Lecture 3: basics of synthetic spectra

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We will now begin our study of synthetic spectra. We do not focus on any constructions for the moment, but defer that to a later lecture. After listing some of the basic categorical properties, we import all of the previous results about τ and spectral sequences into the synthetic setting. This in particular lets us think of synthetic spectra as having an underlying spectral sequence. The main goal of this lecture is to show that for many synthetic spectra (the synthetic analogues of spectra), this agrees with the Adams spectral sequence as it is usually defined. However, this is merely to demonstrate how to work with synthetic spectra: as we will see later, there are many benefits to working with synthetic spectra rather than with just their underlying spectral sequences.

We do not assume any previous knowledge on what the Adams spectral sequence is, and so include a brief introductory section on this. To those who have worked with the Adams spectral sequence before, this serves as a reminder of some of the basic properties. Those who do not yet have experience with it should read this section to get a feeling for what to expect from the Adams spectral sequence: everything, and more, we will derive from synthetic spectra.

1 Motivation: the Adams spectral sequence

Let E be a fixed spectrum. We can use E-homology to study maps between spectra. If the map is nonzero on E-homology, then it was nonzero to begin with. Surprisingly, the story does not end if the map is zero on E-homology.

Definition 1.1. Let X and Y be spectra. The ***E*-based Adams filtration** on $[Y, X]$ is the classical filtration where, for $s \geqslant 1$, a map $f\colon Y \to X$ is in $F^s[Y, X]$ if it can be written as a composite of s maps, each of which is zero on E_*-homology. We define $F^0[Y, X] := [Y, X]$, leading to a classical filtration

$$\cdots \subseteq F^2[Y, X] \subseteq F^1[Y, X] \subseteq F^0[Y, X] = [Y, X].$$

This definition naturally extends to a filtration on the graded abelian group $[Y, X]_* = [\Sigma^* Y,\, X]$.

some remark about E-homology vs E-null

Usually we apply this when Y is the sphere, leading to a filtration on the homotopy groups of X.

If $f\colon Y \to X$ is zero on E-homology, then it may appear that there is not much we can do with it. We can however perform the following trick: completing f into a cofibre sequence

$$Y \longrightarrow X \longrightarrow Z \longrightarrow \Sigma Y,$$

we see that on E_*-homology, the long exact sequence splits up into a short exact sequence

$$0 \longrightarrow E_*(X) \longrightarrow E_*(Z) \longrightarrow E_{*-1}(Y) \longrightarrow 0. \tag{1.2}$$

In other words, the fact that f is zero on E_*-homology realises $E_*(Z)$ as an extension of $E_{*-1}(Y)$ by $E_*(X)$. This extension could be nontrivial.

In an example however, we see that we have to be careful what type of extension we mean. If E is a homotopy ring spectrum, then one natural guess is to talk about an extension of E_*-modules, but this can be very uninteresting.

Example 1.3. If $E = \mathbf{F}_p$, then every extension of E_*-modules splits. We can drastically improve the situation by observing that the $\mathbf{F}_p$-homology of a spectrum has a natural (co)action by the *dual Steenrod algebra*, i.e., by $\mathcal{A}_* := \pi_*(\mathbf{F}_p \otimes \mathbf{F}_p)$. At least if X and Y are finite spectra, then this is dual to the action of the Steenrod powers on their $\mathbf{F}_p$-cohomology; if X and Y are not finite, then it is better to not dualise and work with this homological version.

