mathrm{Sym}_{\mathcal{C}}(X)) \simeq X.$$

Remark 4.16. The second property is an analogue of the algebraic fact that if R is a commutative ring, the indecomposables of $R[x_1, \dots, x_n]$ is the R-module $\langle x_1, \dots, x_n \rangle$.

If R is a commutative ring spectrum, then we write TAQ_R for $\mathrm{TAQ}_{\mathbf{Mod}_R}$ and call it **topological André–Quillen homology** over R. The R-linear dual

$$\mathrm{TAQ}_R(-)^\vee := \mathrm{map}_R(\mathrm{TAQ}_R(-),\ R)$$

we call **topological André–Quillen cohomology** over R. Note that if R is E-local for some spectrum E, then TAQ-cohomology is automatically E-local, while the same is not necessarily true for TAQ-homology. In the E-local setting, it is more natural to work with $L_E\,\mathrm{TAQ}_R$ instead, because this is the functor $\mathrm{TAQ}_{\mathcal{C}}$ for $\mathcal{C} = \mathbf{Mod}_{R,E}$.

The two properties of $\mathrm{TAQ}_{\mathcal{C}}$ show that if one has a presentation for an object $X \in \mathcal{C}$ as a cofibre of symmetric objects $\mathrm{Sym}_{\mathcal{C}}(Y)$, then $\mathrm{TAQ}_{\mathcal{C}}$ is immediately computable from this. In the case of TAQ_R, the TAQ-cohomology also follows from this, because the R-linear dual functor is an exact functor (see Example 1.63).

The right adjoint to $\mathrm{TAQ}_{\mathcal{C}}$ is the *trivial algebra functor*, which informally is given by

$$\mathrm{triv}_{\mathcal{C}} \colon \mathbf{Mod}_{\mathcal{C}} \longrightarrow \mathbf{CAlg}^{\mathrm{aug}}(\mathcal{C}), \quad M \longmapsto \mathbf{1} \oplus M,$$

with M carrying the 'zero multiplication'. (This is also called the *trivial square-zero extension* by M.) See [HA, §7.1.4] for a precise definition. We collect some basic properties of the trivial algebra functor in the case of commutative ring spectra. We write triv_R for $\mathrm{triv}_{\mathbf{Mod}_R}$, and sometimes write $R \oplus M$ for $\mathrm{triv}_R(M)$.

Lemma 4.17 *Let E be a spectrum and let R be an E-local commutative ring spectrum. The functor* triv_R *sends E-local modules to E-local commutative algebra spectra, i.e., it restricts to a functor*

$$\mathrm{triv}_R \colon \mathbf{Mod}_{R,E} \longrightarrow \mathbf{CAlg}^{\mathrm{aug}}_{R,E}.$$

Lemma 4.18 *Let R and S be commutative ring spectra, and $R \to S$ a morphism. If A is a commutative R-algebra spectrum and M is an S-module spectrum, then we have a natural equivalence*

$$\mathrm{Map}_{\mathbf{CAlg}^{\mathrm{aug}}_S}(S \otimes A,\ S \oplus M) \simeq \mathrm{Map}_{\mathbf{CAlg}^{\mathrm{aug}}_R}(A,\ R \oplus M).$$

Informally, Lemma 4.18 says that the functors triv_R and triv_S are compatible with restriction of scalars.

We end the section with some properties of topological André–Quillen cohomology that will be useful for computations.

Proposition 4.19 *Let E be a spectrum and let R be an E-local commutative ring spectrum. If X is a spectrum, then we have a natural equivalence*

$$L_E\,\mathrm{TAQ}_R(L_E(R \otimes \mathrm{Sym}(X))) \simeq L_E(R \otimes X).$$

Proof. Recall from Proposition 1.57 that we have a natural equivalence $R \otimes \mathrm{Sym}(X) \simeq \mathrm{Sym}_R(R \otimes X)$. The functor $L_E\,\mathrm{TAQ}_R$ is the functor $\mathrm{TAQ}_{\mathcal{C}}$ for the ∞-category $\mathcal{C} = \mathbf{Mod}_{R,E}$. The functor $L_E\,\mathrm{Sym}_R$ is the functor $\mathrm{Sym}_{\mathcal{C}}$ (in the sense of Theorem 1.44) for $\mathcal{C} = \mathbf{Mod}_{R,E}$. Thus the result follows from Theorem 4.15(b). ■

Proposition 4.20 *Let E be a spectrum and let R be an E-local commutative ring spectrum. Let A be an augmented $\mathbf{S}_E$-algebra spectrum. Then we have a natural equivalence*

$$\mathrm{TAQ}_R(R \otimes A) \simeq R \otimes \mathrm{TAQ}_{\mathbf{S}_E}(A).$$

Proof. Let M be an R-module spectrum. Using Lemma 4.18, we have a chain of natural equivalences

$$\begin{aligned} \mathrm{Map}_R(\mathrm{TAQ}_R(R \otimes A),\ M) &\simeq \mathrm{Map}_{\mathbf{CAlg}_R^{\mathrm{aug}}}(R \otimes A,\ R \oplus M) \\ &\simeq \mathrm{Map}_{\mathbf{CAlg}_{\mathbf{S}_E}^{\mathrm{aug}}}(A,\ \mathbf{S}_E \oplus M) \\ &\simeq \mathrm{Map}_{\mathbf{S}_E}(\mathrm{TAQ}_{\mathbf{S}_E}(A),\ M) \\ &\simeq \mathrm{Map}_R(R \otimes \mathrm{TAQ}_{\mathbf{S}_E}(A),\ M). \end{aligned}$$ ■

We need to adapt Proposition 1.75 to a localised setting.

Proposition 4.21 *Let E be a spectrum and let R be an E-local commutative ring spectrum. Let X be a space such that $L_E\,\Sigma^\infty_+ X$ is E-locally dualisable. Then the natural map*

$$L_E(R \otimes \mathbf{S}_E^{X_+}) \longrightarrow R^{X_+}$$

is an equivalence.

Proof. It suffices to check that the map of underlying spectra

$$L_E(R \otimes \mathrm{map}(L_E\,\Sigma^\infty_+ X,\ \mathbf{S}_E)) \longrightarrow \mathrm{map}(L_E\,\Sigma^\infty_+ X,\ R)$$

is an equivalence. This is implied by $L_E\,\Sigma^\infty_+ X$ being E-locally dualisable. ■

Recall from §1.4.1 that if X is pointed, then R^{X_+} is an augmented commutative R-algebra.

Corollary 4.22 *Let E be a spectrum and let R be an E-local commutative ring spectrum. Let X be a pointed space such that $L_E\,\Sigma^\infty_+ X$ is E-locally dualisable. Then we have a natural equivalence*

$$L_E\,\mathrm{TAQ}_R(R^{X_+}) \simeq L_E(R \otimes \mathrm{TAQ}_{\mathbf{S}_E}(\mathbf{S}_E^{X_+})).$$

4.3 Descent

The usefulness of K-theory for chromatic homotopy theory at height one originates with the following result.

Theorem 4.23 *Let p be an odd prime, and let r be a topological generator of $\mathbf{Z}_p^\times$. There is a fibre sequence in $\mathbf{Sp}$ of the form*

$$\mathbf{S}_{K(1)} \longrightarrow KU_p \xrightarrow{\psi^r - 1} KU_p.$$

Proof. See, e.g., [Lur10, Cor. 4]. ∎

Because r is a topological generator of $\mathbf{Z}_p^\times$, we think of taking the fibre of $\psi^r - 1$ as taking "$\mathbf{Z}_p^\times$-fixed points".

This result one to study $K(1)$-local phenomena by using K-theory. We will use the term *descent* to refer to this approach.

Lemma 4.24 *Let p be an odd prime. If X is a $K(1)$-local spectrum, then the sequence*

$$X \longrightarrow L_{K(1)} KU_p \otimes X \xrightarrow{\psi^r - 1} L_{K(1)} KU_p \otimes X$$

is a fibre sequence in $\mathbf{Sp}_{K(1)}$.

