The map τ on filtered spectra and the deformation picture

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Last time, we discussed how filtered spectra naturally give rise to a spectral sequence. We formulated this in terms of a formal variable τ, and gave an impression of how differentials lead to τ-power torsion in the bigraded homotopy groups $\pi_{*,*}X$ of a filtered spectrum. So far however, we only viewed τ as a device to record the effect of the transition maps on these homotopy groups. This time, we will consider τ as a map on the level of filtered spectra, and discuss all the shifts in perspective that comes with this. The result will be that the ∞-category behaves as an “interface” for spectral sequences. At the end, we will discuss how one can move this interface to other ∞-categories; next time, this is how we will view synthetic spectra as being a certain type of spectral sequence.

1 Examples of spectral sequences

The goal of this section is twofold. The first is to give various examples of spectral sequences constructed via filtered spectra; our hope that this will make the process of constructing a spectral sequence less mysterious. The second is a natural consequence of the first, and is to give various functors and tools that one needs to work with filtered spectra. The most notable tools are: is there more?

- the diagonal t-structure; see Proposition 1.2,
- the Whitehead and Postnikov functors $\mathrm{Sp} \to \mathrm{FilSp}$; see Definition 1.4,
- the stable Dold–Kan correspondence; see. ref

While most examples play no big role in the rest of the mini-course, the p-Bockstein spectral sequence of Sections 1.5 and 1.6 serves as a warm-up for the τ-Bockstein spectral sequence of Section 5.

1.1 Cofibre sequences

The simplest example of a spectral sequence comes from a cofibre sequence: this becomes a spectral sequence with only a d_1-differential.

Example 1.1. Let $f\colon X \to Y$ be a map of spectra. This gives rise to a filtered spectrum

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{f} & Y & = & Y = \cdots, \\ & & & & \| & & \downarrow & & \\ & & & & X & & \operatorname{cofib} f & & \end{array}$$

where for convenience we place X in filtration 1. We have also drawn the nonzero terms of the associated graded. The first page of the resulting spectral sequence is

$$\mathrm{E}_1^{n,s} = \begin{cases} \pi_n X & \text{if } s = 1, \\ \pi_n \operatorname{cofib} f & \text{if } s = 0, \\ 0 & \text{else.} \end{cases}$$

Moreover, working through the diagram chase defining the differential, we see that the map

$$d_1\colon \mathrm{E}_1^{n,s} \longrightarrow \mathrm{E}_1^{n-1,s+1}$$

is only nonzero for $s = 0$, in which case it is given by the boundary map

$$\partial_n\colon \pi_n \operatorname{cofib} f \longrightarrow \pi_n(\Sigma X) = \pi_{n-1} X.$$

All higher differentials are zero for degree reasons. This results in identifications

$$\mathrm{E}_\infty^{n,0} = \ker \partial_n \qquad \text{and} \qquad \mathrm{E}_\infty^{n,1} = \operatorname{coker} \partial_{n+1}.$$

Strong convergence is trivial in this case, and boils down to the short exact sequences

$$0 \longrightarrow \operatorname{coker} \partial_{n+1} \longrightarrow \pi_n Y \longrightarrow \ker \partial_n \longrightarrow 0. \qquad ▲$$

1.2 The Whitehead filtration and the Atiyah–Hirzebruch spectral sequence

If X is a spectrum, then the **Whitehead filtration** of X is the diagram

make spacing consistent with later things

$$\cdots \longrightarrow \tau_{\geqslant 1} X \longrightarrow \tau_{\geqslant 0} X \longrightarrow \tau_{\geqslant -1} X \longrightarrow \cdots,$$

and its **Postnikov tower** is the diagram

$$\cdots \longrightarrow \tau_{\leqslant 1} X \longrightarrow \tau_{\leqslant 0} X \longrightarrow \tau_{\leqslant -1} X \longrightarrow \cdots.$$

We begin with a small digression to make these functorial in X. The trick for doing this is to use a t-structure on filtered spectra.

Proposition 1.2. *There exists a unique accessible t-structure on* FilSp *with the following properties.*

(a) *A filtered spectrum X is connective if and only if*

$$\pi_{n,s} X = 0 \qquad \textit{whenever } n < s.$$

(b) *A filtered spectrum X is* 0*-truncated if and only if*

$$\pi_{n,s} X = 0 \qquad \textit{whenever } n > s.$$

(c) *The connective cover $\tau_{\geqslant 0}^{\mathrm{diag}} X \to X$ induces an isomorphism*

$$\pi_{n,s}(\tau_{\geqslant 0}^{\mathrm{diag}} X) \xrightarrow{\cong} \pi_{n,s}(X) \qquad \textit{whenever } n \geqslant s.$$

Likewise, the 0*-truncation $X \to \tau_{\leqslant 0}^{\mathrm{diag}} X$ induces an isomorphism*

$$\pi_{n,s}(X) \xrightarrow{\cong} \pi_{n,s}(\tau_{\leqslant 0}^{\mathrm{diag}} X) \qquad \textit{whenever } n \leqslant s.$$

(d) *The t-structure is compatible with the monoidal structure.*

We refer to this t-structure as the ***diagonal t-structure*** *on filtered spectra.*

proof will be in appendix someday

Note that since (de)suspensions on FilSp are computed levelwise, it follows that the analogous versions of (a) and (b) hold for n-connective and n-truncated filtered spectra as well. For instance, X is n-connective if and only if $\mathrm{Gr}^s X$ is $(n+s)$-connective, for all s.

It follows that if X is a filtered spectrum, then its connective cover $\tau_{\geqslant 0}^{\mathrm{diag}} X$ is the filtered spectrum

$$\cdots \longrightarrow \tau_{\geqslant 1} X^1 \longrightarrow \tau_{\geqslant 0} X^0 \longrightarrow \tau_{\geqslant -1} X^{-1} \longrightarrow \cdots,$$

and its 0-truncation $\tau_{\leqslant 0}^{\mathrm{diag}} X$ is the filtered spectrum

$$\cdots \longrightarrow \tau_{\leqslant 1} X^1 \longrightarrow \tau_{\leqslant 0} X^0 \longrightarrow \tau_{\geqslant -1} X^{-1} \longrightarrow \cdots.$$

Definition 1.3. The **constant** functor Const: Sp $\to$ FilSp is the functor given by precomposition with $\mathbf{Z}^{\mathrm{op}} \to \Delta^0$:

$$\mathrm{Sp} = \mathrm{Fun}(\Delta^0, \mathrm{Sp}) \longrightarrow \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{FilSp}.$$

On objects, the functor Const sends a spectrum X to the constant filtered spectrum

$$\cdots = X = X = X = \cdots.$$

Note that Const is canonically a symmetric monoidal functor.

Definition 1.4.

(1) The **Whithead filtration** is the functor $\mathrm{Wh}\colon \mathrm{Sp} \to \mathrm{FilSp}$ given by $\tau_{\geqslant 0}^{\mathrm{diag}} \circ \mathrm{Const}$.

(2) The **Postnikov tower** is the functor $\mathrm{Post}\colon \mathrm{Sp} \to \mathrm{FilSp}$ given by $\tau_{\leqslant 0}^{\mathrm{diag}} \circ \mathrm{Const}$.

We get even more than we set out for at first: because the diagonal t-structure on FilSp is compatible with the symmetric monoidal structure, the functor $\tau_{\geqslant 0}^{\mathrm{diag}}\colon \mathrm{FilSp} \to \mathrm{FilSp}$ is canonically lax symmetric monoidal, making Wh a lax symmetric monoidal functor as well.

Having set up these constructions functorially, we now turn to the spectral sequences one can make with them.

Warning 1.5. Although Wh and Post are functors between stable ∞-categories, they are *not* exact functors. One can check that the functor Wh send a cofibre sequence of spectra

$$X \longrightarrow Y \longrightarrow Z$$

to a cofibre sequence of filtered spectra if and only if the induced sequence

$$0 \longrightarrow \pi_* X \longrightarrow \pi_* Y \longrightarrow \pi_* Z \longrightarrow 0$$

is a short exact sequence of graded abelian groups.

Example 1.6. Let X be a spectrum. Then the first page of the spectral sequence associated to $\mathrm{Wh}(X)$ is given by

$$\mathrm{E}_1^{n,s} = \pi_n(\tau_{\leqslant s}\, \tau_{\geqslant s}\, X) = \pi_n(\Sigma^s\, \pi_s X) = \begin{cases} \pi_s X & \text{if } s = n, \\ 0 & \text{else.} \end{cases}$$

It looks slightly easier if we reindex this to start on the second page instead: ref to remark about reindexing

$$\mathrm{E}_2^{n,s} = \begin{cases} \pi_s X & \text{if } s = 0, \\ 0 & \text{else.} \end{cases}$$

The spectral sequence associated to $\mathrm{Post}(X)$ is the same: $\mathrm{Wh}(X)$ and $\mathrm{Post}(X)$ are related by the flipping of **??**. ▲ more precise reference

This spectral sequence is not very interesting. However, we can use it to build more interesting ones, by levelwise tensoring with a spectrum.[1]

Definition 1.7. Let A and X be spectra. The **Atiyah–Hirzebruch filtration** of $A \otimes X$ is the filtered spectrum $\mathrm{Wh}(A) \otimes X$ (meaning, levelwise tensoring with the spectrum X). The **Atiyah–Hirzebruch spectral sequence** (AHSS) for $A_*(X)$ is its associated spectral sequence.

[1] Levelwise tensoring with a spectrum X is the same as tensoring with the filtered spectrum that is zero in positive filtration, and is constantly X in filtrations 0 and onwards.

Note that we can take X to be a suspension spectrum, yielding the Atiyah–Hirzebruch spectrum for the homology of a space.