Since the maps in the short exact sequence (1.2) naturally respect this action, it is an extension of $\mathcal{A}_*$-comodules, and this is often nontrivial, even if X and Y are both spheres. For instance, consider the Hopf map $\eta\colon \mathbf{S}^1 \to \mathbf{S}$ and $E = \mathbf{F}_2$. On cohomology, we have a short exact sequence

$$0 \longrightarrow \mathrm{H}^*(\mathbf{S}^2; \mathbf{F}_2) \longrightarrow \mathrm{H}^*(\operatorname{cofib}\eta;\, \mathbf{F}_2) \longrightarrow \mathrm{H}^*(\mathbf{S}; \mathbf{F}_2) \longrightarrow 0.$$

This extension does not split as a module over the Steenrod algebra: the middle term is $\mathbf{F}_2$ in degrees 0 and 2, but these have a nontrivial Sq^2 between them. (Indeed, it is a shfit of $\mathrm{H}^*(\mathbf{RP}^2;\, \mathbf{F}_2)$.) ▲

We can do something similar for more general E, although we do need a number of restrictions on E to ensure that the appropriate notion of comodules is well behaved.

Definition 1.4. Let E be a homotopy associative ring spectrum. We say that E is of **Adams type** if E can be written as a filtered colimit

do we wanna say more? compare to flatness? (see earlier drafts)

$$E = \operatorname*{colim}_\alpha E_\alpha$$

where each E_α is a finite spectrum such that $E_*(E_\alpha)$ is projective as a (left) E_*-module.

ALSO: add discussion of how restrictive it is (devinatz's theorem on landweber ones, but also that KO isn't, etc)

In the end, if E is of Adams type, then there is an abelian category $\mathrm{Comod}_{E_*E}(\mathrm{grAb})$ of E_*E-comodules in graded abelian groups, and E_* naturally lifts to a homology theory valued in these comodules. We will usually abbreviate this category by Comod_{E_*E}. In the end, the short exact sequence (1.2) defines an element in

$$\mathrm{Ext}^1_{E_*E}(E_{*-1}(Y),\, E_*(X)).$$

It may turn out that this extension is trivial. It turns out that we can continue this process, ending up with classes in higher and higher Ext groups. At this point, it is useful to introduce some notation.

Notation 1.5. Let s and t be integers, and M and N be E_*E-comodules. We write

$$\mathrm{Ext}^{s,t}_{E_*E}(M,N) := \mathrm{Ext}^s(M[t],\, N),$$

where square brackets indicate a grading shift: $(M[t])_n = M_{n-t}$.

Remark 1.6. In the above indexing convention, s is indexed cohomologically (being an Ext degree), while t is indexed homologically. We use homological indexing on t so that shifts in homology correspond to shifts of spectra: $E_*(\Sigma^t X) \cong E_*(X)[t]$.

With this notation, the extension (1.2) is an element of $\mathrm{Ext}^{1,1}(E_*(Y), E_*(X))$. It turns out that this class is the obstruction to f being the composite of two maps that are both zero on E-homology, i.e., the obstruction to having E-Adams filtration at least 2. If it does have filtration at least 2, then there is a class in

$$\mathrm{Ext}^{2,2}_{E_*E}(E_*(Y), E_*(X)),$$

and this pattern continues.

With the idea that this homological algebra can see a lot about maps between spectra, one can ask to go the other way: starting with a class in an Ext group as above, we can ask if it comes from a map of spectra. This will lead to the *E-based Adams spectral sequence*, which is of the form

$$\mathrm{E}_2^{n,s} = \mathrm{Ext}^{s,n+s}_{E_*E}(E_*(Y), E_*(X)) \implies [Y, X]_n.$$

We do not mean to say that this always converges: at the very least, this spectral sequence will never be able to see more than maps between the E-localisations of Y and X. If we replace $[Y, X]_n$ by $[Y, L_E X]_n$, then in most cases we get (at least conditional) convergence. In general we only see an approximation to the E-localisation, known as the *E-nilpotent completion*.

Remark 1.7. The original case introduced by Adams is where $E = \mathbf{F}_p$, and is often simply referred to as the *Adams spectral sequence*. For $E = \mathrm{MU}$ or BP, the resulting spectral sequence is known as the *Adams–Novikov spectral sequence*.