Proof. The functors $- \otimes X \colon \mathbf{Sp} \to \mathbf{Sp}$ and $L_{K(1)} \colon \mathbf{Sp} \to \mathbf{Sp}_{K(1)}$ are left adjoints, so they preserve colimits. Thus $L_{K(1)}(- \otimes X)$ is a colimit-preserving functor between stable ∞-categories, so by Proposition 1.12 it is an exact functor. Therefore it also preserves finite limits, in particular fibre sequences. Because X is $K(1)$-local, we have $L_{K(1)}(\mathbf{S}_{K(1)} \otimes X) \simeq X$, yielding the claimed fibre sequence. ∎

Proposition 4.25 *Let p be an odd prime. The functor $L_{K(1)} KU_p \otimes - \colon \mathbf{Sp}_{K(1)} \to \mathbf{Mod}_{KU_p,\, K(1)}$ is conservative.*

Proof. Let $f \colon X \to Y$ be a map of $K(1)$-local spectra such that $L_{K(1)} KU_p \otimes f$ is an equivalence. Choose an inverse equivalence $h \colon L_{K(1)} KU_p \otimes Y \to L_{K(1)} KU_p \otimes X$. Using the above lemma, we have a diagram in $\mathbf{Sp}_{K(1)}$ of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & L_{K(1)} KU_p \otimes X \arrow[r, "\psi^r - 1"] \arrow[d, "L_{K(1)} KU_p \otimes f"] & L_{K(1)} KU_p \otimes X \arrow[d, "L_{K(1)} KU_p \otimes f"] \\
Y \arrow[r] \arrow[d, "g"'] & L_{K(1)} KU_p \otimes Y \arrow[r, "\psi^r - 1"] \arrow[d, "h"] & L_{K(1)} KU_p \otimes Y \arrow[d, "h"] \\
X \arrow[r] & L_{K(1)} KU_p \otimes X \arrow[r, "\psi^r - 1"] & L_{K(1)} KU_p \otimes X,
\end{tikzcd}$$

where the rows are fibre sequences. Write g for the map $Y \to X$ induced by h by taking fibres. Since all three rows are fibre sequences and the last two vertical composites are equivalences, we find that the composite $g \circ f$ is an equivalence. It follows similarly that the composite $f \circ g$ is an equivalence, and thus f is an equivalence. ∎

Remark 4.26. We can actually show something stronger, namely that $L_{K(1)} KU_p \otimes -$ sets up an equivalence between $\mathbf{Sp}_{K(1)}$ and the ∞-category of KU_p-modules "with a continuous action of $\mathbf{Z}_p^\times$". One way to formalise this is to say that the adjunction involving

$L_{K(1)}KU_p \otimes -$ and the forgetful functor is *comonadic*. (Showing that $L_{K(1)}KU_p \otimes -$ is conservative is the first small step toward proving this: see §4.7, and specifically Theorem 4.7.3.5, of [HA] for more information.) This comonadicity result can properly be called a 'descent' statement, but we only need the above weaker version.

Remark 4.27. There is an analogous version of Theorem 4.23 for the case $p = 2$, which involves *real K-theory* instead of complex K-theory. Roughly speaking, this is because $\mathbf{Z}_2^\times$ is not topologically cyclic, but rather splits as $\mathbf{Z}/2$ times a topologically cyclic group. Taking fixed points of K-theory with respect to the $\mathbf{Z}/2$-summand passes from complex to real K-theory.

4.4 The Behrens–Rezk comparison map

Let R be a $K(1)$-local commutative ring spectrum. In [BR20b, §6], Behrens and Rezk construct a natural transformation

$$c_R^X\colon\ \mathrm{TAQ}_R(R^{X_+}) \longrightarrow R^{\Phi_1 X}$$

of functors $\mathscr{S}_*^{\mathrm{op}} \to \mathbf{Mod}_R$. Taking the R-linear dual and precomposing with the natural map

$$L_{K(1)}\, R \otimes \Phi_1 X \longrightarrow \mathrm{map}_R(R^{\Phi_1 X}, R)$$

yields the ***R*-theoretic comparison map**

$$c_X^R\colon\ L_{K(1)}\, R \otimes \Phi_1 X \longrightarrow \mathrm{TAQ}_R(R^{X_+})^\vee,$$

which is a natural transformation of functors $\mathscr{S}_* \to \mathbf{Mod}_{R,\,K(1)}$. We write c_X for the comparison map where $R = \mathbf{S}_{K(1)}$, and simply refer to this as the *comparison map*. For general R, we may refer to c_R^X as the (R-theoretic) *dual comparison map*. Note that the prime p is (again) left implicit in the notation for these maps.

Remark 4.28. Benhrens and Rezk give the construction at a general height $n \geqslant 1$.

Definition 4.29 Let X be a pointed space such that $L_{K(1)}\,\Sigma_+^\infty X$ is $K(1)$-locally dualisable. We say X is **Φ_1-good** if the comparison map c_X is an equivalence.

Remark 4.30. The above definition is nonstandard. In the literature, a pointed space X is called Φ_n-good if the Goodwillie tower for Φ_n converges at X. If X is such that the spectrum $L_{K(n)}\,\Sigma_+^\infty X$ is $K(n)$-locally dualisable, then by [Heu19, Cor. 7.15] this is equivalent to the above definition. Our aim is to show Φ_1-goodness of a class of spaces satisfying this dualisability condition for $n = 1$, and to do so without the use of Goodwillie calculus, so the above definition suffices.

If X is Φ_1-good, we think of $\mathbf{S}_{K(1)}^{X_+}$ as being a 'good model' for the v_1-periodic homotopy type of X. Unlike rational homotopy theory, even for finite spaces, this algebra need not always be a good model. (Though even if the comparison map is not an equivalence, this does not mean that the cochains are useless.) A discussion of Φ_1-good spaces is given by Behrens and Rezk [BR20a, §8]. Examples include the spheres, $\mathrm{SU}(k)$, and

$\mathrm{Sp}(k)$ (these spaces are even Φ_n-good for all $n \geqslant 1$). Not all spaces are Φ_1-good: Brantner and Heuts [BH20] prove that the wedge of two spheres of dimension greater than one, and mod p Moore spaces, are not Φ_n-good for any $n \geqslant 1$.

Remark 4.31. In rational homotopy theory, there is also an approach using *differential graded Lie algebras* (*dgla's* for short). More precisely, there is an equivalence

$$\mathscr{S}_{\mathbf{Q}}^{\geqslant 2} \simeq \mathbf{Lie}(\mathbf{Ch}_{\mathbf{Q}})^{\geqslant 1}$$

between simply-connected rational spaces and connected rational dgla's. This approach does generalise to height 1, and even to height n for any $n \geqslant 1$. Heuts [Heu21] proved that $\Phi_n X$ has the structure of a *spectral Lie algebra*, and that this sets up an equivalence between v_n-periodic spaces and spectral Lie algebras in $T(n)$-local spectra. The survey by Heuts [Heu19] gives an introduction to spectral Lie algebras and their application to v_n-periodic homotopy theory, including this result. We will not use spectral Lie algebras in this work.

Remark 4.32. The approach with cochain algebras is only suited to deal with spaces that are suitably finite (more precisely, spaces X for which the spectrum $L_{K(n)}\Sigma^\infty_+ X$ is $K(n)$-locally dualisable). To rid oneself of the finiteness condition, one should work with commutative *coalgebras* in the spectral sense. This approach is described by Heuts [Heu21, §5]. For spaces that are finite in the earlier sense, this approach is equivalent to working with commutative algebras. The comparison map generalises to coalgebras, and this comparison map is an equivalence if and only if X is Φ_n-good (without finiteness assumptions).

If we work over an odd prime p, then using descent we can detect Φ_1-goodness via the K-theoretic comparison map. This result is specific to height 1.

Proposition 4.33 *Let p be an odd prime. Let X be a pointed space such that the spectra $L_{K(1)}\Sigma^\infty_+ X$ and $L_{K(1)}\,\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})$ are $K(1)$-locally dualisable. Then X is Φ_1-good if and only if the K-theoretic comparison map*

$$c_X^{KU_p}\colon\ L_{K(1)}\,KU_p \otimes \Phi_1 X \longrightarrow \mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee$$

is an equivalence.

Proof. By Proposition 4.25 the functor $L_{K(1)}\,KU_p \otimes -$ is conservative, so X is Φ_1-good if and only if

$$L_{K(1)}\,KU_p \otimes c_X\colon\ L_{K(1)}\,KU_p \otimes \Phi_1 X \longrightarrow L_{K(1)}\,KU_p \otimes \left(\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})^\vee\right)$$

is an equivalence. We now rewrite the target of this map via a number of identifications. The natural map

$$L_{K(1)}\,KU_p \otimes \left(\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})^\vee\right) \longrightarrow \mathrm{map}\left(\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+}),\ KU_p\right)$$

is an equivalence because $L_{K(1)}\,\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})$ is $K(1)$-locally dualisable. Corollary 4.22 gives us an equivalence

$$L_{K(1)}\,KU_p \otimes \mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+}) \simeq L_{K(1)}\,\mathrm{TAQ}_{KU_p}(KU_p^{X_+}).$$

Using this, we find an equivalence

$$\begin{aligned}
\operatorname{map}\Big(\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+}),\, KU_p\Big) &\simeq \operatorname{map}_{KU_p}\Big(L_{K(1)}\, KU_p \otimes \mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+}),\, KU_p\Big) \\
&\simeq \operatorname{map}_{KU_p}\Big(L_{K(1)}\, \mathrm{TAQ}_{KU_p}(KU_p^{X_+}),\, KU_p\Big) \\
&= \mathrm{TAQ}_{KU_p}(KU_p^{X_+})^{\vee}.
\end{aligned}$$

Postcomposing with these identifications, we obtain a map

$$L_{K(1)}\, KU_p \otimes \Phi_1 X \longrightarrow \mathrm{TAQ}_{KU_p}(KU_p^{X_+})^{\vee},$$

and this map agrees with the K-theoretic Behrens–Rezk comparison map $c_X^{KU_p}$. (Specifically, we use that these identifications are compatible with the equivalence (4.1) of [BR20b].) This proves the claim. ■

5 Evaluation of the comparison map

Let p be an odd prime, and let X be a pointed space satisfying the conditions of Theorem 3.12, and such that $L_{K(1)}\Sigma^\infty_+ X$ is $K(1)$-locally dualisable. In this chapter we use the presentation of $KU_p^{X_+}$ from Theorem 3.12 to show that, if X satisfies some additional conditions, then X is Φ_1-good. We do this by showing that the Behrens–Rezk comparison map

$$c_X\colon \Phi_1 X \longrightarrow \mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})^\vee$$

is an equivalence. By a descent argument, this is equivalent to the KU_p-theoretic comparison map being an equivalence. The topological André–Quillen cohomology of $KU_p^{X_+}$ follows directly from from a presentation for $KU_p^{X_+}$ (see Theorem 4.15), which is exactly what we gave in Theorem 3.12. This way we can compute the target of the comparison map.