Proposition 1.8. *Let A and X be spectra.*

(1) *The colimit of the Atiyah–Hirzebruch filtration is $A \otimes X$.*

(2) *The Atiyah–Hirzebruch spectral sequence for $A_*(X)$ is of signature*

$$\mathrm{E}_1^{n,s} = \mathrm{H}_{n-s}(X;\, \pi_s A) \implies A_n(X).$$

(3) *If X is bounded below (e.g., if it is a suspension spectrum), then the AHSS converges strongly to $A_*(X)$.*

Proof. The colimit of $\mathrm{Wh}(A)$ is A. Using that the tensor product of spectra preserves colimits, this implies the first claim. The associated graded of $\mathrm{Wh}(A) \otimes X$ at level s is given by

Spectra are right complete

$$\Sigma^s\, \pi_s(A) \otimes X,$$

so the first page of the spectral sequence is

$$\mathrm{E}_1^{n,s} = \pi_n(\Sigma^s\, \pi_s(A) \otimes X) = \mathrm{H}_{n-s}(X;\, \pi_s A).$$

To prove convergence, it suffices by **??** to show that for fixed n, the group

$$\pi_n(\tau_{\geqslant s} A \otimes X)$$

vanishes for $s \gg 0$. We assumed that X is bounded below; choose an N such that X is N-connective. Then $\tau_{\geqslant s} A \otimes X$ is $(s+N)$-connective. In particular, it follows that for $s > n - N$, its n-th homotopy group vanishes. ■

It is more common to reindex this spectral sequence via $(n,s) \mapsto (n,\, n-s)$. This does two things: first, it flips the filtration to let it be indexed homologically (recall that our standard convention on filtered spectra is to index them cohomologically; see **??**), and second, it makes d_r-differentials look like d_{r+1}-differentials. The resulting expression for the second page is

$$\mathrm{E}^2_{n,s} = \mathrm{H}_s(X;\, \pi_{n-s} A),$$

with differentials d_r of bidegree $(-1, -r)$.

Remark 1.9. Often, the AHSS is presented as a spectral sequence whose second page is given by the homology groups as in the above proposition. It arises in that way by using a different construction, one based on a skeletal filtration on X instead of the Whitehead filtration of A. The resulting first page will depend on this choice of filtration, and the second page is a reindexed version of the first page from Proposition 1.8. By contrast, the above construction does not depend on any choices, so it does not have this choice-dependent page.

Remark 1.10. There is also a cohomological version of the Atiyah–Hirzebruch spectral sequence. It is based on the filtered spectrum

$$\operatorname{map}_{\mathrm{Sp}}(X, \mathrm{Wh}(A)),$$

i.e., levelwise taking the mapping spectrum out of X. The resulting first page is

$$\mathrm{E}_1^{n,s} = \pi_n(\operatorname{map}(X, \Sigma^s \pi_s A)) = \mathrm{H}^{s-n}(X; \pi_s A).$$

In this case, convergence becomes a more subtle point. The colimit is $\operatorname{map}(X, A)$, and if X is bounded below, then the limit vanishes. However, in general we cannot do better than this, so we only get conditional convergence. If X is a finite spectrum (e.g., the suspension spectrum of a finite space), then we get strong convergence: in that case, the functor $\operatorname{map}(X, -)$ is naturally equivalent to $X^\vee \otimes -$.

1.3 The Serre spectral sequence

Todo (see commented)

1.4 The Tot spectral sequence

Todo (see commented)

1.5 The Bockstein spectral sequence

Definition 1.11. Let X be a spectrum. The ***p*-Bockstein filtration** of X, denoted $\mathrm{BF}_p\, X$, is the filtered spectrum

$$\cdots \xrightarrow{p} X \xrightarrow{p} X =\!=\!= \cdots,$$

which we index to be constant from filtration 0 onwards. The ***p*-Bockstein spectral sequence** of X is the underlying spectral sequence.

remark about other way to set it up? (mirror Palmieri, and probably also cite him right now, or right at the start of the section)

If p is clear from the context, then we will denote $\mathrm{BF}_p\, X$ simply by $\mathrm{BF}\, X$. We use the notation $\mathrm{BF}_p^s\, X$ for the spectrum in filtration s in $\mathrm{BF}_p\, X$. These are of course all isomorphic to X, but since the transition maps are not the identity, one should think of them as playing different roles. The notation helps distinguish between the different filtrations.

We immediately see that the induced filtered abelian group $\pi_n\, \mathrm{BF}_p\, X$ is the p-adic filtration on $\pi_n X$ from **??**. As such, this equips $\pi_n X$ with the classical p-adic filtration

$$F^s\, \pi_n X = p^s \cdot \pi_n X.$$

Going forward, we will use the notation X/p^k to mean $\mathbf{S}/p^k \otimes X$.

From the definition of the Bockstein filtration, it is immediate that the first page of the spectral sequence is given, as a bigraded abelian group, by

$$\mathrm{E}_1^{n,s} = \begin{cases} \pi_n(X/p) & \text{if } s \geqslant 0, \\ 0 & \text{else.} \end{cases} \tag{1.12}$$

It will be very convenient to introduce some notation to distinguish and manipulate these groups more easily.

Notation 1.13. We write $\overline{p}$ for a formal element of bidegree $(0,1)$. By definition, we say that an element $a \in \mathrm{E}_1^{n,s}$ is denoted by $\overline{p}^s \cdot a$, where on the right-hand side we consider a as an element of $\pi_n(X/p)$ placed in filtration 0. In other words, we define $\overline{p}$ so that the isomorphism of (1.12) refines to an isomorphism of bigraded $\mathbf{Z}[\overline{p}]$-algebras

$$\mathrm{E}_1^{*,*} \cong \pi_*(X/p) \otimes \mathbf{Z}[\overline{p}],$$

where $\pi_n(X/p)$ is placed in bidegree $(n,0)$ and $\overline{p}$ has bidgree $(0,1)$.

Remark 1.14. We use the symbol $\overline{p}$ instead of p to avoid confusion with the integer p. The source of this potential confusion is that multiplication by p is still a sensible operation on $\pi_n(X/p)$ (simply because it is an abelian group), but it is not particularly useful. The element $\overline{p}$ on the other hand will be our model for multiplication by p in the homotopy groups of $\pi_* X$. There is, of course, no hope that multiplication by p on $\pi_*(X/p)$ can model this: in good cases it is zero, but even if it is not, then p^2 is zero.

not sure if you should include this remark

Warning 1.15. The formal variable $\overline{p}$ plays a slightly different role than the formal variable $\widetilde{p}$ from .

clarify

ex from prev lecture

In this way, we think of the first page of the p-Bockstein spectral sequence as consisting of formal p-multiples. However, the group $\pi_* X$ we are trying to converge to can contain p-power torsion, so if there is any meaning to this intuition, then we should kill the appropriate formal p-multiples. This is precisely what the differentials do. In preparation for later applications, we include here a discussion of p-Bockstein maps; as we will see, these describe the differentials in the p-Bockstein spectral sequence.

Notation 1.16. Let X be a spectrum, and let $n \geqslant 1$. Write ∂_n^∞ for the boundary map

$$X \xrightarrow{p^n} X \longrightarrow X/p^n \xrightarrow{\partial_n^\infty} \Sigma X.$$

For $N \geqslant n$, write ∂_n^N for the boundary map

$$X/p^{N-n} \xrightarrow{p^n} X/p^N \longrightarrow X/p^n \xrightarrow{\partial_n^N} \Sigma X/p^{N-n}.$$

We call $\partial_n^{n+1} \colon X/p^n \to \Sigma X/p$ the **length n Bockstein** of X.

The motivation for this notation is as follows: ∂_n^N contains the information of the differentials $d_n, d_{n+1}, \ldots, d_{N-1}$ in the p-Bockstein spectral sequence. We single out ∂_n^{n+1} by a special name because in a precise sense, it *is* the length n differential: see Theorem 1.25.

These different Bocksteins are related as follows. We leave the proof as an exercise.

Proposition 1.17. *Let X be a spectrum, let $n \geqslant k \geqslant 1$, and let $\infty \geqslant N \geqslant n$. Then we have commutative diagrams*

$$\begin{array}{ccc} X/p^n & \xrightarrow{\partial_n^\infty} & \Sigma X \\ & \searrow^{\partial_n^N} & \downarrow \\ & & \Sigma X/p^{N-n} \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} X/p^n & \xrightarrow{\partial_n^\infty} & \Sigma X \\ \downarrow & & \downarrow{\scriptstyle p^{n-k}} \\ X/p^k & \xrightarrow{\partial_k^\infty} & \Sigma X, \end{array}$$

where the unlabelled maps are the reduction maps. In words: ∂_n^N is the mod p^{N-n} reduction of ∂_n^∞, and $p^{n-k} \cdot \partial_n^\infty = \partial_k^\infty$.

The following describes the structure of the p-Bockstein spectral sequence. The formulation of the following result is taken from Palmieri [Pal05].

Theorem 1.18. *Let X be a spectrum.*

(1) *There is a natural isomorphism of bigraded $\mathbf{Z}[\overline{p}]$-algebras*

$$\mathrm{E}_1^{*,*} \cong \pi_*(X/p) \otimes \mathbf{Z}[\overline{p}],$$

where $\pi_n(X/p)$ is placed in bidegree $(n, 0)$ and $\overline{p}$ has bidgree $(0, 1)$.

(2) *The differentials are $\overline{p}$-linear: if $a, b \in \mathrm{E}_r^{*,*}$ are such that*

$$d_r(a) = b,$$

then for all $t \geqslant 0$, there is a differential

$$d_r(\overline{p}^t a) = \overline{p}^t b.$$

In particular, the $\mathbf{Z}[\overline{p}]$-module structure on the first page induces a $\mathbf{Z}[\overline{p}]$-module structure on later pages.

