TALK SUMMARIES - ORIENTATION SEMINAR

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1. Introduction - Lennart Meier - 01/03/2022

2. Prerequisites - Miguel Barrero - 08/03/2022

Cover the ∞-categorical background presented in [And+14a, Section 1 and 2]. In particular, explain parametrised spaces and spectra; another reference is [ABG18, Section 3]. Finally, explain the construction of the units functors GL_1 and gl_1 of ring spectra [And+14b], and prove the adjunction between gl_1 and $\Sigma^\infty_+\Omega^\infty$.

3. Thom spectra - Max Blans - 15/03/2022

Define the Thom spectrum associated to a local system of R-modules, and describe the universal property of this Thom spectrum. This is done in the $\mathbf{A}_\infty$-case by [And+14a], which can be extended to the $\mathbf{E}_\infty$-case using the result of [AB19] and [ABG18]. (Only the $\mathbf{E}_\infty$-case is needed for later talks.) Discuss twisted cohomology in the sense of [ABG18, Defition 1.2] or [And+14a, Corollary 1.11]. Finally, give a rough impression for the comparison given in [And+14a, Section 3] between the ∞-categorical and the model-categorical approach, explaining why ∞-categories are useful in this situation.

4. $\hat{A}$-genus - Yuqing Shi - 22/03/2022

The goal of this talk is to construct the $\hat{A}$-genus $\hat{A}\colon \pi_*(\mathrm{MSpin}) \to \pi_*(\mathrm{KO})$. Define the map α_P [ABS64, Section 11] and introduce its product formula [ABS64, Proposition 11.3]. Prove the KO-Thom isomorphism [ABS64, Theorem 12.3.i)]. You also need to introduce relevant prerequisites from the previous chapters of [ABS64]. It would be interesting to explore whether the above can be presented with the modern language from Talk 3.

Construct the map $\hat{A}\colon \pi_*(\mathrm{MSpin}) \to \pi_*(\mathrm{KO})$ by expanding the summary from [DG18, Section 3.3.1]. For this, you need to explain the pushforward map in KO-theory [DG18, Definition 3.25] which is a baby example of the Umkehr map introduced in the next talk.

5. Atiyah–Bott–Shapiro orientation as map of spectra - Miguel Barata 29/03/2022

This talk is concerned with constructing the map of (E_∞-ring) spectra $\mathrm{MSpin} \to \mathrm{KO}$ which refines the $\hat{A}$-genus. You shall present this construction in two ways. The first approach continues the last part of the previous talk. Construct a map of ring spectra $\mathrm{MSpin} \to \mathrm{KO}$, following the explanation in [Mei, Section 2.5].

In the second part of the talk you follow [AHR] to show that ABS is a map of E_∞-ring spectra. Give a sketch of [AHR, Theorem 6.1].

6. Elliptic curves and elliptic genera - 05/04/2022

The goal of this talk is to cover [Mei, Section 3.3, 3.7, 3.8 and 3.9] in detail. This gives an introduction on elliptic curves and elliptic genera, and provides some examples. You should also sketch the proofs (if it is given in the lecture notes) of the theorems in [Mei, Section 3.3, 3.7]

7. Calculation with elliptic genus and properties - 12/04/2022

This talk covers [Mei, Section 3.10, 3.11 and 3.12]. Explain elliptic integrals and exponential with examples [Mei, Section 3.10, 3.11]. For this purpose you may need to recall briefly the invariant differential of an elliptic curve [Mei, p.27]. State the results about the evaluation of Ochanine's genus on complex projective spaces [Mei, Example 3.38]. Explain the relation of Ochanine's genus with the signature and the $\hat{A}$-genus [Mei, Example 3.39, 3.40]. Introduce modular forms [Mei, Section 3.11]. Give a sketch of the proof of [Mei, Proposition 3.45]. Explain properties of elliptic genus [Mei, Section 3.11]. An alternative reference is [HBJ92, Section 2].

8. Introduction to TMF - Jack Davies - Jack - 19/04/2022

Introduce the two of the variants of topological modular forms: the periodic version TMF, and the connective version tmf. Explain briefly the origin of TMF, black-boxing the existence of the sheaf of $\mathbf{E}_\infty$-rings $\mathcal{M}_{\mathrm{ell},\text{ét}} \to \mathrm{CAlg}$ on the (small) étale site of elliptic curves. Next, outline how the periodic variant tmf is obtained, and highlight the main differences between the two. Describe what their homotopy groups look like (when inverting 6, and the differences that arise when 6 is not inverted). References for topological modular forms are [Beh20; Hop02; Tal14]. The final aim of this talk is to cover the main results in [AHR, Section 11].

9. Introduction to Witten genus - 26/04/2022

10. TMF orientation on string bordism - 03/05/2022

The goal is to motivate, and then prove, the existence of $\mathbf{E}_\infty$-map MString $\to$ tmf refining the Witten genus. The relevant papers include [AHR], [AHS01] and [AHS04]; the book [Tal14] is an additional source. The introduction to [AHR] gives a brief overview of the history and outlines the proof strategy.

Discuss elliptic spectra in the sense of [AHR], and roughly discuss how the σ-orientation on these comes about. Then motivate why it is natural to look for a strengthened version, namely an orientation of tmf. Then give an impression for the proof described in [AHR, Section 12 - 15]. Recall results from §5 if necessary. Be light on the technicalities (e.g., do not prove results from [AHR, Section 10]). Note that unlike the Spin orientation on KO, there is no uniqueness statement about the $\mathbf{E}_\infty$ structure on the map MString $\to$ tmf.

References

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