We do not give a new computation of $\Phi_1 X$. Rather, we rely on work done by Bousfield [Bou99] for our knowledge of $\Phi_1 X$. Working over an odd prime, he computed the spectrum $\Phi_1 X$ for X as above. More specifically, he described $\Phi_1 X$ as the fibre of a map between K-cohomology Moore spectra. These Moore spectra turn out to be dual versions of the spectra $\mathscr{M}(G)$ from §3.1. We introduce these Moore spectra and summarise Bousfield's results in §5.1, slightly rephrasing them in our terminology. The main takeaway of §5.1 is Theorem 5.11. The proof of Φ_1-goodness is given in §5.2, with the exception of the construction of a crucial diagram, which is done in §5.3.

Previously Kjaer [Kja19] did something very similar: he also used Bousfield's results to conclude that such spaces are Φ_1-good, but with a different approach to computing topological André–Quillen cohomology. There does not seem to be another source in the literature that computes the topological André–Quillen cohomology of these cochains from a presentation of the cochains.

Throughout this chapter, p denotes a fixed odd prime. We work at height 1, and shall refer to the height 1 Bousfield–Kuhn functor by Φ instead of Φ_1. Unlike the previous chapter, the results in this chapter are very specifically tied to the height 1 assumption. Like in other chapters, we follow Convention 2.1 and view K-theory as $\mathbf{Z}/2$-graded.

5.1 Bousfield's infinite loop space approach

Recall the following from §3.1: for a Morava module G concentrated either in even or odd degree which is finitely generated as $\mathbf{Z}_p$-module, Bousfield constructed a spectrum

$\mathscr{M}_{(p)}(G)$ with an isomorphism

$$(KU_{(p)})_*(\mathscr{M}_{(p)}(G)) \cong G$$

respecting p-local Adams operations. We defined $\mathscr{M}(G) := \mathscr{M}_{(p)}(G)^\wedge_p$ and showed in Lemma 3.2 that the above isomorphism induces an isomorphism of Morava modules

$$(KU_p)^\wedge_*(\mathscr{M}(G)) \cong G.$$

Recall also that $\mathscr{M}(G)$ is $K(1)$-local by construction. We will now use a dual version of this spectrum. In this chapter, if Y is a $K(1)$-local spectrum, we write $Y^\vee$ for its $K(1)$-local Spanier–Whitehead dual:

$$Y^\vee = \mathrm{map}(Y,\, \mathbf{S}_{K(1)}).$$

Definition 5.1 Let G be a Morava module concentrated either in even or odd degree which is finitely generated as $\mathbf{Z}_p$-module. We write $\mathscr{M}^\vee(G)$ for the $K(1)$-local Spanier–Whitehead dual of $\mathscr{M}(G)$.

Proposition 5.2 *Let G be a Morava module concentrated either in even or odd degree which is finitely generated as $\mathbf{Z}_p$-module. The spectrum $\mathscr{M}(G)$ is $K(1)$-locally dualisable.*

Proof. By Theorem 2.13, this is equivalent to $(KU_p)^\wedge_*(\mathscr{M}(G))$ being finitely generated. We have an isomorphism $(KU_p)^\wedge_*(\mathscr{M}(G)) \cong G$, so the result follows. ■

Corollary 5.3 *We have an isomorphism of Morava modules*

$$KU_p^*(\mathscr{M}^\vee(G)) \cong G.$$

Remark 5.4. In [Bou99], Bousfield denotes the spectrum $\mathscr{M}^\vee(G)$ by $\mathscr{M}(G,1)$ when G is concentrated in odd degree, and $\mathscr{M}(G,0)$ when concentrated in even degree. He gives a different construction of this spectrum (see 10.3 of op. cit.), but he notes that (up to equivalence) it is the unique $K(1)$-local spectrum such that its KU_p-cohomology is G.

Using this spectrum, Bousfield in [Bou99] gave a computation of ΦX for a wide variety of spaces X. We require some results about the K-theory of infinite loop spaces before we can discuss this. In 3.5 of op. cit., Bousfield defines a modification of $\mathscr{M}^\vee(G)$, denoted by $\widetilde{\mathscr{M}}^\vee(G)$, together with a map $\widetilde{\mathscr{M}}^\vee(G) \to \mathscr{M}^\vee(G)$, such that

(i) $\pi_n \widetilde{\mathscr{M}}^\vee(G) = 0$ for $n < 2$;

(ii) $\pi_2 \widetilde{\mathscr{M}}^\vee(G)$ is a modified version of $\pi_2 \mathscr{M}^\vee(G)$;

(iii) the map $\widetilde{\mathscr{M}}^\vee(G) \to \mathscr{M}^\vee(G)$ induces an isomorphism $\pi_n \widetilde{\mathscr{M}}^\vee(G) \cong \pi_n \mathscr{M}^\vee(G)$ for $n > 2$;

(iv) the map $\widetilde{\mathscr{M}}^\vee(G) \to \mathscr{M}^\vee(G)$ is a $K(1)$-localisation.

If G is concentrated in odd degree, then the K-theory of the zeroth space $\Omega^\infty \widetilde{\mathscr{M}}^\vee(G)$ turns out to be a free p-adic θ-algebra in the sense of Bousfield (recall from Remark 2.32

that this is slightly different from the free θ-algebras from §2.4). This algebra can be constructed as follows. Write $\overline{F}G$ for the module $G \times G \times \cdots$ (cf. the notation $FG = G \oplus G \oplus \cdots$ from Example 2.31). Given a $\mathbf{Z}_p$-module N, define

$$\hat{\Lambda}(N) := \lim_{\alpha} \Lambda_{\mathbf{Z}_p}(N_\alpha),$$

where the N_α are the finite quotients of N. If G is concentrated in odd degree, then the algebra $\hat{\Lambda}\overline{F}G$ is the underlying algebra of Bousfield's free p-adic θ-algebra on G. The module $\overline{F}G$ has an operation θ^p given by shifting each copy of G one to the right. This induces an operation θ^p on $\hat{\Lambda}\overline{F}G$ satisfying the conditions for a $\mathbf{Z}/2$-graded θ-algebra. The Adams operations on G induce Adams operations $\hat{\Lambda}\overline{F}G$. Based on previous results from [Bou96a], Bousfield concludes the following.

Theorem 5.5 ([Bou99], Thm. 3.7) *Let G be a Morava module concentrated in odd degree, which is finitely generated and free as $\mathbf{Z}_p$-module. Then we have an isomorphism of θ-algebras*

$$KU_p^*(\Omega^\infty \widetilde{\mathscr{M}}^\vee(G)) \cong \hat{\Lambda}\overline{F}G.$$

Remark 5.6. This result is a first hint that the KU_p-algebra $KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(G)_+}$ is closely related to the symmetric algebra $L_{K(1)}KU_p \otimes \mathrm{Sym}(\mathscr{M}(G))$. This connection will play a further role later in this chapter.

5.1.1 Construction of the fibre sequence

Let M be a Morava module concentrated in odd degree, and which is finitely generated and free as $\mathbf{Z}_p$-module, and let $\theta_M^p\colon M \to M$ be a homomorphism. Let X be a connected space such that $KU_p^*(X)$ is an exterior algebra on (M, θ_M^p) (see Theorem 3.12). If X satisfies some additional conditions, then Bousfield realises ΦX as the fibre of a map $\mathscr{M}^\vee(M) \to \mathscr{M}^\vee(M)$. We now review his construction. (Bear in mind the differences in notation: see Remarks 2.21 and 2.22.)

For technical reasons, one must assume that (M, θ_M^p) satisfies the *regularity condition* of Bousfield (see Definition 4.4 of [Bou99], and see Remark 5.12 below for a discussion of when this happens), and that $H^1(X; \mathbf{Z}_p)$ and $H^2(X; \mathbf{Z}_p)$ both vanish. Then in Theorem 4.8 of [Bou99], Bousfield constructs a triangle

$$\begin{tikzcd} L_{K(1)}X \arrow[r, "h'"] \arrow[d] & \Omega^\infty \widetilde{\mathscr{M}}^\vee(M) \arrow[d, "f"] \\ * \arrow[r] & \Omega^\infty \widetilde{\mathscr{M}}^\vee(M) \end{tikzcd} \tag{5.7}$$

in $\mathscr{S}_*$. There is some freedom in the choosing of the maps h' and f. The requirements are as follows. One must start with a map $h\colon X \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ which induces the natural quotient

$$KU_p^*(\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)) \cong \hat{\Lambda}\overline{F}M \longrightarrow \Lambda M \cong KU_p^*(X)$$

on K-theory. Bousfield proves such an h always exists: see Theorem 3.10 of op. cit. The K-theory of the cofibre $\operatorname{cofib} h$ injects into $\hat{\Lambda}\overline{F}M$. Next one must choose a map $k\colon \operatorname{cofib} h \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ which induces the map

$$\theta^p - \overline{F}\theta^p_M\colon \hat{\Lambda}\overline{F}M \longrightarrow KU_p^*(\operatorname{cofib} h) \subseteq \hat{\Lambda}\overline{F}M$$

on K-theory. (This exists by the same Theorem 3.10 of op. cit.) The map f is defined to be the composite

$$\Omega^\infty \widetilde{\mathscr{M}}^\vee(M) \longrightarrow \operatorname{cofib} h \xrightarrow{k} \Omega^\infty \widetilde{\mathscr{M}}^\vee(M).$$

On K-theory, this induces

$$\theta^p - \overline{F}\theta^p_M\colon \hat{\Lambda}\overline{F}M \longrightarrow \hat{\Lambda}\overline{F}M.$$

Bousfield shows that h factors to a map $h'\colon L_{K(1)}X \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ that sits in a triangle of the form (5.7).