(3) *The target of a d_r-differential is a $\overline{p}^r$-multiple: for every $a \in \mathrm{E}_r^{n,s}$, there is an element $b \in \mathrm{E}_r^{n-1,s}$ such that*

element is not unique; add warning!

$$d_r(a) = \overline{p}^r \cdot b,$$

where the multiplication denotes the $\mathbf{Z}[\overline{p}]$-module structure from (2).

(4) *The p-Bockstein spectral sequence for X converges conditionally to $\pi_* X$ if and only if the spectrum X is p-adically complete.*

Proof. Item (1) is true by definition of $\overline{p}$ in Notation 1.13.

Item (2) is a diagram chase after unwrapping the definitions. Indeed, suppose we have a differential $d_r(a) = b$. The diagram that defines this differential is the same as the diagram that defines $d_r(\overline{p} \cdot a)$. In more detail: $d_r(a)$ is obtained by dividing $\partial a \in \pi_n X$ by p^{r-1}, and projecting the result to $\mathrm{E}_r^{n-1,s+r}$. A choice of p^{r-1}-division for ∂a is also a valid choice for a p^{r-1}-division for $\partial(\overline{p}a)$, and the projection maps are also the same.

Item (3) will follow from (2), combined with the following claim: for every r, n and s, multiplication by $\overline{p}$ induces a surjection

$$\overline{p}\colon \mathrm{E}_r^{n,s} \longrightarrow\!\!\!\!\!\rightarrow \mathrm{E}_r^{n,s+1}.$$

This claim, in turn, we prove by induction on r using item (2). For $r = 1$ it is clear. Assume that for some $r \geqslant 1$, multiplication by $\overline{p}$ induces a surjection $\mathrm{E}_r^{n,s} \twoheadrightarrow \mathrm{E}_r^{n,s+1}$. Write $(\ker d_r)^{n,s}$ for the kernel of the d_r-differential out of $\mathrm{E}_r^{n,s}$, and write $(\operatorname{im} d_r)^{n,s}$ for the image of d_r into $\mathrm{E}_r^{n,s}$. Then we have

$$\mathrm{E}_{r+1}^{n,s} = \frac{(\ker d_r)^{n,s}}{(\operatorname{im} d_r)^{n,s}} \qquad \text{and} \qquad \mathrm{E}_{r+1}^{n,s+1} = \frac{(\ker d_r)^{n,s+1}}{(\operatorname{im} d_r)^{n,s+1}}.$$

The $\overline{p}$-linearity of the differentials from (2) implies that multiplication by $\overline{p}$ restricts to a surjection $(\ker d_r)^{n,s} \to (\ker d_r)^{n,s+1}$, which implies the desired statement.

Item (4) follows similarly to **??**. ∎

We will now make precise why we think of $\overline{p}$ as a formal p-multiple. Roughly speaking, if $\overline{p}^s \cdot a$ is the target of a differential, this means that a represents a class α such that $p^s \cdot \alpha = 0$. The differentials are organised nicely: a d_r-differential hits a $\overline{p}^r$-multiple. In other words, a d_r-differential keeps track of p^r-torsion.

Making this precise runs into the issues of having to choose these lifts carefully. This is the purpose of the *Omnibus Theorem*, which describes the homotopy groups of X in terms of its the p-Bockstein spectral sequence. The phrasing and the name are copied from the synthetic version of Burklund–Hahn–Senger; we discuss this more in Section 4.

Theorem 1.19 (Omnibus, part 1). *Let X be a spectrum such that multiplication by p is nullhomotopic on X/p. Assume that the p-Bockstein spectral sequence for X converges strongly.*

Let $x \in \pi_n(X/p)$. Then the following are equivalent.

(1a) *The element x, considered as a class in $\mathrm{E}_1^{n,0}$, is a permanent cycle.*

(1b) *The element x lifts to an element in $\pi_n X$.*

For any such lift α to $\pi_n X$, the following are true.

(2a) *The element α is detected by x.*

(2b) *If $\overline{p}^s \cdot x$ survives to page ∞ (equivalently, to page s), then $p^s \cdot \alpha$ is nonzero and is detected by $\overline{p}^s \cdot x$.*

Moreover, we can choose a lift α such that it satisfies the following properties.

(3a) *If $\overline{p}^s \cdot x$ does not survive to page ∞, then $p^s \cdot \alpha = 0$.*

(3b) *If $\theta \in \pi_n X$ is nonzero, p^s-divisible, and detected by $\overline{p}^s \cdot x$, then $p^s \cdot \alpha = \theta$.*

Finally, we have the following generation statement.

(4) *Let $\{ \alpha_i \}$ be a collection of elements in $\pi_n X$ such that their mod p reductions generate $\mathrm{E}_\infty^{n,0}$. Then the p-adic completion of the subgroup generated by the $\{ \alpha_i \}$ is equal to $\pi_n X$.*

Proof.

TBD

The strong convergence hypothesis says that the p-adic filtration on $\pi_n X$ is derived complete. By **??**, this means that the abelian group $\pi_n X$ is p-adically complete. Let M denote the submodule generated by the α_i. As $\pi_n X$ is p-complete, this means the module $M_p^\wedge$ is naturally a submodule of $\pi_n X$ as well. To check that the inclusion $M_p^\wedge \subseteq \pi_n X$ is an equality, it suffices to check that it is surjective after modding out by p. The map then becomes

$$\langle x_i \rangle = (M_p^\wedge)/p \longrightarrow \pi_n(X)/p = \mathrm{E}_\infty^{n,0},$$

and this map is surjective by assumption. ■

To illustrate the way this spectral sequence works, we give an example where we understand the object we are trying to compute, and compute from that what the Bockstein spectral sequence looks like.

Example 1.20. Let X be the Eilenberg–MacLane spectrum $\mathbf{Z}_p \oplus \mathbf{Z}/p$. It will be helpful to label the generators: we write $X = \alpha \cdot \mathbf{Z}_p \oplus \beta \cdot \mathbf{Z}/p$.

To compute the p-Bockstein spectral sequence for X, we first compute its mod p homotopy groups. This is the same as tensoring X with $\mathbf{S}/p$. Tensoring the cofibre sequence

$$\mathbf{Z} \xrightarrow{p} \mathbf{Z} \longrightarrow \mathbf{Z}/p$$

with $\mathbf{S}/p$ results in the split cofibre sequence

$$\mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \longrightarrow (\mathbf{Z}/p) \otimes (\mathbf{S}/p),$$

because the endomorphism p is zero on $\mathbf{Z}/p$.[2] The cofibre $(\mathbf{Z}/p) \otimes (\mathbf{S}/p)$ therefore splits as $\mathbf{Z}/p \oplus \Sigma\mathbf{Z}/p$. Note that the first generator is the reduction of β, while the generator of

[2]This is why we work with $\mathbf{Z}$ instead of $\mathbf{S}$ in this example: the spectrum $\mathbf{S}/p \otimes \mathbf{S}/p$ is more complicated.

$\Sigma\mathbf{Z}/p$ does not come from α or β. Write x for the generator of $\Sigma\mathbf{Z}/p$. In this way, we have computed the homotopy of X/p: writing a for the reduction of α, and b for the reduction of β, we have

$$\pi_n(X/p) = \begin{cases} a \cdot \mathbf{Z}/p \oplus b \cdot \mathbf{Z}/p & \text{if } n = 0, \\ x \cdot \mathbf{Z}/p & \text{if } n = 1, \\ 0 & \text{else.} \end{cases}$$

The Bockstein is only nonzero in the case

$$x \cdot \mathbf{Z}/p = \pi_1(X/p) \longrightarrow \pi_0(X/p) = a \cdot \mathbf{Z}/p \oplus b \cdot \mathbf{Z}/p,$$

in which case it is the map $(0, \mathrm{id})$. In other words, we have a d_1-differential $d_1(x) = b$.

The first page of the spectral sequence therefore looks as follows.

These are in fact all the differentials.

add discussion about choosing lifts, and call back to the omnibus

▲

1.6 The truncated Bockstein spectral sequence

This approach is also useful to understand the intermediate stages X/p^k.

Definition 1.21. Let $k \geqslant 1$. The **k-truncated p-Bockstein filtration** on X is the filtered object

$$\cdots \longrightarrow 0 \longrightarrow X/p \xrightarrow{p} X/p^2 \xrightarrow{p} \cdots \xrightarrow{p} X/p^k = \cdots,$$

indexed to be constant from filtration 0 onwards (i.e., X/p appears in filtration $k-1$). The resulting spectral sequence we call the **k-truncated p-Bockstein spectral sequence**.

Theorem 1.22. *Let X be a spectrum such that multiplication by p is nullhomotopic on X/p, and let $k \geqslant 1$.*

(1) *There is a natural isomorphism of bigraded* $\mathbf{Z}[\overline{p}]$*-algebras*

$$\mathrm{E}_1^{*,*} \cong \pi_*(X/p) \otimes \mathbf{Z}[\overline{p}]/\overline{p}^k,$$

where $\pi_n(X/p)$ *is placed in bidegree* $(n, 0)$ *and* $\overline{p}$ *has bidegree* $(0, 1)$.

(2) *There is a natural morphism of spectral sequences from the p-BSS to the k-truncated p-BSS for X which on page 1 is given by the natural map*

$$\pi_*(X/p) \otimes \mathbf{Z}[\overline{p}] \twoheadrightarrow \pi_*(X/p) \otimes \mathbf{Z}[\overline{p}]/\overline{p}^k.$$

In particular, the differentials in the truncated BSS are precisely those differentials occurring in the non-truncated BSS between bidegrees in the range $0 \leqslant s \leqslant k-1$.