Remark 5.8. There seems to be a missing assumption in Theorem 4.8 of [Bou99]. The theorem ends with the conclusion that $H^1(X;\mathbf{Z}_p)$ and $H^2(X;\mathbf{Z}_p)$ vanish under the assumptions posed there, but $X = S^1$ is a counterexample to this. (The module $KU_p^1(S^1)$ is indeed regular — see Remark 5.12.) Because the conclusion that $H^1(X;\mathbf{Z}_p)$ and $H^2(X;\mathbf{Z}_p)$ vanish is used in later results of [Bou99], we have included it as a separate assumption. Requiring M to be *reduced* in the sense of Definition 2.8 of op. cit. would imply $H^1(X;\mathbf{Z}_p) = 0$ (see also 2.9 of op. cit.).

One of the main results of [Bou99] is that the triangle (5.7) becomes a fibre sequence after application of Φ. To state the precise assumptions on X that are necessary for this, we require some terminology. Recall from Definition 4.7 that a map of pointed spaces is a v_1-periodic equivalence if it induces an isomorphism on $v_1^{-1}\pi_*$, or equivalently (see §4.1.2) if it becomes an equivalence after application of Φ.

Definition 5.9 ([Bou99], §7) A pointed space X is called $\boldsymbol{K(1)}$**-durable** if its $K(1)$-localisation map $X \to L_{K(1)}X$ is a v_1-periodic equivalence.

In other words, X is $K(1)$-durable when the map $X \to L_{K(1)}X$ induces an equivalence $\Phi X \simeq \Phi(L_{K(1)}X)$.

Proposition 5.10 ([Bou99], Cor. 7.8) *If X is a pointed nilpotent space whose p-completion $X_p^\wedge$ is an H-space, then X is $K(1)$-durable.*

Note that this includes the odd spheres, since we are working over an odd prime p.

Recall from §4.1.2 that for a spectrum Y, we have a natural equivalence

$$\Phi(\Omega^\infty Y) \simeq L_{K(1)}Y.$$

Theorem 5.11 ([Bou99], Thm. 8.1) *Let p be an odd prime number. Let X be a pointed space, let M be a p-adic Morava module, and $\theta^p_M\colon M \to M$ a morphism of Morava modules, such that* (*i*) *M is finitely generated and free as a $\mathbf{Z}_p$-module;*

(*ii*) *M is concentrated in degree 1;*

(*iii*) *the θ-algebra $KU_p^*(X)$ is an exterior algebra on (M, θ_M^p);*

(*iv*) *the pair (M, θ_M^p) is regular in the sense of Bousfield (see* [Bou99, Def. 4.4]*);*

(*v*) *the space X is connected and $K(1)$-durable;*

(*vi*) *the groups $H^1(X; \mathbf{Z}_p)$ and $H^2(X; \mathbf{Z}_p)$ vanish.*

Then any triangle of the form (5.7) *constructed by the above method becomes, after application of Φ, a pullback square*

$$\begin{tikzcd} \Phi X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{M}^\vee(M) \arrow[d] \\ 0 \arrow[r] & \mathscr{M}^\vee(M) \end{tikzcd}$$

in $\mathbf{Sp}_{K(1)}$.

Note that the conditions of Theorem 5.11 imply the conditions of Theorem 3.12.

In §8 and §9 of [Bou99], Bousfield discusses how to compute $v_1^{-1}\pi_*(X)$ from this result, relying on the equivalence between $L_{K(1)}$ and M_1 (see Theorem 4.10). He also gives some examples of such a computation, including for $\mathrm{SU}(n)$.

Remark 5.12. The most technical assumption is that of the regularity of (M, θ^p). Recall that Bousfield calls (M, θ^p) regular if it is an extension of a *strictly nonlinear* submodule and a *linear* quotient module: see Definitions 4.2 and 4.4 of [Bou99]. This is often satisfied in practise. It is for instance satisfied for odd spheres, and for the H-spaces that we considered in §3.3.2. For the n-sphere S^n this is evident because then (M, θ^p) is either linear (if $n = 1$) or strictly nonlinear (if $n > 1$). Bousfield shows that if X satisfies the conditions of Theorem 3.16, then (M, θ_M^p) is regular: see [Bou99, Lem. 6.1] (we in fact remarked this in the proof overview we gave of Theorem 3.16).

Combining Theorem 3.17, Proposition 5.10, and Remark 5.12, we see that simply-connected compact Lie groups satisfy the conditions of Theorem 5.11 (since π_2 of a compact Lie group always vanishes). For the odd spheres we have the following result.

Example 5.13 Let n be an odd natural number, and consider $X = S^n$. Recall from §3.3.1 that $KU_p^*(S^n)$ is an exterior algebra on $M = \widetilde{KU}_p^*(S^n)$. The map $\theta_M^p \colon M \to M$ is given by multiplication by $p^{(n-1)/2}$. Lastly, recall that we have an equivalence $\mathscr{M}(M) \simeq \mathbf{S}_{K(1)}^{-n}$.

If $n > 1$, then the additional conditions of Theorem 5.11 are satisfied. The dual spectrum $\mathscr{M}^\vee(M)$ is now $\mathbf{S}_{K(1)}^n$. Theorem 5.11 expresses $\Phi(S^n)$ as sitting in a fibre sequence

$$\Phi(S^n) \longrightarrow \mathbf{S}_{K(1)}^n \xrightarrow{p^{(n-1)/2}} \mathbf{S}_{K(1)}^n.$$

This fibre is easily computed, yielding

$$\Phi(S^n) \simeq \mathbf{S}_{K(1)}^{n-1} \,/\, p^{(n-1)/2}.$$ ▲

Remark 5.14. If the map $\theta^p_M\colon M \to M$ is injective, then the fibre sequence of Theorem 5.11 simplifies to say that ΦX is equivalent to $\mathscr{M}(M/\theta^p_M)$. Bousfield shows that if X satisfies the conditions of Theorem 3.17, then θ^p_M is injective: see the proof of Theorem 9.2 in [Bou99].

5.2 Proof overview

We now come to the main result of this chapter. The triangle (5.7) (from which the fibre sequence of Theorem 5.11 followed) resembles a dual version of the cofibre sequence of Theorem 3.12: on KU_p^*, the triangle (5.7) induces a presentation

$$\hat{\Lambda}\overline{F}M \xrightarrow{\theta^p - \overline{F}\theta^p_M} \hat{\Lambda}\overline{F}M \longrightarrow \Lambda M,$$

similar to how on $(KU_p)^\wedge_*$, the presentation of Theorem 3.12 induces

$$\mathrm{Free}_{\theta,p}(M) \xrightarrow{\theta^p - F\theta^p_M} \mathrm{Free}_{\theta,p}(M) \longrightarrow \Lambda M.$$

This was in fact the inspiration behind Theorem 3.12. We now study this resemblance in more detail, and in the process prove that the spaces X to which this applies are Φ-good.

Note that the condition that $L_{K(1)}\Sigma^\infty_+ X$ should be $K(1)$-locally dualisable is a type of finiteness condition on X; this is in particular satisfied if X is finite.

Theorem 5.15 *Let p be an odd prime. Let X be a pointed space satisfying the conditions of Theorem 5.11, and such that $L_{K(1)}\Sigma^\infty_+ X$ is $K(1)$-locally dualisable. Then the Behrens–Rezk comparison map c_X is an equivalence, i.e., the space X is Φ-good.*

Proof. Applying $K(1)$-local topological André-Quillen homology over KU_p to the presentation of $KU_p^{X_+}$ from Theorem 3.12 yields a fibre sequence

$$L_{K(1)}\mathrm{TAQ}_{KU_p}(KU_p^{X_+}) \longrightarrow L_{K(1)}KU_p \otimes \mathscr{M}(M) \longrightarrow L_{K(1)}KU_p \otimes \mathscr{M}(M)$$

in $\mathbf{Mod}_{KU_p,\,K(1)}$ (see Proposition 4.19). Because $\pi_*(L_{K(1)}KU_p \otimes \mathscr{M}(M)) \cong M$ is finitely generated, we find that $\pi_*(L_{K(1)}\mathrm{TAQ}_{KU_p}(KU_p^{X_+}))$ is also finitely generated. Using the equivalence

$$L_{K(1)}\mathrm{TAQ}_{KU_p}(KU_p^{X_+}) \simeq L_{K(1)}KU_p \otimes \mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})$$

from Corollary 4.22, this shows that

$$(KU_p)^\wedge_*(\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+}))$$

is finitely generated. By Theorem 2.13, this means that $L_{K(1)}\mathrm{TAQ}_{\mathbf{S}_{K(1)}}(\mathbf{S}_{K(1)}^{X_+})$ is $K(1)$-locally dualisable. Thus we are in the setting of Proposition 4.33, meaning that it suffices to show that the K-theoretic comparison map

$$c_X^{KU_p}\colon L_{K(1)}KU_p \otimes \Phi X \longrightarrow \mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee$$

is an equivalence. Applying the functor $L_{K(1)}KU_p \otimes -$ to the fibre sequence of Bousfield's Theorem 5.11 yields a fibre sequence

$$L_{K(1)}\, KU_p \otimes \Phi X \longrightarrow L_{K(1)}\, KU_p \otimes \mathscr{M}^\vee(M) \longrightarrow L_{K(1)}\, KU_p \otimes \mathscr{M}^\vee(M)$$

in $\mathbf{Mod}_{KU_p,\, K(1)}$. Applying TAQ-cohomology over KU_p to the presentation of Theorem 3.12 yields a fibre sequence

$$\mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee \longrightarrow \mathrm{map}(\mathscr{M}(M),\, KU_p) \longrightarrow \mathrm{map}(\mathscr{M}(M),\, KU_p)$$

in $\mathbf{Mod}_{KU_p,\, K(1)}$. In §5.3, we construct a diagram of the form

$$\begin{tikzcd}
L_{K(1)}\, KU_p \otimes \Phi X \arrow[r] \arrow[d, "c_X^{KU_p}"] & L_{K(1)}\, KU_p \otimes \mathscr{M}^\vee(M) \arrow[r] \arrow[d, "\simeq"] & L_{K(1)}\, KU_p \otimes \mathscr{M}^\vee(M) \arrow[d, "\simeq"] \\
\mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee \arrow[r] & \mathrm{map}(\mathscr{M}(M),\, KU_p) \arrow[r] & \mathrm{map}(\mathscr{M}(M),\, KU_p)
\end{tikzcd}$$

between these two fibre sequences, such that the last two vertical maps are equivalences. Thus the left vertical map is an equivalence, finishing the proof. ∎

5.3 Construction of the diagram

In this section we finish the proof of Theorem 5.15 by constructing the advertised diagram. The construction centres around the construction of the last two vertical maps. This is mostly a formal endeavour, and so is the proof that these maps are equivalences. The construction can be done in more generality, so to make the argument more transparent we work with a general spectrum Y, and specialise to $Y = \mathscr{M}^\vee(M)$ at the end.

We begin with a construction given by Behrens and Rezk in [BR20b, §6]. If Y is a spectrum, let us write KU_p^Y for the KU_p-module $\mathrm{map}(Y,\, KU_p)$. The counit $\Sigma^\infty\Omega^\infty Y \to Y$ induces a natural transformation

$$KU_p^Y \longrightarrow KU_p^{\Omega^\infty Y_+}$$

of functors $\mathbf{Sp}^{\mathrm{op}} \to \mathbf{Mod}_{KU_p}$. By the universal property of the symmetric algebra, this induces a natural transformation

$$\varepsilon_Y\colon \mathrm{Sym}_{KU_p}(KU_p^Y) \longrightarrow KU_p^{\Omega^\infty Y_+}$$

of functors $\mathbf{Sp}^{\mathrm{op}} \to \mathbf{CAlg}^{\mathrm{aug}}_{KU_p}$.

This turns out to be a right inverse to the K-theoretic dual comparison map of the space $\Omega^\infty Y$. The proof of the following lemma was originally given when working over $\mathbf{S}_{K(1)}$, but it applies to KU_p as well (or any $K(1)$-local commutative ring spectrum), mutatis mutandis. Specifically, we use that the equivalence (4.1) of [BR20b] is given in this generality. The proof is formal.

Lemma 5.16 **([BR20b], Lem. 6.1)** *If Y is a spectrum, then the composite*

$$KU_p^Y \simeq \mathrm{TAQ}_{KU_p}(\mathrm{Sym}_{KU_p}(KU_p^Y)) \xrightarrow{\varepsilon_Y^*} \mathrm{TAQ}_{KU_p}(KU_p^{\Omega^\infty Y_+}) \xrightarrow{c_{KU_p}^{\Omega^\infty Y}} KU_p^{\Phi(\Omega^\infty Y)} \simeq KU_p^Y$$

is the identity.

Taking KU_p-linear duals, precomposing with the natural map

$$L_{K(1)} KU_p \otimes Y^\vee \longrightarrow \mathrm{map}_{KU_p}(KU_p^Y, KU_p)$$

and postcomposing with the natural map

$$\mathrm{map}_{KU_p}(KU_p^Y, KU_p) \longrightarrow \mathrm{map}(Y^\vee, KU_p)$$

yields the composite

$$\begin{tikzcd}
L_{K(1)} KU_p \otimes Y^\vee \arrow[d, "\simeq"] & & \\
L_{K(1)} KU_p \otimes \Phi(\Omega^\infty Y) \arrow[r, "c_{\Omega^\infty Y}^{KU_p}"] & \mathrm{TAQ}_{KU_p}(KU_p^{\Omega^\infty Y_+})^\vee \arrow[r, "\varepsilon^\vee"] & \mathrm{TAQ}_{KU_p}(\mathrm{Sym}_{KU_p}(KU_p^Y))^\vee \arrow[d, "\simeq"] \\
& & \mathrm{map}_{KU_p}(KU_p^Y, KU_p) \arrow[d] \\
& & \mathrm{map}(Y^\vee, KU_p).
\end{tikzcd}$$

By Lemma 5.16, the 'middle' falls out, and this composite is equal to the composite of natural maps

$$L_{K(1)} KU_p \otimes Y^\vee \longrightarrow \mathrm{map}_{KU_p}(KU_p^Y, KU_p) \longrightarrow \mathrm{map}(Y^\vee, KU_p).$$

The proof of the following lemma is immediate from the definition of $K(1)$-local dualisability.

Lemma 5.17 *Let Y be a $K(1)$-locally dualisable spectrum. Then*

(a) *the composite of natural maps*

$$L_{K(1)} KU_p \otimes Y^\vee \longrightarrow \mathrm{map}_{KU_p}(KU_p^Y, KU_p) \longrightarrow \mathrm{map}(Y^\vee, KU_p)$$

is an equivalence;

(b) *we have a natural equivalence*

$$\mathrm{Sym}_{KU_p}(KU_p^Y) \simeq KU_p \otimes \mathrm{Sym}(Y^\vee).$$

By Proposition 5.2, the spectrum $\mathscr{M}(M)$ is $K(1)$-locally dualisable, so the above lemma applies to $Y = \mathscr{M}^\vee(M)$.

We now construct a diagram of the form

$$\begin{tikzcd}
L_{K(1)} KU_p \otimes \Phi X \arrow[r] \arrow[d, equal] & L_{K(1)} KU_p \otimes \mathscr{M}^\vee(M) \arrow[r] \arrow[d, "\simeq"] & L_{K(1)} KU_p \otimes \mathscr{M}^\vee(M) \arrow[d, "\simeq"] \\
L_{K(1)} KU_p \otimes \Phi X \arrow[r] \arrow[d, "c_X^{KU_p}"] & L_{K(1)} KU_p \otimes \Phi(\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)) \arrow[r] \arrow[d, "c^{KU_p}"] & L_{K(1)} KU_p \otimes \Phi(\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)) \arrow[d, "c^{KU_p}"] \\
\mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee \arrow[r, "h"] \arrow[d, equal] & \mathrm{TAQ}_{KU_p}(KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+})^\vee \arrow[r, "f"] \arrow[d, "\varepsilon^\vee"] & \mathrm{TAQ}_{KU_p}(KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+})^\vee \arrow[d, "\varepsilon^\vee"] \\
\mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee \arrow[r] & \mathrm{map}(\mathscr{M}(M),\, KU_p) \arrow[r] & \mathrm{map}(\mathscr{M}(M),\, KU_p),
\end{tikzcd}$$

as follows.

- The top horizontal row is obtained by applying $L_{K(1)}KU_p \otimes -$ to the fibre sequence of Bousfield of Theorem 5.11.
- The first triple of vertical maps are the equivalences from §4.1.2, keeping in mind that $L_{K(1)}\widetilde{\mathscr{M}}^\vee(M) \simeq \mathscr{M}^\vee(M)$.
- The second triple of vertical maps is the K-theoretic Behrens–Rezk comparison map. For the left vertical map, it is the comparison map for the space X, and for the last two it is for the space $\Omega^\infty\widetilde{\mathscr{M}}^\vee(M)$.
- In the third row, we use the notation h and f to reflect the notation in the triangle (5.7).
- In the third triple of vertical maps, the last two are given by applying TAQ-cohomology to the natural transformation ε constructed above. We have abbreviated $\mathrm{TAQ}(\varepsilon)^\vee$ by $\varepsilon^\vee$ in the diagram, and have made the identification

  $$\mathrm{TAQ}_{KU_p}\Big(L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}^\vee(M))\Big)^\vee \simeq \mathrm{map}(\mathscr{M}(M), KU_p)$$

  by using Lemma 5.17.