(3) *The k-truncated p-Bockstein spectral sequence for X converges strongly to* $\pi_*(X/p^k)$.

Proof. The first claim is clear. To construct the map of the second claim, we construct a map of filtered spectra of the form

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{p} & X & \xrightarrow{p} & X & \xrightarrow{p} & X & \xrightarrow{p} & \cdots & \xrightarrow{p} & X & = & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X/p & \xrightarrow{p} & X/p^2 & \xrightarrow{p} & \cdots & \xrightarrow{p} & X/p^k & = & \cdots,
\end{array}$$

where the vertical morphisms are the canonical maps. We do this by letting each nontrivial square be the canonical pushout square. To see that these pushouts are of the claimed form, first observe that this is clear for the first nontrivial square. Inductively, using the two-out-of-three property for pushout squares, we see that each square is a pushout.

add thing about left Kan extension?

It is straightforward to see that this map induces an isomorphism on associated graded in degrees between 0 and $k-1$. The resulting morphism of spectral sequences is therefore an isomorphism on bidegrees (n, s) with $0 \leqslant s \leqslant k-1$. In particular, it identifies the differentials happening between classes of this bidegree. For degree reasons there are no other differentials in the truncated Bockstein spectral sequence.

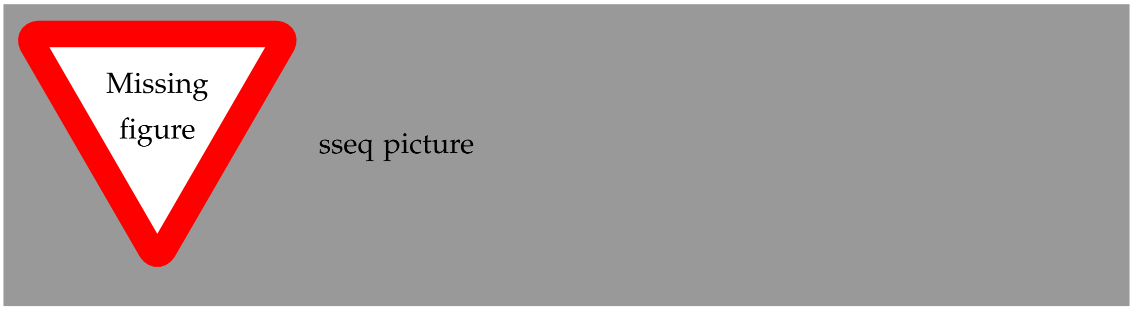
Lastly, the claim about strong convergence is clear, as the k-truncated Bockstein filtration is zero for $s \ll 0$. ■

In summary then, the k-truncated Bockstein spectral sequence is obtained by removing the classes in filtration $s \geqslant k$. This can result in more classes surviving the spectral sequence, as we will now see.

Example 1.23. Let $X = \mathbf{Z}_p \oplus \mathbf{Z}/p$; we wish to compute the 2-truncated p-Bockstein spectral sequence for X. Again we already know the abutment: we have $\mathbf{Z}/p \otimes_{\mathbf{Z}} \mathbf{Z}/p^2 = \mathbf{Z}/p \oplus \Sigma\,\mathbf{Z}/p$, and there is a length 1 Bockstein connecting these two.

explain why, and also make (or repeat) remark that all tensor products are derived

give the two generators in X names. Then argue that the new class that is created "wants to be p times β", but β itself does not exist there (as it supports a differential).



▲

Remark 1.24.

comment how all $\overline{p}^k$-multiples are zero on page 1, even though a $\overline{p}^m$-multiple need not represent a p^m-multiple in the htpy groups. for example, if a $\overline{p}$-multiple on page 1 is a permanent cycle that survives to page infty of truncated one, then its k-th power is zero, even though it need not be a p-multiple in $\pi_*(X/p^k)$ (this is also what happens in the above example).

There is also a version of the Omnibus theorem that describes the homotopy of X/p^r for $r > 1$ in terms of the r-truncated p-Bockstein spectral sequence.

Theorem 1.25 (Omnibus, part 2). *Let X be a spectrum such that multiplication by p is nullhomotopic on X.*

Let $x \in \pi_n(X/p)$. For any $r \geqslant 1$, the following are equivalent.

(1a) *Considering x as a class in $\mathrm{E}_1^{n,0}$ in the p-Bockstein spectral sequence, the differentials $d_1(x), \ldots, d_{r-1}(x)$ vanish.*

(1b) *The element x lifts to $\pi_n(X/p^r)$.*

For any such lift $\widetilde{x}$ to $\pi_n(X/p^r)$, the following is true.

(2a) *For $s \leqslant r-1$, if $\overline{p}^s \cdot x$ survives to page ∞ of the k-truncated p-Bockstein spectral sequence (equivalently, to page s), then $p^s \cdot \widetilde{x}$ is nonzero.*

(2b) *The image of $\widetilde{x}$ under $\partial_r^{r+1}\colon\ \pi_n(X/p^r) \to \pi_{n-1}(X/p)$ is a representative for $d_r(x)$.*

Moreover, we can choose a lift $\widetilde{x}$ such that it satisfies either of the following additional properties (but not necessarily both).

(3a) *For $s \leqslant r-1$, if $\overline{p}^s \cdot x$ does not survive to page ∞, then $p^s \cdot \widetilde{x} = 0$.*

(3b) *If $y \in \pi_{n-1}(X/p)$ is a representative of $d_r(x)$, then $\partial_r^{r+1}(\widetilde{x}) = y$.*

We also have the following generation statement.

(4) *Let $\{\, y_i \,\}$ be a collection of elements of $\pi_{n,*}X/\tau^r$ such that their mod τ reductions generate $\mathrm{E}_r^{n,0}$. Then the subgroup generated by the $\{\, y_i \,\}$ is equal to $\pi_{n,*}X/\tau^r$.*

2 Lifting τ to filtered spectra

Back when discussing filtered abelian groups, we introduced the letter τ as a way of translating a filtered abelian group to a graded $\mathbf{Z}[\tau]$-module. In particular, the homotopy groups of a filtered spectrum naturally form a $\mathbf{Z}[\tau]$-module.

Definition 2.1. Recall the lax symmetric monoidal functor $\pi_{*,*}\colon \mathrm{FilSp} \to \mathrm{FilgrAb}$ from **??**, given by $\pi_{n,s}X = \pi_n(X^s)$. Using the equivalence of **??**, we view this as a lax symmetric monoidal functor

$$\pi_{*,*}\colon \mathrm{FilSp} \longrightarrow \mathrm{Mod}_{\mathbf{Z}[\tau]}(\mathrm{bigrAb}).$$

Our goal now is to show that the τ on $\pi_{*,*}X$ comes from a self-map τ on X itself. This is simply a matter of rewriting, but it ends up leading a powerful, more homotopical way to interact with the spectral sequence of a filtered spectrum.

To accomplish this, we need to write down a few basic filtered spectra. While we could define these by hand, it is convenient to use the following functor. In Section 6 we will see the further importance of this functor: FilSp is the universal stable ∞-category with a diagram of this shape.

Definition 2.2. We write $i\colon \mathbf{Z} \to \mathrm{FilSp}$ for the functor given by

different name?

$$s \longmapsto \Sigma^\infty_+ \mathrm{Hom}_{\mathbf{Z}}(-, s).$$

Note that $\mathrm{Hom}_{\mathbf{Z}}(k, s)$ is a singleton set when $k \leqslant s$, and is empty when $k > s$. Using this, we can give a concrete expression for the values of the functor i:

$$i(s)^k = \begin{cases} 0 & \text{if } k > s, \\ \mathbf{S} & \text{if } k \leqslant s. \end{cases}$$

Moreover, since the connecting map $i(s)^k \to i(s)^{k-1}$ is given by Σ^∞_+ applied to the map of sets $\mathrm{Hom}_{\mathbf{Z}}(k, s) \to \mathrm{Hom}_{\mathbf{Z}}(k-1, s)$, we see that this map is the identity when $k \leqslant s$, and zero otherwise. In summary then, the filtered spectrum $i(s)$ is given by

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{S} = \mathbf{S} = \cdots,$$

where the first $\mathbf{S}$ appears in position s.

We can now find corepresenting objects for the bigraded homotopy groups.

Definition 2.3 (Filtered bigraded spheres). Let n and s be integers.

(1) The **filtered (n, s)-sphere** is

$$\mathbf{S}^{n,s} := \Sigma^n\, i(s).$$

We refer to n as the **stem**, and to s as the **filtration**.

(2) We write $\Sigma^{n,s}\colon \mathrm{FilSp} \to \mathrm{FilSp}$ for the functor given by tensoring with $\mathbf{S}^{n,s}$ on the left.

Example 2.4.

(1) In the above notation, the unit of the monoidal structure on FilSp is $\mathbf{S}^{0,0}$. We refer to it as the *filtered sphere*, and may abusively denote it simply by $\mathbf{S}$. Context will make clear whether we mean the filtered or the ordinary sphere.

(2) Categorical suspension is given by $\Sigma^{1,0}$. ▲

(3) Since FilSp is a functor category, limits and colimits therein are computed levelwise. In particular, we see that $\mathbf{S}^{n,s}$ is the filtered spectrum given by

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{S}^n =\!=\!= \mathbf{S}^n =\!=\!= \cdots,$$

where the first $\mathbf{S}^n$ appears in position s. Moreover, we see that the functor $\Sigma^{n,s}$ shifts a filtered spectrum s units to the left, and applies Σ^n levelwise.

The spectral sequence associated to $\mathbf{S}^{n,s}$ is not very helpful: its associated graded is $\mathbf{S}^n$ in filtration s, and vanishes elsewhere. The use of the bigraded spheres rather is that they corepresent the bigraded homotopy groups. While we could show this by hand, it is again convenient to use a formal argument.