All squares except the bottom left are obtained by applying a natural transformation, so these squares come with homotopies witnessing their commutativity. Two things remain to be done:

(1) the bottom-left square has to be constructed;

(2) we ought to show the bottom row agrees with the fibre sequence coming from the presentation of the algebra $KU_p^{X_+}$.

To this end, it will be useful to introduce some labels. We will also make use of the same lemmas and methods that we used in §3.2 and Lemma 3.8 to classify maps of the form $Y \to KU_p^{W_+}$ with Y a spectrum and W a space.

If Y is a $K(1)$-locally dualisable spectrum, then the map of KU_p-algebras

$$\varepsilon\colon \operatorname{Sym}_{KU_p}(KU_p^Y) \simeq KU_p \otimes \operatorname{Sym}(Y^\vee) \longrightarrow KU_p^{\Omega^\infty Y_+}$$

corresponds to a map of spectra

$$Y^\vee \longrightarrow KU_p^{\Omega^\infty Y_+},$$

which by abuse of notation we will also denote by ε. The map

$$\mathrm{TAQ}_{KU_p}(\varepsilon)^\vee\colon \operatorname{map}(Y^\vee, KU_p) \longrightarrow \mathrm{TAQ}_{KU_p}(KU_p^{\Omega Y_+})^\vee$$

as indicated in the diagram above is the KU_p-linear dual $\varepsilon^\vee$ of ε.

In §3.2, we constructed a map $\mathscr{M}(M) \to KU_p^{X_+}$ corresponding to the inclusion $M \to \Lambda M$ in

$$KU_p^*(\mathscr{M}(M) \otimes \Sigma_+^\infty X) \cong KU_p^*(\mathscr{M}(M)) \otimes_{(KU_p)_*} KU_p^*(X) \cong \operatorname{Hom}_{\mathbf{Z}_p}(M, \Lambda M).$$

We now use the notation ξ for this map $\mathscr{M}(M) \to KU_p^{X_+}$.

Lastly, recall from §5.1 that the map $h'\colon L_{K(1)}X \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ from the triangle (5.7) comes from a map $h\colon X \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ which on KU_p^* induces the natural quotient

$$\hat{\Lambda}\overline{F}M \longrightarrow \Lambda M.$$

Lemma 5.18 *The diagram*

$$\begin{tikzcd} \mathscr{M}(M) \arrow[r, "\varepsilon"] \arrow[dr, "\xi"'] & KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+} \arrow[d, "h^*"] \\ & KU_p^{X_+} \end{tikzcd}$$

is commutative up to homotopy.

Proof. The composite $h^* \circ \varepsilon\colon \mathscr{M}(M) \to \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$ is classified by the corresponding element in

$$KU_p^*\Big(\mathscr{M}(M) \otimes \Sigma_+^\infty \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)\Big) \cong \operatorname{Hom}_{\mathbf{Z}_p}(M,\ \hat{\Lambda}\overline{F}M).$$

The natural transformation ε and the isomorphism $KU_p^*(\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)) \cong \hat{\Lambda}\overline{F}M$ of Theorem 5.5 are such that the map ε corresponds to the natural inclusion $M \to \hat{\Lambda}\overline{F}M$. Thus $h^* \circ \varepsilon$ corresponds to the map

$$M \longrightarrow \hat{\Lambda}\overline{F}M \longrightarrow \Lambda M$$

that is the composite of the inclusion and the quotient; more plainly, it corresponds to the natural inclusion $M \to \Lambda M$. The map ξ by definition corresponds to $M \to \Lambda M$, and thus ξ is homotopic to $h^* \circ \varepsilon$. ∎

Thus we may construct the desired bottom-left square by putting $\mathrm{TAQ}(\xi)^\vee$ at the bottom. We therefore arrive at a diagram of the form

$$\begin{tikzcd}
L_{K(1)} KU_p \otimes \Phi X \arrow[r] \arrow[d, "c_X^{KU_p}"] & L_{K(1)} KU_p \otimes \mathscr{M}^\vee(M) \arrow[r] \arrow[d, "\simeq"] & L_{K(1)} KU_p \otimes \mathscr{M}^\vee(M) \arrow[d, "\simeq"] \\
\mathrm{TAQ}_{KU_p}(KU_p^{X_+})^\vee \arrow[r, "\mathrm{TAQ}(\xi)^\vee"] & \mathrm{map}(\mathscr{M}(M), KU_p) \arrow[r] & \mathrm{map}(\mathscr{M}(M), KU_p).
\end{tikzcd}$$

It remains to be shown that the second map in the bottom row is homotopic to the map coming from the presentation of $KU_p^{X_+}$. Recall from §3.2 that this map was induced by a map

$$\eta\colon \mathscr{M}(M) \longrightarrow L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M))$$

that corresponds to the homomorphism $\theta^p - F\theta^p_M\colon M \to \mathrm{Free}_\theta(M)$ on K-theory. Recall from §5.1.1 that in the triangle (5.7), the map

$$f\colon \Omega^\infty \widetilde{\mathscr{M}}^\vee(M) \longrightarrow \Omega^\infty \widetilde{\mathscr{M}}^\vee(M)$$

induces the map $\theta^p - \overline{F}\theta^p_M$ on KU_p^*.

Lemma 5.19 *In the diagram*

$$\begin{tikzcd}
L_{K(1)} KU_p \otimes \mathscr{M}(M) \arrow[r] \arrow[d, "\varepsilon"'] & L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \arrow[d, "\varepsilon"] \\
KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+} \arrow[r, "f^*"] & KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+}
\end{tikzcd}$$

the top map is homotopic to η.

Proof. The natural transformation ε and the isomorphism from Theorem 5.5 are such that after taking homotopy groups, the square

$$\begin{tikzcd}
L_{K(1)} KU_p \otimes \mathscr{M}(M) \arrow[r] \arrow[d, "\varepsilon"'] & L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M)) \arrow[d, "\varepsilon"] \\
KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+} \arrow[r, "f^*"] & KU_p^{\Omega^\infty \widetilde{\mathscr{M}}^\vee(M)_+}
\end{tikzcd}$$

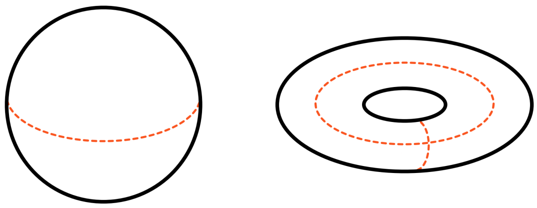
becomes

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \mathrm{Free}_\theta(M) \arrow[d] \\
\hat{\Lambda}\overline{F}M \arrow[r, "\theta^p - \overline{F}\theta^p_M"] & \hat{\Lambda}\overline{F}M
\end{tikzcd}$$

with the vertical maps the natural inclusions. Thus the map $M \to \mathrm{Free}_\theta(M)$ in this diagram is $\theta^p - F\theta^p_M\colon M \to \mathrm{Free}_\theta(M)$, and therefore (using Proposition 3.5) the map $\mathscr{M}(M) \to L_{K(1)} KU_p \otimes \mathrm{Sym}(\mathscr{M}(M))$ is homotopic to the map η. ∎

Popular summary (Dutch)

De topologie heeft een kenmerkende vraag: hoe kunnen we ruimtes van elkaar onderscheiden? Neem bijvoorbeeld de tweedimensionale bol en de torus.



Voor onze ogen zijn dit overduidelijk verschillende ruimtes: je kan de één niet in de ander omvormen zonder te scheuren, plakken, of iets anders raars te doen. Om dit idee om te zetten in een wiskundig bewijs, is erg lastig. De formele term voor 'hetzelfde' is *homeomorf*; de vraag is dus waarom de bol niet homeomorf is aan de torus.

Algebraïsche topologie biedt een oplossing voor dit probleem. De techniek die gebruikt wordt lijkt misschien omslachtig, maar het idee is ongelofelijk krachtig. Het idee is om een functie te construeren die aan elke ruimte een getal toekent, namelijk het aantal gaten. De formele term voor het aantal gaten is het *geslacht*. Het geven van de constructie kost wat tijd, maar de crux is dat het relatief eenvoudig is om uit de constructie af te leiden dat homeomorfe ruimtes hetzelfde geslacht hebben. Om deze reden wordt het geslacht een *invariant* van de ruimte genoemd: als je de ruimte omvormt zonder haar wezenlijk te veranderen (dat wil zeggen, een homeomorfe ruimte bekijken), verandert het geslacht niet. Nu blijkt dat, zoals je zou verwachten, de bol geslacht 0 heeft, en de torus geslacht 1. Conclusie: de bol en de torus zijn niet homeomorf.

Het geslacht is slechts het eerste voorbeeld van een invariant. Algebraïsch topologen hebben vele invarianten bedacht en bestudeerd in de afgelopen honderd jaar. De studie van invarianten wordt pas echt serieus zodra je kijkt naar invarianten die geen getallen, maar *groepen* aan ruimtes toekennen. Het woord 'invariant' betekent nu dat homeomorfe ruimtes naar isomorfe groepen moeten worden gestuurd. Meestal kijkt men naar invarianten die alleen abelse groepen gebruiken. Groepen zijn natuurlijk lastigere objecten dan getallen, maar dat maakt dat ze meer informatie kunnen 'onthouden'. Daarnaast zijn topologen erg goed geworden in het berekenen van dit soort groepen.

Deze invarianten komen in verschillende soorten en maten. Er zijn twee hoofdrolspelers. De *homotopiegroepen*, aangeduid met π_n (voor n een natuurlijk getal), zijn zeer complex en moeilijk uit te rekenen, maar geven ongelofelijk veel informatie. De *homologiegroepen*, aangeduid met H_n (voor n een natuurlijk getal), zijn in zekere opzichten makkelijkere versies van π_n (maar zijn in meer exotische gevallen ook erg verschillend). Zowel π_n als H_n zeggen iets over het aantal 'n-dimensionale gaten' in een ruimte. Het geslacht van een oppervlak X kan bijvoorbeeld afgeleid worden uit $\pi_1(X)$ of $H_1(X)$. Vaak probeert men uit H_n informatie af te leiden over π_n.

Toch bleek het voor sommigen nog niet genoeg om deze invarianten te bekijken. Als een generalisatie van deze invarianten is in de afgelopen decennia het vakgebied *hogere algebra* ontwikkeld. Dit is een zeer vreemde, maar prachtige combinatie van topologie en algebra. De cruciale observatie is dat achter de homologiegroepen H_n een heel bijzonder soort ruimte schuilgaat: een *spectrum*. Omdat H_n een functie is die abelse groepen toekent aan ruimtes, krijgt dit spectrum ook een soort algebraïsche structuur. Je zou een spectrum een topologische versie van een abelse groep kunnen noemen. Hogere algebra probeert zoveel mogelijk resultaten over abelse groepen te generaliseren voor spectra. Dit lijkt ongelofelijk ingewikkeld: het doel van invarianten was om ruimtes te begrijpen, maar nu zijn we weer terug bij af en moeten we een ruimte proberen te begrijpen! In de praktijk zijn spectra gelukkig een stuk makkelijker dan gewone ruimtes: er zijn genoeg stellingen bewezen om, voor een redelijk groot deel, te kunnen doen alsof spectra zich gedragen als gewone abelse groepen. (Deze resultaten zijn in detail uitgewerkt door Lurie [HA] in een boek van meer dan 1500 bladzijden.) Net zoals abelse groepen meer informatie kunnen onthouden dan getallen, kunnen spectra meer informatie onthouden dan abelse groepen. Gepaard met al deze stellingen, maakt dit spectra tot nog betere objecten om invarianten mee te bouwen.

In deze scriptie doe ik een berekening met hogere algebra, geïnspireerd door een artikel van Bousfield [Bou99] uit 1999. Dit heeft meerdere doelen: allereerst is het goed om te zien hoe gerekend kan worden met dit soort objecten, en hoe makkelijk (of moeilijk) dat gaat. Ook is deze specifieke berekening nog niet eerder gedaan. Het tweede doel brengt ons weer terug bij de oude invarianten. Zelfs vandaag de dag kunnen we voor maar heel weinig ruimtes de homotopiegroepen π_n uitrekenen. In de afgelopen paar jaar is gebleken dat hogere algebra hierbij kan helpen: de uitkomst van de hogere algebra berekening geeft inzichten in hoe de homotopiegroepen werken. De aanpak van deze scriptie geeft niet zozeer nieuwe berekening van homotopiegroepen, maar levert een bijdrage aan een nieuw vakgebied dat de patronen achter de homotopiegroepen probeert te begrijpen. Omdat deze berekening nog niet eerder is gedaan, is het extra interessant om te zien hoe dit leidt tot meer begrip van de homotopiegroepen.

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A

Derived completeness

Let p be a fixed prime number. In this chapter we describe the theory of derived p-completion of $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-modules. Originally Greenlees and May [GM92] studied this in a much more general setting. Hovey and Strickland [HS99, App. A] work in the setting of $\mathbf{Z}/2$-graded modules over $\mathbf{Z}_p[\![u_1, \ldots, u_{n-1}]\!]$, for some fixed $n \geqslant 1$. In the language of chromatic homotopy theory, they work at height n, whereas we work at height 1. In §A.1 and §A.2 we summarise parts of the expositions by Hovey and Strickland [HS99, App. A] and Barthel and Frankland [BF15, App. A], simplified to the $\mathbf{Z}_p$-setting.

Throughout this appendix, we use an asterisk to indicate $\mathbf{Z}/2$-graded objects: $\mathbf{Mod}^*_{\mathbf{Z}_p}$ denotes the category of $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-modules, and $\mathbf{CAlg}^*_{\mathbf{Z}_p}$ the category of graded-commutative $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebras.

A.1 Derived complete modules

Consider the functor of **classical p-completion** of $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-modules:

$$(-)^\wedge_p \colon \mathbf{Mod}^*_{\mathbf{Z}_p} \longrightarrow \mathbf{Mod}^*_{\mathbf{Z}_p}, \quad M \longmapsto M^\wedge_p := \lim_n M/p^n M.$$

This is an exact functor when restricting to finitely generated modules, but in general it is neither left nor right exact. This is a problem because modules coming from topology are often not finitely generated. To remedy this, one can instead consider the zeroth left-derived functor of p-completion, and regard this as the 'correct' version of p-completion. In the main text we will indeed see that this is the correct notion for topology: derived p-completion is the shadow of p-completion of spectra (or more specifically, in the present $\mathbf{Z}/2$-graded context, p-completion of KU_p-module spectra).

Definition A.1 For a natural number $s \geqslant 0$, let L_s denote the s-th left derived functor of the classical p-completion functor $\mathbf{Mod}^*_{\mathbf{Z}_p} \to \mathbf{Mod}^*_{\mathbf{Z}_p}$.

More concretely, if M is a graded $\mathbf{Z}_p$-module and $F_1 \to F_0$ a free resolution of M, then

$$L_0 M \cong (F_0)^\wedge_p / (F_1)^\wedge_p,$$

and this is independent of the free resolution.

Usually when working with left derived functors, one can prove that the zeroth derived functor is equal to the functor itself, but this requires *right exactness*. (E.g., the tensor product is right exact, and Tor_0 retrieves the tensor product functor.) Classical p-adic completion is not right exact and so L_0 is truly a distinct functor from $(-)^\wedge_p$.

For a graded $\mathbf{Z}_p$-module M, the natural map $M \to M^\wedge_p$ factors canonically to

$$M \longrightarrow L_0 M \longrightarrow M^\wedge_p.$$

Definition A.2 Let M be a $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module. The module $L_0 M$ is called the **derived p-completion** of M. The module M is called **derived p-complete** (or *L-complete*, or *Ext-p-complete*) if the natural map $M \to L_0 M$ is an isomorphism. Write $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ for the full subcategory of $\mathbf{Mod}^*_{\mathbf{Z}_p}$ on the derived p-complete modules.

It turns out that only the functors L_0 and L_1 are interesting, and moreover these have an explicit description.

Theorem A.3 *The functor L_s vanishes if $s > 1$. (In particular, L_0 is right exact, and L_1 is left exact.) There are natural isomorphisms*

$$L_0 \cong \mathrm{Ext}^1_{\mathbf{Z}_p}(\mathbf{Z}/p^\infty, -) \qquad \text{and} \qquad L_1 \cong \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Z}/p^\infty, -),$$

where $\mathbf{Z}/p^\infty$ denotes $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$.

Proof. See [HS99], Theorem A.2(d). ■

Classical p-completion is a quotient of derived p-completion.

Theorem A.4 *For $M \in \mathbf{Mod}^*_{\mathbf{Z}_p}$, there is a short exact sequence*

$$0 \longrightarrow \lim_i{}^1 \mathrm{Hom}(\mathbf{Z}/p^i,\ M) \longrightarrow L_0 M \longrightarrow M^\wedge_p \longrightarrow 0.$$

Proof. See [HS99], Theorem A.2(b). ■

As special cases, we find that if M is either a finitely generated or a flat module, that derived p-completion coincides with classical p-completion. (The flat case follows because $\mathrm{Hom}(\mathbf{Z}/p^i, M) = \mathrm{Tor}_1^{\mathbf{Z}_p}(\mathbf{Z}/p^i, M)$.)

Proposition A.5 *Let $M \in \mathbf{Mod}^*_{\mathbf{Z}_p}$. Then $M^\wedge_p$ and $L_0 M$ are derived p-complete. In particular, derived p-completion is idempotent: the natural map $L_0 M \to L_0 L_0 M$ is an isomorphism.*

Proof. See [HS99], Theorem A.6(a). ■

A.2 The derived complete category

We investigate the structure and basic properties of the category $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ of derived p-complete modules. This category is defined as a full subcategory of $\mathbf{Mod}^*_{\mathbf{Z}_p}$, so let us first review the relevant structure and properties of $\mathbf{Mod}^*_{\mathbf{Z}_p}$. The tensor product of $\mathbf{Z}/2$-graded modules turns $\mathbf{Mod}^*_{\mathbf{Z}_p}$ into a symmetric monoidal category, where the switch map incorporates the Koszul sign rule:

$$M \otimes N \longrightarrow N \otimes M, \quad x \otimes y \longmapsto (-1)^{|x||y|} y \otimes x \tag{A.6}$$

on homogeneous $x \in M$ and $y \in N$. The category $\mathbf{Mod}^*_{\mathbf{Z}_p}$ has all limits and colimits, and has enough projectives.