Proposition 2.5. *We have a natural isomorphism*

$$\mathrm{map}_{\mathrm{FilSp}}(\mathbf{S}^{n,s}, X) \cong \Sigma^{-n} X^s.$$

In particular, we have a natural isomorphism

$$\pi_{n,s} X \cong \pi_n(X^s).$$

Proof. This follows from the Yoneda lemma. In more detail: observe that FilSp is the stabilisation of $\mathrm{PSh}(\mathbf{Z}) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{S})$, via the functor that applies Σ^∞_+ levelwise. It follows that fishy pointed-ness stuff

$$\begin{aligned}\mathrm{Map}_{\mathrm{FilSp}}(\Sigma^n i(s), X) &= \mathrm{Map}_{\mathrm{FilSp}}(\Sigma^\infty_+ \mathrm{Hom}_{\mathbf{Z}}(-,s), \Omega^n X)\\ &= \mathrm{Map}_{\mathrm{PSh}(\mathbf{Z})}(\mathrm{Hom}_{\mathbf{Z}}(-,s), \Omega^{\infty+n} X)\\ &= \Omega^{\infty+n} X^s.\end{aligned}$$

The statement on mapping spectra now follows. ■

By the Yoneda lemma, the natural transformation $\tau\colon \pi_{n,s} \to \pi_{n,s-1}$ is induced from a map $\mathbf{S}^{n,s-1} \to \mathbf{S}^{n,s}$.

Definition 2.6. The map $\tau\colon \mathbf{S}^{0,-1} \to \mathbf{S}$ is the image of the morphism $-1 \to 0$ in $\mathbf{Z}$ under the functor i. If X is a filtered spectrum, then tensoring $\tau\colon \mathbf{S}^{0,-1} \to \mathbf{S}$ with X results in a map $\Sigma^{0,-1} X \to X$; we will abuse notation and denote this by τ as well.

In a diagram, writing $\mathbf{S}^{0,-1}$ in the top row and $\mathbf{S}$ in the bottom row, the map τ looks like

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{S} & = & \cdots \\ & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbf{S} & = & \mathbf{S} & = & \cdots . \end{array}$$

If X is a filtered spectrum, then the map $\tau\colon \Sigma^{0,-1}X \to X$ looks like

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^2 & \xrightarrow{f_1} & X^1 & \xrightarrow{f_0} & X^0 & \xrightarrow{f_0} & \cdots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\ \cdots & \longrightarrow & X^1 & \xrightarrow{f_0} & X^0 & \xrightarrow{f_{-1}} & X^{-1} & \longrightarrow & \cdots . \end{array}$$

In words: the components of the map $\tau\colon \Sigma^{0,-1}X \to X$ are the transition maps in the filtered spectrum X. As a result, the map τ induces the $\mathbf{Z}[\tau]$-module structure on the bigraded homotopy groups.

Remark 2.7. Using the above notation, the functor $i\colon \mathbf{Z} \to \mathrm{FilSp}$ can be depicted as the diagram in FilSp given by

$$\cdots \xrightarrow{\tau} \mathbf{S}^{0,-1} \xrightarrow{\tau} \mathbf{S} \xrightarrow{\tau} \mathbf{S}^{0,1} \xrightarrow{\tau} \cdots .$$

3 Reinterpreting the spectral sequence

Now that we have realised the map τ on the level of filtered spectra, we should understand what structure it gives to the ∞-category of FilSp. We can invert τ, we can "mod out" by τ, and consider τ-adic completion. These notions turn out not to be new, and in fact lead to a reinterpretation of what the spectral sequence associated to a filtered spectrum is. The category theory here is very standard and closely parallels the case of p-inversion and p-completion, but we include it here for completeness. However, τ behaves better than p in one crucial aspect: modding out by τ can be done in a monoidal way, which is far from true in the case of p.

bad way of phrasing it.

3.1 Inverting τ

todo: fix math symbol in section title (also elsewhere)

Definition 3.1. A filtered spectrum X is called **τ-invertible** if the map $\tau\colon \Sigma^{0,-1}X \to X$ is an isomorphism. We write $\mathrm{FilSp}[\tau^{-1}]$ for the full subcategory of FilSp on the **τ-invertible** filtered spectra, i.e., those X for which τ acts as an isomorphism on X. Write $\tau^{-1}\colon \mathrm{FilSp} \to \mathrm{FilSp}$ for the functor sending X to the colimit

$$X[\tau^{-1}] := \operatorname{colim}\big(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \cdots \big).$$

This functors lands in τ-invertible filtered spectra, and participates in an adjunction

$$\mathrm{FilSp} \underset{\longleftarrow}{\overset{\tau^{-1}}{\longrightarrow}} \mathrm{FilSp}[\tau^{-1}].$$

Inverting τ is a particularly good kind of localisation: it is a *smashing localisation*. We refer to [GGN16, Section 3] for an introduction to such localisations. The practical upshot is that the localisation functor preserves both limits *and* colimits (or in other words, τ-invertible objects are closed under limits and colimits), and τ-local objects get an essentially unique structure of a $\mathbf{S}[\tau^{-1}]$-module structure.

Proposition 3.2. *The functor of τ-localisation is a smashing localisation, i.e., it is given by tensoring with the idempotent object $\mathbf{S}[\tau^{-1}]$. In particular, the inclusion functor $\mathrm{FilSp}[\tau^{-1}] \subseteq \mathrm{FilSp}$ preserves colimits and has a further right adjoint.*

say somewhere that this right adjoint is taking limit over mult by τ

Proof. The tensor product of filtered spectra preserves colimits. It follows that

$$\begin{aligned} X[\tau^{-1}] &= \operatorname{colim}\left(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \cdots \right) \\ &= \operatorname{colim}\left(\mathbf{S} \xrightarrow{\tau} \mathbf{S}^{0,1} \xrightarrow{\tau} \cdots \right) \otimes X \\ &= \mathbf{S}[\tau^{-1}] \otimes X. \end{aligned}$$

∎

The notion of a τ-invertible filtered spectrum is not new: it is the same as a filtered spectrum whose transition maps are invertible. More precisely, we have the following identifications.

Proposition 3.3. *The symmetric monoidal functor $\mathrm{Const}\colon \mathrm{Sp} \to \mathrm{FilSp}$ restricts to an equivalence onto the τ-invertible filtered spectra:*

$$\mathrm{Const}\colon \mathrm{Sp} \xrightarrow{\simeq} \mathrm{FilSp}[\tau^{-1}].$$

Under this equivalence, the adjunctions

arrows are ugly

$$\begin{array}{ccc} & \xrightarrow{\tau^{-1}} & \\ \mathrm{FilSp} & \hookleftarrow & \mathrm{FilSp}[\tau^{-1}] \\ & \longrightarrow & \end{array}$$

can be identified with

$$\begin{array}{ccc} & \xrightarrow{X \mapsto X^{-\infty}} & \\ \mathrm{FilSp} & \xleftarrow{\mathrm{Const}} & \mathrm{Sp}. \\ & \xrightarrow[X \mapsto X^{\infty}]{} & \end{array}$$

Proof TBD

Paralleling **??**, we will sometimes use the following notation to distinguish between the two equivalent ∞-categories Sp and $\mathrm{FilSp}[\tau^{-1}]$.

Notation 3.4. We write $(-)^{\tau=1}$ for the composite

$$\mathrm{FilSp} \xrightarrow{\tau^{-1}} \mathrm{FilSp}[\tau^{-1}] \simeq \mathrm{Sp}.$$

3.2 Modding out by τ

Notation 3.5. For $k \geqslant 1$, we write $C\tau^k$ for the cofibre of the map $\tau^k\colon \mathbf{S}^{0,-k} \to \mathbf{S}$.

Concretely, $C\tau^k$ is the filtered spectrum

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{S} = \cdots = \mathbf{S} \longrightarrow 0 \longrightarrow \cdots,$$

where the nonzero terms are in filtration $0, \ldots, k-1$.

Theorem 3.6 ([Lur15], Proposition 3.2.5). *For every $k \geqslant 0$, the filtered spectrum $C\tau^k$ admits (uniquely up to contractible choice) the structure of a filtered $\mathbf{E}_\infty$-ring such that its unit map $\mathbf{S} \to C\tau^k$ is an isomorphism in degree zero.*

is this also correct for $k > 1$?

Proof sketch.

to write; copy Lurie

■

We can now speak of modules over $C\tau$ in FilSp; this notion turns out to be equivalent to the notion of a *graded* spectrum. To prove this, we first define a functor

$$\mathrm{grSp} \longrightarrow \mathrm{FilSp}$$

by left Kan extension along the functor $\mathbf{Z}^{\mathrm{discr}} \to \mathbf{Z}^{\mathrm{op}}$, with $\mathbf{Z}^{\mathrm{discr}}$ denoting $\mathbf{Z}$ as a discrete category. Informally, this functor is given by sending a graded spectrum $(X_n)_n$ to the filtered spectrum

$$\cdots \longrightarrow \bigoplus_{n\geqslant 1} X_n \longrightarrow \bigoplus_{n\geqslant 0} X_n \longrightarrow \bigoplus_{n\geqslant -1} X_n \longrightarrow \cdots,$$

with maps the natural inclusions. Being defined as a left Kan extension, this is naturally a symmetric monoidal functor.

Theorem 3.7 ([Lur15], Proposition 3.2.7). *The composite*

$$\mathrm{grSp} \longrightarrow \mathrm{FilSp} \xrightarrow{C\tau\otimes -} \mathrm{Mod}_{C\tau}(\mathrm{FilSp})$$

is a symmetric monoidal equivalence. Moreover, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{FilSp} & \xrightarrow{\mathrm{Gr}} & \mathrm{grSp} \\ & {\scriptstyle C\tau\otimes -}\searrow & \downarrow \simeq \\ & & \mathrm{Mod}_{C\tau}(\mathrm{FilSp}). \end{array}$$

Proof sketch?