Loosely speaking, the category $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ inherits all of this structure from $\mathbf{Mod}^*_{\mathbf{Z}_p}$. The main result from which this follows is the adjunction between L_0 and the inclusion functor.

Proposition A.7 *The functor L_0 is left adjoint to the inclusion $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p} \to \mathbf{Mod}^*_{\mathbf{Z}_p}$.*

Proof. See [HS99], Theorem A.6(f). ■

Let us now consider the symmetric monoidal structure of $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$. In general the tensor product of two derived p-complete modules need not be derived p-complete. It is therefore only natural to consider the **completed tensor product**: for $M, N \in \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$, define

$$M \hat{\otimes} N := L_0(M \otimes N).$$

The structure maps of the symmetric monoidal category $\mathbf{Mod}^*_{\mathbf{Z}_p}$ induce structure maps for the completed tensor product.

If $M, N \in \mathbf{Mod}^*_{\mathbf{Z}_p}$, then we have a natural map

$$L_0M \otimes L_0N \longrightarrow L_0(M \otimes N).$$

The target is a derived p-complete module, so by the adjunction of Proposition A.7 this yields a natural map

$$L_0(L_0M \otimes L_0N) = L_0M \hat{\otimes} L_0N \longrightarrow L_0(M \otimes N).$$

Theorem A.8 *The completed tensor product, together with the structure maps described above, gives $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ the structure of a symmetric monoidal category. The natural map $L_0M \hat{\otimes} L_0N \to L_0(M \otimes N)$ described above is an isomorphism, giving L_0 the structure of a symmetric monoidal functor.*

Proof. See the proof of Corollary A.7 in [HS99]. ■

Our next objective is to show that $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ has all limits and colimits. This is a formal consequence of $\mathbf{Mod}^*_{\mathbf{Z}_p}$ having all limits and colimits, and the following result.

Proposition A.9 *The category $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is an abelian subcategory of $\mathbf{Mod}^*_{\mathbf{Z}_p}$, and is closed under extensions formed in $\mathbf{Mod}^*_{\mathbf{Z}_p}$. In particular, submodules and quotient modules of derived complete modules are again derived complete.*

Proof. See [HS99], Theorem A.6(e). ■

Corollary A.10

(a) *The limit of a diagram in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is computed by taking the limit in $\mathbf{Mod}^*_{\mathbf{Z}_p}$.*

(b) *The colimit of a diagram in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is computed by applying L_0 to the colimit in $\mathbf{Mod}^*_{\mathbf{Z}_p}$.*

In particular, the category $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ has all limits and colimits.

Proof. To prove the assertion about limits, we make two observations.

- The product of derived p-complete modules is derived p-complete. This is immediate from the isomorphism $L_0M \cong \mathrm{Ext}^1_{\mathbf{Z}_p}(\mathbf{Z}/p^\infty, M)$ of Theorem A.3.
- The kernel of a morphism between derived p-complete modules is p-complete. This is a special case of Proposition A.9.

Any limit can be expressed as an equaliser of products, so these two observations imply that the limit in $\mathbf{Mod}^*_{\mathbf{Z}_p}$ of derived p-complete modules is again derived p-complete. This proves the claim about limits.

The case of colimits is a purely formal consequence of Proposition A.7: L_0 is left adjoint and left inverse to the inclusion $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p} \to \mathbf{Mod}^*_{\mathbf{Z}_p}$, so it creates colimits. ∎

As a special case of the above, the coproduct of a collection $\{ M_\alpha \}_\alpha$ in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is given by $L_0 \bigoplus_\alpha M_\alpha$. Forming this derived p-complete direct sum turns out to be an exact procedure.

Proposition A.11 (Direct sums are exact) *Let I be an indexing set, and for $\alpha \in I$ let*

$$0 \longrightarrow M'_\alpha \longrightarrow M_\alpha \longrightarrow M''_\alpha \longrightarrow 0$$

be a short exact sequence in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$. Then the sequence

$$0 \longrightarrow L_0 \bigoplus_\alpha M'_\alpha \longrightarrow L_0 \bigoplus_\alpha M_\alpha \longrightarrow L_0 \bigoplus_\alpha M''_\alpha \longrightarrow 0$$

is exact.

Proof. See [Hov13], Proposition 1.7. ∎

We end our discussion by discussing the projective objects of $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$.

Definition A.12 A module $M \in \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is called **pro-free** when it is of the form L_0F for some free graded $\mathbf{Z}_p$-module F.

It turns out that projective objects in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ are exactly the pro-free modules. See [HS99, §A.4] or [BF15, §A.3] for more infomration. We will not make use of such results, as pro-free objects make only a minor appearance in this work.

A.3 Derived complete algebras

Let $\mathbf{CAlg}^*_{\mathbf{Z}_p}$ denote the category of graded-commutative $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebras. Categorically speaking, this is the category $\mathbf{CAlg}(\mathbf{Mod}^*_{\mathbf{Z}_p})$ of commutative algebra objects in the symmetric monoidal category $\mathbf{Mod}^*_{\mathbf{Z}_p}$. (The graded commutativity follows from the symmetric structure we chose in the previous section: see Equation (A.6).) It is natural to ask what the relevant notion of derived p-completeness is for such algebras. For us, this is the following.

Definition A.13 A graded-commutative $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-algebra is **derived p-complete** when it is so as a $\mathbf{Z}/2$-graded $\mathbf{Z}_p$-module. Write $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ for the full subcategory of $\mathbf{CAlg}^*_{\mathbf{Z}_p}$ on the derived p-complete algebras.

A more categorical way to define the same would be to consider the commutative algebra objects in $\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$, i.e., to consider the category $\mathbf{CAlg}(\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p})$. This category is equivalent to $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ as defined above, for the following reason. In $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$, an object is of the form

$$(A,\ A \otimes A \longrightarrow A), \quad \text{where } A \in \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p},$$

whereas in $\mathbf{CAlg}(\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p})$ an object is of the form

$$(A,\ A \mathbin{\hat{\otimes}} A \longrightarrow A), \quad \text{where } A \in \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}.$$

Because A is required to be derived p-complete in both cases, the adjunction of Proposition A.7 sets up a one-to-one correspondence between maps $A \otimes A \to A$ and maps $A \mathbin{\hat{\otimes}} A \to A$. One can then check that the associativity and commutativity constraints on the maps match up under this correspondence. Furthermore, these assignments are functors in a natural way and are inverse equivalences. Verifying this is a straightforward but tedious endeavour, so we omit the details.

Proposition A.14 *The categories $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ and $\mathbf{CAlg}(\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p})$ are equivalent.*

Recall from Theorem A.8 that the functor $L_0\colon \mathbf{Mod}^*_{\mathbf{Z}_p} \to \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$ is symmetric monoidal. This means that we obtain an induced functor $L_0\colon \mathbf{CAlg}^*_{\mathbf{Z}_p} \to \widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$. The adjunction of Proposition A.7 then implies that $L_0\colon \mathbf{CAlg}^*_{\mathbf{Z}_p} \to \widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}$ is left adjoint to the inclusion $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p} \to \mathbf{CAlg}^*_{\mathbf{Z}_p}$.

To conclude this section, we consider free derived p-complete algebras. In the uncompleted case, the forgetful functor $\mathbf{CAlg}^*_{\mathbf{Z}_p} \to \mathbf{Mod}^*_{\mathbf{Z}_p}$ has a left adjoint, the *free algebra functor*:

$$\mathbf{Mod}^*_{\mathbf{Z}_p} \longrightarrow \mathbf{CAlg}^*_{\mathbf{Z}_p}, \quad M \longmapsto \operatorname{Sym}_{\mathbf{Z}_p}(M^0) \otimes_{\mathbf{Z}_p} \Lambda_{\mathbf{Z}_p}(M^1),$$

with Sym and Λ denoting the symmetric and exterior algebra, respectively. Postcomposing this with L_0 and pre-composing with the inclusion yields the functor

$$\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p} \longrightarrow \widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p}, \quad M \longmapsto L_0\Big(\operatorname{Sym}_{\mathbf{Z}_p}(M^0) \otimes_{\mathbf{Z}_p} \Lambda_{\mathbf{Z}_p}(M^1)\Big).$$

This is left adjoint to the forgetful functor $\widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p} \to \widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p}$. Thus we end up with a square of adjunctions (left adjoints on the outside)

$$\begin{tikzcd}
\widehat{\mathbf{Mod}}^*_{\mathbf{Z}_p} \arrow[r, shift left, "\text{free}"] \arrow[d, shift left] & \widehat{\mathbf{CAlg}}^*_{\mathbf{Z}_p} \arrow[l, shift left, "\text{forget}"] \arrow[d, shift left] \\
\mathbf{Mod}^*_{\mathbf{Z}_p} \arrow[u, shift left, "L_0"] \arrow[r, shift right, "\text{free}"'] & \mathbf{CAlg}^*_{\mathbf{Z}_p}. \arrow[l, shift right, "\text{forget}"'] \arrow[u, shift left, "L_0"']
\end{tikzcd}$$