Remark 3.8. The above in particular puts a symmetric monoidal structure on the associated graded functor $\mathrm{Gr}\colon \mathrm{FilSp} \to \mathrm{grSp}$, because the functor $C\tau \otimes -\colon \mathrm{FilSp} \to \mathrm{Mod}_{C\tau}(\mathrm{FilSp})$ is canonically symmetric monoidal. One could have also done this more directly: see [Hed20, Section II.1.3].

Warning 3.9. Being a module over $C\tau$ is not a property, but additional structure. One can see this by observing that $C\tau$ is not an idempotent, i.e., that $C\tau \otimes C\tau$ is not equivalent to $C\tau$: we have

$$C\tau \otimes C\tau \cong C\tau \oplus \Sigma^{1,-1} C\tau.$$

Indeed, the map τ on $C\tau$ is nullhomotopic (because $C\tau$ is a ring), so the cofibre sequence defining $C\tau$ splits after tensoring with $C\tau$. We could have seen this coming in a different way too: the associated graded of $C\tau$ is concentrated in filtrations 0 and -1, where it is $\mathbf{S}$ and $\mathbf{S}^1$, respectively. This is exactly the bidgrees in the above splitting.

wanna say what the structure consists of, roughly?

Notation 3.10. If X is a filtered spectrum and $k \geqslant 1$, then we write X/τ^k for $C\tau^k \otimes X$.

3.3 Completing at τ

Definition 3.11. The functor of **τ-adic completion** (or *τ-completion* for short) is $C\tau$-localisation of FilSp, i.e., inverting those maps that become an isomorphism after tensoring with $C\tau$.

We write $\mathrm{FilSp}^{\wedge}_{\tau}$ for the full subcategory of FilSp on the τ-adically complete filtered spectra. This results in an adjunction

$$\mathrm{FilSp} \underset{}{\overset{(-)^{\wedge}_{\tau}}{\rightleftarrows}} \mathrm{FilSp}^{\wedge}_{\tau}.$$

This notion, in fact, coincides with the previously introduced notion of completeness from **??**.

Proposition 3.12.

(1) *A filtered spectrum X is τ-adically complete if and only if its limit X^{∞} vanishes.*

(2) *The τ-adic completion functor is given by $X \mapsto \mathrm{cofib}(X^{\infty} \to X)$.*

(3) *A map of filtered spectra $f\colon X \to Y$ induces an isomorphism $X^{\wedge}_{\tau} \cong Y^{\wedge}_{\tau}$ if and only if f induces an isomorphism on associated graded.*

Proof. These are immediate from Proposition 3.3 and Theorem 3.7. ■

The operations of inverting and completing at τ are related in the following way.

Proposition 3.13. *For a filtered spectrum X, there is a natural pullback square*

$$\begin{array}{ccc} X & \longrightarrow & X^{\wedge}_{\tau} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X[\tau^{-1}] & \longrightarrow & (X^{\wedge}_{\tau})[\tau^{-1}]. \end{array}$$

In particular, a map of filtered spectra $X \to Y$ is an isomorphism if and only if the maps

$$X[\tau^{-1}] \longrightarrow Y[\tau^{-1}] \qquad \textit{and} \qquad C\tau \otimes X \longrightarrow C\tau \otimes Y$$

are both an isomorphism.

Proof. This is a standard result which follows from the fact that $(X[\tau^{-1}])^{\wedge}_{\tau} = 0$ for all X; see. ∎

cite somebody. TMF book? Lawson?

Concretely, this says that a map of filtered spectra is an isomorphism if and only if it is an isomorphism on the limit and on the associated graded. As such, Proposition 3.13 can be thought of as the filtered analogue of **??**.

3.4 Total differentials

So far, we have seen how to recover the notions of the colimit, associated graded, and the (conditional) convergence criterion in terms of τ. The key missing piece is the differentials. These can also be lifted to more coherent objects, namely the *total differentials* of a filtered spectrum. Working with these instead of the ordinary differentials is one of the big benefits of working with filtered spectra, and leads to strengthened versions of the usual Leibniz rule.

Notation 3.14. Let X be a filtered spectrum, and let $n \geqslant 1$. Write ∂_n^∞ for the boundary map in the cofibre sequence

$$\Sigma^{0,-n} X \xrightarrow{\tau^n} X \longrightarrow X/\tau^n \xrightarrow{\partial_n^\infty} \Sigma^{1,-n} X.$$

For $N \geqslant n$, write ∂_n^N for the boundary map

$$\Sigma^{0,-n} X/\tau^{N-n} \xrightarrow{\tau^n} X/\tau^N \longrightarrow X/\tau^n \xrightarrow{\partial_n^N} \Sigma^{1,-n} X/\tau^{N-n}.$$

We call ∂_1^∞ the **total differential** on X, and call ∂_1^N the ***N*-truncated total differential** on X.

The map δ_n^N captures information about the $d_n, \ldots, d_{N-1}$-differentials in X. Decreasing N results in a loss of information. Meanwhile, increasing the lower index n should be thought of as a lift of δ_1^N rather than a loss of information: we will see that, roughly speaking, δ_n^N is only defined on elements on which the differentials $d_1, \ldots, d_{n-1}$ vanish.

We will now make these ideas precise. We begin with the relationship between the various total differentials.

Proposition 3.15. *Let X be a filtered spectrum, let $n \geqslant k \geqslant 1$, and let $\infty \geqslant N \geqslant n$. Then we have commutative diagrams*

$$\begin{array}{ccc} X/\tau^n & \xrightarrow{\partial_n^\infty} & \Sigma^{1,-n} X \\ & \searrow^{\partial_n^N} & \downarrow \\ & & \Sigma^{1,-n} X/\tau^{N-n} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X/\tau^n & \xrightarrow{\partial_n^\infty} & \Sigma^{1,-n} X \\ \downarrow & & \downarrow{\scriptstyle \tau^{n-k}} \\ X/\tau^k & \xrightarrow{\partial_k^\infty} & \Sigma^{1,-k} X, \end{array}$$

where the unlabelled maps are the reduction maps. In words: ∂_n^N is the mod τ^{N-n} reduction of ∂_n^∞, and $\tau^{n-k} \cdot \partial_n^\infty = \partial_k^\infty$.

Proof. For readability, we omit the bigraded suspensions in this proof. We start with the commutative diagram

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \tau^{N-n}}\downarrow & & \downarrow{\scriptstyle \tau^N} \\ X & \xrightarrow{\tau^n} & X. \end{array}$$

Taking pushouts in the vertical direction once, and repeatedly in the horizontal direction, we arrive at a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\tau^n} & X & \longrightarrow & X/\tau^n & \xrightarrow{\partial_n^\infty} & X \\ \downarrow & & \downarrow & & \| & & \downarrow \\ X/\tau^{N-n} & \xrightarrow{\tau^n} & X/\tau^N & \longrightarrow & X/\tau^n & \xrightarrow{\partial_n^N} & X/\tau^{N-n}. \end{array}$$

The right-most square is the first claimed diagram.

The second diagram comes from the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\tau^n} & X & \longrightarrow & X/\tau^n & \xrightarrow{\delta_n^\infty} & X \\ {\scriptstyle \tau^{n-k}}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle \tau^{n-k}} \\ X & \xrightarrow{\tau^k} & X & \longrightarrow & X/\tau^k & \xrightarrow{\delta_k^\infty} & X \end{array}$$

obtained by taking horizontal pushouts of the left-most square. ■

Proposition 3.16. *Let X be a filtered spectrum, and let $x \in \mathrm{E}_1^{n,s}$. View x as an element of $\pi_{n,s} X/\tau$.*

(1) *If $\delta_1^\infty(x)$ is τ^r-divisible, then $d_1(x) = \cdots = d_r(x) = 0$.*

(2) *For any $\alpha \in \pi_{n-1,s+r+1} X$ such that $\tau^r \cdot \alpha = \delta_1^\infty(x)$, the image of α under the mod τ reduction $\pi_{n-1,s+r+1} X \to \pi_{n-1,s+r+1} X/\tau = \mathrm{E}_1^{n-1,s+r+1}$ is a representative for $d_{r+1}(x)$.*

Proof. This is a rephrasing of the definition of the differentials in the spectral sequence associated to X. Indeed, evaluating the cofibre sequence of filtered spectra

$$\Sigma^{0,-r} X \xrightarrow{\tau^r} X \longrightarrow X/\tau^r \xrightarrow{\partial_r^\infty} \Sigma^{1,-r} X.$$

at filtration s is exactly the cofibre sequence of spectra

$$X^{s+r} \longrightarrow X^s \longrightarrow \mathrm{Gr}^s X \longrightarrow \Sigma X^{s+r}.$$

■

Warning 3.17. The converse of item (1) is not true in general. The reason is that the r-th differential is only well defined up to the images of shorter differentials. As a result, one cannot in general use $d_r(x) = 0$ to deduce that $\delta_1^\infty(x)$ is τ^r-divisible if $r > 1$. This can be done, of course, if previous differentials vanish in the appropriate range: more specifically, if $(n-1, s+r)$ receives no differentials of length shorter than r.

Example of differential stretching?

4 The Omnibus Theorem

We can summarise the previous section as follows: if X is a filtered spectrum, then its associated spectral sequence is of signature

$$\mathrm{E}_1^{n,s} = \pi_{n,s}(C\tau \otimes X) \implies \pi_n(X^{\tau=1}).$$

In the earlier informal discussion on spectral sequences, we argued that this spectral sequence should lead to a description of the bigraded homotopy groups of X as a $\mathbf{Z}[\tau]$-module. We will now make this precise. This runs into the issue of having to choose lifts of elements in the spectral sequence, so the precise statement requires a very careful formulation. ref

We refer to this result as the *Omnibus Theorem*. It was proved by Burklund–Hahn–Senger [BHS23, Theorem 9.19, Appendix A] in the context of synthetic spectra (specifically for synthetic analogues). Their proof generalises in a straightforward fashion to the filtered setting by using the τ-Bockstein spectral sequence as the proving device. For the sake of completeness, we include a proof of their theorem in this more general setting, but it does not feature any new ideas compared to the synthetic one. We will see next time how to recover the synthetic version from this filtered one. forward ref

Theorem 4.1 (Omnibus, Burklund–Hahn–Senger). *Let X be a filtered spectrum. Let $x \in \mathrm{E}_1^{n,s} = \pi_{n,s} X/\tau$ be a class. For every $r \geqslant 1$, the following are equivalent.*

(1a) *The differentials $d_1(x), \ldots, d_{r-1}(x)$ vanish.*

(1b) *The element $x \in \pi_{n,s} X/\tau$ lifts to an element of $\pi_{n,s} X/\tau^r$.*

For any such lift $\widetilde{x}$ *to* $\pi_{n,s}\, X/\tau^r$*, the following is true.*

(2a) *If* x *survives to page* k *for* $k < r$*, then* $\tau^{k-1} \cdot \widetilde{x}$ *is nonzero.*

(2b) *The image of* $\widetilde{x}$ *under* $\partial_r^{r+1} \colon\ \pi_{n,s}\, X/\tau^r \to \pi_{n-1,s+r}\, X/\tau$ *is a representative for* $d_r(x)$.

Moreover, we can choose a lift $\widetilde{x}$ *such that it satisfies either of the following additional properties (but not necessarily both).*

(3a) *If* x *is the target of a* d_k*-differential for* $k < r$*, then* $\tau^k \cdot \widetilde{x} = 0$.

(3b) *If* $y \in \pi_{n-1,s+r}\, X/\tau$ *is a representative of* $d_r(x)$*, then* $\partial_r^{r+1}(\widetilde{x}) = y$.

We also have the following generation statement.

(4) *Let* $\{ y_i \}$ *be a collection of elements of* $\pi_{n,*}X/\tau^r$ *such that their mod* τ *reductions generate the* $(r-1)$*-cycles in stem* n*. Then the* $\mathbf{Z}[\tau]$*-submodule generated by the* $\{ y_i \}$ *is equal to* $\pi_{n,*}X/\tau^r$.

is my numbering correct?

Assume from now on that the spectral sequence underlying X *converges strongly to* $\pi_* X^{-\infty}$*. Then the following are equivalent.*

(5a) *The element* x *is a permanent cycle.*

(5b) *The element* $x \in \pi_{n,s}\, X/\tau$ *lifts to an element of* $\pi_{n,s}\, X$.

For any such lift α *to* $\pi_{n,s}\, X$*, the following are true.*

(6a) *If* x *survives to page* k*, then* $\tau^{k-1} \cdot \alpha$ *is nonzero.*

(6b) *If* x *survives to page* ∞*, then* α *maps to a nonzero element in* $\pi_n X^{-\infty}$*, and this element is detected by* x.

Moreover, we can choose a lift α *such that it satisfies the following properties.*

(7a) *If* x *is the target of a* d_r*-differential, then* $\tau^r \cdot \alpha = 0$.

(7b) *If* $\theta \in \pi_n X^{-\infty}$ *is nonzero and is detected by* x *(in particular,* x *is a permanent cycle), then the image of* α *under* $\pi_{n,s}\, X \to \pi_{n,s}\, X[\tau^{-1}] = \pi_n X^{-\infty}$ *is* θ.

Finally, we have the following generation statement.

(8) *Let* $\{ \alpha_i \}$ *be a collection of elements of* $\pi_{n,*}X$ *such that their mod* τ *reductions generate the permanent cycles in stem* n*. Then the* τ*-adic completion of the* $\mathbf{Z}[\tau]$*-submodule of* $\pi_{n,*}X$ *generated by the* $\{ \alpha_i \}$ *is equal to* $\pi_{n,*}X$.

Proof TBD.

The big instrument in proving this is the τ-*Bockstein spectral sequence*.

5 The τ-Bockstein spectral sequence

To prove the Omnibus Theorem, we will use the *τ-Bockstein spectral sequence* of a filtered spectrum. It turns out that this captures exactly the spectral sequence associated to the filtered spectrum. The setup and the proofs are the exact same as in the case of the p-Bockstein spectral sequence of Sections 1.5 and 1.6. Because the indexing conventions are more confusing here, we describe how it is set up.

Definition 5.1. Let X be a filtered spectrum. The **τ-Bockstein filtration** of X, denoted $\mathrm{BF}_\tau X$, is the bifiltered spectrum $\mathbf{Z}^{\mathrm{op}} \to \mathrm{FilSp}$ given by

$$\cdots \xrightarrow{\tau} \Sigma^{0,-2} X \xrightarrow{\tau} \Sigma^{0,-1} X \xrightarrow{\tau} X = \!= \cdots.$$

This leads to a spectral sequence in the same way as before. To help us index it, we use the philosophy from **??**. Roughly speaking, the filtration variable s records the location in the diagram as depicted above. The homotopy groups in FilSp are naturally bigraded, so we would like to understand $\pi_{n,w}$ of the colimit. Accordingly, we apply $\pi_{n,w}$ to the above diagram, and study the resulting behaviour using the long exact sequences involving the associated graded.

Formally, we obtain a trigraded exact couple

$$\mathrm{A}^{n,w,s} = \pi_{n,w}(\mathrm{BF}^s_\tau X) \qquad \text{and} \qquad \mathrm{E}^{n,w,s} = \pi_{n,w}(\mathrm{Gr}^s \mathrm{BF}_\tau X).$$

In this indexing, the differential d_r will have tridegree $(-1, 0, r)$: indeed, suspension on FilSp is $\Sigma^{1,0}$, and every differential arises by applying a boundary map exactly once. The resulting spectral sequence is of signature

$$\mathrm{E}_1^{n,w,s} \implies \pi_{n,w} X.$$

The induced filtration is the τ-adic filtration on $\pi_{n,w} X$; the resulting classical filtration is

$$F^s \ \pi_{n,w} X = \mathrm{im}(\tau^s \colon \ \pi_{n,\,w+s} X \to \pi_{n,w} X).$$

For a general bifiltered spectrum, we would not be able to simplify this further. In the specific case of the τ-Bockstein filtration, the expressions for the exact couple simplifies: we have

$$\pi_{n,w}(\mathrm{BF}^s_\tau X) = \pi_{n,w}(\Sigma^{0,-s} X) = \pi_{n,\,w+s} X.$$

We also have

$$\mathrm{Gr}^s \mathrm{BF}_\tau X = \Sigma^{0,-s} X/\tau,$$

so that the first page is of the form

$$\mathrm{E}_1^{n,w,s} = \begin{cases} \pi_{n,\,w+s}(X/\tau) & \text{if } s \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

Like before, it will be useful to think of these as formal τ-multiples. However, since τ on a filtered spectrum has nonzero (n,w)-bidegree $(0,-1)$, this formal variable has a slightly different grading too.

Notation 5.2. We define a formal variable $\overline{\tau}$ to have tridegree $(0,-1,1)$. We define a $\mathbf{Z}[\overline{\tau}]$-module structure on $\mathrm{E}_1^{*,*,*}$ in such a way that we have an isomorphism of trigraded $\mathbf{Z}[\overline{\tau}]$-modules

$$\mathrm{E}_1^{*,*,*} \cong \pi_{*,*}(X/\tau) \otimes \mathbf{Z}[\overline{\tau}]$$

where $\pi_{n,w} X$ is placed in tridegree $(n,w,0)$.

Remark 5.3. Even though it is a trigraded spectral sequence, one can depict it as follows. Since the d_r-differential has tridegree $(-1,0,s)$, by fixing a constant value for w, we get a bigraded spectral sequence trying to converge to $\pi_{*,w} X$. An element in filtration s will be a formal τ^s-multiple, the only catch being that the class it is a formal multiple of lives in a spectral sequence for a different w-value (namely $w+s$).

The behaviour of the τ-Bockstein spectral sequence is analogous to the p-Bockstein spectral sequence from before: differentials are $\overline{\tau}$-linear and hit $\overline{\tau}$-multiples. A feature of the τ-Bockstein spectral sequence for X which does not have a counterpart in the version based on p, is that the differentials capture exactly the differentials in the spectral sequence underlying X.

something explaining how the τ-power torsion in the htpy of X arises from the tau BSS.

Making this claim about the differentials precise takes a little bit of care, because the differentials in the τ-BSS only hit $\overline{\tau}$-multiples. A consequence of this is that elements in filtration 0 can never be hit by any differentials. Although this makes the following result slightly harder to phrase, one should regard this as a big feature, and *not* as a bug. Indeed, it is the very reason that differentials lead to τ-power torsion in $\pi_{*,*}X$, instead of killing the elements directly.

Theorem 5.4. *Let X be a filtered spectrum. Let $\{\mathrm{E}_r^{*,*}\}$ denote the underlying spectral sequence of X, and let $\{\mathrm{E}_r^{*,*,*}\}$ denote the τ-Bockstein spectral sequence of X.*

(1) *There is a natural isomorphism of trigraded $\mathbf{Z}[\overline{\tau}]$-algebras*

$$\mathrm{E}_1^{*,*,*} \cong \pi_{*,*}(X/\tau) \otimes \mathbf{Z}[\overline{\tau}] = \mathrm{E}_1^{*,*} \otimes \mathbf{Z}[\overline{\tau}],$$

where $\pi_{n,w}(X/\tau)$ is placed in tridegree $(n,w,0)$ and $\overline{\tau}$ has tridgree $(0,-1,1)$.

(2) *The differentials are $\overline{\tau}$-linear: if $a, b \in \mathrm{E}_r^{*,*,*}$ are such that*

$$d_r(a) = b,$$

then for all $t \geqslant 0$, there is a differential

$$d_r(\overline{\tau}^t a) = \overline{\tau}^t b.$$

In particular, the $\mathbf{Z}[\overline{\tau}]$*-module structure on the first page induces a* $\mathbf{Z}[\overline{\tau}]$*-module structure on later pages.*

(3) *The target of a* d_r*-differential is a* $\overline{\tau}^r$*-multiple: for every* $a \in \mathrm{E}_r^{n,w,s}$*, there is an element* $b \in \mathrm{E}_r^{n-1,w+r,s}$ *such that*

element is not unique; add warning!

$$d_r(a) = \overline{\tau}^r \cdot b,$$

where the multiplication denotes the $\mathbf{Z}[\overline{\tau}]$*-module structure from* (2).

(4) *For all* r*, there is a surjective map*

$$\mathrm{E}_r^{n,w,s} \twoheadrightarrow \mathrm{E}_r^{n,s} \tag{5.5}$$

which is an isomorphism if $s \geqslant r-1$*. If* $x \in \mathrm{E}_r^{n,w,s}$ *and* $y \in \mathrm{E}_r^{n-1,w+r,s}$*, then there is a differential*

$$d_r(x) = \overline{\tau}^r \cdot y$$

if and only if there is a differential

$$d_r([x]) = [y],$$

in the spectral sequence underlying X*, where square brackets indicate the image under the map* (5.5)*. (Note that on the tridegree of* $\overline{\tau}^r y$*, the map* (5.5) *is an isomorphism.)*

(5) *The* τ*-Bockstein spectral sequence for* X *converges conditionally to* $\pi_{*,*} X$ *if and only if the filtered spectrum* X *is* τ*-adically complete, or in other words, if the limit* X^∞ *vanishes.*

Proof. Item (1) is true by definition of $\overline{\tau}$ in Notation 5.2.

write the rest. see comments

■

6 Deformations

So far, we viewed spectral sequences as naturally associated to filtered spectra, and have reformulated various aspects of spectral sequences in terms of the homotopy theory of the ∞-category FilSp. This choice of working in FilSp was not an arbitrary one: it is, in fact, the universal ∞-category with an endomorphism τ. Proving this will allow us to export all of the homotopy theory of spectral sequences to other categories; later, synthetic spectra will be the main example of this.

Recall from Remark 2.7 that the functor $i\colon \mathbf{Z} \to \mathrm{FilSp}$ of Definition 2.2 is given by

$$\cdots \xrightarrow{\tau} \mathbf{S}^{0,-1} \xrightarrow{\tau} \mathbf{S} \xrightarrow{\tau} \mathbf{S}^{0,1} \xrightarrow{\tau} \cdots.$$

Informally, the purpose of this diagram is to encode the endomorphism τ. Since τ is not a true endomorphism, but only after the "twist" $\Sigma^{0,-1}$, it must do this by recording all iterates of this twist as well. The formal way in which FilSp is the universal ∞-category with an endomorphism τ is to say that it is the universal (stable, cocomplete) ∞-category with a diagram of this form.

In the following, we view $\mathbf{Z}$ as a symmetric monoidal ∞-category under addition.

Theorem 6.1 (Universal property of filtered spectra).

(1) *Let $\mathcal{C}$ be a presentable stable ∞-category. Then left Kan extension along i induces an equivalence*

$$\mathrm{Fun}(\mathbf{Z}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathrm{FilSp}, \mathcal{C}),$$

where $\mathrm{Fun}^{\mathrm{L}}$ *denotes the colimit-preserving functors.*

(2) *Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Then left Kan extension along i induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Z}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L},\otimes}(\mathrm{FilSp}, \mathcal{C}),$$

where $\mathrm{Fun}^{\otimes}$ *denotes symmetric monoidal functors, and* $\mathrm{Fun}^{\mathrm{L},\otimes}$ *denotes the full subcategory thereof on the colimit-preserving functors.*

Proof. Notice that FilSp is the stabilisation of $\mathrm{PSh}(\mathbf{Z}) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{S})$, or in other words, we have $\mathrm{Sp} \otimes \mathrm{PSh}(\mathbf{Z}) \simeq \mathrm{FilSp}$. As $\mathcal{C}$ is stable, any colimit-preserving functor $\mathrm{PSh}(\mathbf{Z}) \to \mathcal{C}$ canonically factors over $\mathrm{Sp} \otimes \mathrm{PSh}(\mathbf{Z}) \to \mathcal{C}$. The first claim now follows from the universal property of presheaf-∞-categories [HTT, Theorem 5.1.5.6]. The second claim is analogous, using the universal property of Day convolution on $\mathrm{PSh}(\mathbf{Z})$; see [HA, Section 2.2.6]. ■

Our earlier motivation suggests a formula for the functor $\mathrm{FilSp} \to \mathcal{C}$ induced by $\mathbf{Z} \to \mathcal{C}$; we now verify that this is correct.

Proposition 6.2. *Let $\mathcal{C}$ be a stable presentable ∞-category. If $f\colon \mathbf{Z} \to \mathcal{C}$ is a functor, then the functor $\mathrm{FilSp} \to \mathcal{C}$ induced by the above universal property satisfies*

$$\mathbf{S}^{n,s} \longmapsto \Sigma^n f(s),$$

and sends $\tau\colon \mathbf{S}^{0,-1} \to \mathbf{S}$ to the map $f(-1 \to 0)$.

Proof. Write F for the induced functor $\mathrm{FilSp} \to \mathcal{C}$. Then F is characterised by preserving small colimits and satisfying

$$F \circ i = f.$$

By definition we have $i(s) = \mathbf{S}^{0,s}$. Since F preserves colimits, it is an exact functor of stable ∞-categories, so it preserves arbitrary suspensions and desuspensions. The first claim now follows from the definition $\mathbf{S}^{n,s} = \Sigma^n \mathbf{S}^{0,s}$. The second claim follows from the definition of τ as $i(-1 \to 0)$. ■

In the presentable setting, a colimit-preserving functor is automatically a left adjoint. The right adjoint to the functor FilSp $\to \mathcal{C}$ will be useful as well; we will later think of such a functor as the "underlying spectral sequence functor". (We will work this out in detail in the case of synthetic spectra; see.) ref

Remark 6.3. Let $f\colon \mathbf{Z} \to \mathcal{C}$ be a functor as in the previous remark, let $F\colon \mathrm{FilSp} \to \mathcal{C}$ denote the induced functor, and let $G\colon \mathcal{C} \to \mathrm{FilSp}$ denote the right adjoint to F. We have a concrete expression for G as well. Write $\mathrm{map}_{\mathcal{C}}(-,-)$ for the mapping spectrum functor of the stable ∞-category $\mathcal{C}$. Then for $X \in \mathcal{C}$, the filtered spectrum GX is given by

$$\cdots \longrightarrow \mathrm{map}_{\mathcal{C}}(f(1), X) \longrightarrow \mathrm{map}_{\mathcal{C}}(f(0), X) \longrightarrow \mathrm{map}_{\mathcal{C}}(f(-1), X) \longrightarrow \cdots,$$

where the connecting maps are induced by f.

Being a stable presentably symmetric monoidal ∞-category, we can view FilSp as an $\mathbf{E}_\infty$-algebra in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. We copy the following formulation from Barkan [Bar23, Definition 2.2].

Definition 6.4. A **(1-parameter) deformation** is a left module over FilSp in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

In later lectures, we will have more structure than a left module structure over FilSp: the following kind of example of a deformation suffices for the rest of this lecture.

Example 6.5. If $\mathcal{C}$ is presentably symmetric monoidal and stable, then a symmetric monoidal left adjoint FilSp $\to \mathcal{C}$ in particular determines a FilSp-module structure on $\mathcal{C}$, turning $\mathcal{C}$ into a deformation. In particular, we learn that a symmetric monoidal functor $\mathbf{Z} \to \mathcal{C}$ also determines the structure of a deformation on $\mathcal{C}$. ▲

If $\mathcal{C}$ is a deformation, then we think of $\mathcal{C}$ as categorifying a particular kind of spectral sequence for spectra. Using the structure of a deformation, we can use the abstract tools of the bigraded spheres and the $\mathbf{E}_\infty$-algebra $C\tau$ in $\mathcal{C}$ as well. In this sense, we can work with $\mathcal{C}$ as if it consists of a certain type of spectral sequences.

Meanwhile, we think of the universal case, FilSp, as categorifying *all* spectral sequences.[3] This makes FilSp somewhat unwieldy and difficult to work with concretely. For instance, the ∞-category $\mathrm{Mod}_{C\tau}(\mathrm{FilSp})$ is equivalent to grSp, which is still topological in nature. If we work with a more specialised deformation, then $C\tau$-modules might become simpler. Since $C\tau$-modules form the starting page of the spectral sequence, making $C\tau$-modules simpler should make the entire category simpler as well.

The rest of these lectures will be devoted to essentially one example of this phenomenon. Namely, the ∞-category Syn_E of *E-based synthetic spectra* categorifies the *E-based Adams spectral sequence*. It comes with a symmetric monoidal left adjoint FilSp $\to \mathrm{Syn}_E$, realising it as a deformation. Meanwhile, it is simpler in that $\mathrm{Mod}_{C\tau}(\mathrm{Syn}_E)$ is no longer topological, but *algebraic* in nature.

[3]At least, if we are agnostic about spectral sequences not arising from filtered spectra.

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