Remark 1.8. For our purposes, the reason we index this spectral sequence to start on page 2 is for convenience. Usually, the Adams spectral sequence comes about through a choice of resolution for X, leading to a first page whose cohomology groups are the above Ext groups. This first page however depends on the choice of resolution. Our later approach will be independent of choices, and as a result does not see this first page.

We could at this point define the Adams spectral sequence by defining a filtered spectrum that is the analogue of Definition 1.1. While knowing this is useful, we will take a shortcut and go straight to the modern approach: *synthetic spectra*. This recovers the other approach (and more) as follows: there are functors $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$ and $\sigma\colon \mathrm{Syn}_E \to \mathrm{FilSp}$. Their composite $\mathrm{Sp} \to \mathrm{FilSp}$ will turn out to be the Adams spectral sequence, but one can get a lot more mileage out of it by studying the object we get before applying σ. The reason for this is that synthetic spectra are a categorification of the Adams spectral sequence: the Adams spectral sequence for $[Y, X]$ is naturally captured by maps from νY to (certain shifts of) νX.

2 Categorical properties

Construction 2.1. Let E be a homotopy associative ring spectrum of Adams type. In [Pst22], Pstrągowski constructs a symmetric monoidal ∞-category Syn_E of ***E*-based synthetic spectra**, together with a unital lax symmetric monoidal functor $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$. We call ν the **synthetic analogue functor**.

We may refer to E-based synthetic spectra as *E-synthetic spectra*, or even simply by *synthetic spectra* if E is clear from the context. On the opposite end, when we want to vary the variable E, we write ν_E for ν, emphasising it as the *E-synthetic analogue*.

Remark 2.2. Although the notation seems to suggest otherwise, the symmetric monoidal ∞-category Syn_E depends on much less data than the ring spectrum E. This is because the E-based Adams spectral sequence depends only on the E_*-epimorphisms, and not on E itself. In the case of synthetic spectra, the precise statement is that the symmetric monoidal ∞-category Syn_E only depends on the class of finite spectra that are E_*-projective, together with the E_*-epimorphism class on these. (This is evident from the construction of Syn_E; see.) In particular, Syn_E is not sensitive to a potential coherent multiplicative structure on reference
E (nor does it require it for its construction as a symmetric monoidal ∞-category).

Notation 2.3. For the rest of this lecture, E denotes a fixed choice of a homotopy associative ring spectrum of Adams type.

Before we dive into the specifics of synthetic spectra, we discuss some general properties of this ∞-category. The following terminology will be helpful.

Definition 2.4. A spectrum P is called **finite *E*-projective** if it is a finite spectrum and if

E_*P is projective as a left E_*-module.[1] We write $\mathrm{Sp}_E^{\mathrm{fp}}$ for the full subcategory of Sp on the finite E-projective spectra.

For example, for every E, every sphere is finite E-projective. Beware that $\mathrm{Sp}_E^{\mathrm{fp}}$ is not a stable subcategory of Sp.

Proposition 2.5.

(1) *The* ∞*-category* Syn_E *is stable.*

(2) *The* ∞*-category* Syn_E *is presentable, and the symmetric monoidal structure preserves colimits in each variable separately; that is to say,* Syn_E *is presentably symmetric monoidal.*

(3) *If P is a finite E-projective spectrum, then νP is a compact and dualisable object in* Syn_E*. In particular, the monoidal unit is compact.*

(4) *As a stable* ∞*-category,* Syn_E *is generated under colimits by the synthetic analogues of the spheres $\nu(\mathbf{S}^n)$ for $n \in \mathbf{Z}$. In particular,* Syn_E *is compactly generated by dualisables.*

(5) *The monoidal* ∞*-category is* rigid *in the sense that an object is compact if and only if it is dualisable.*

Proof. ■ References

Remark 2.6 (Cellularity). Currently, what we denote by Syn_E is a slight modification of Pstrągowski's category: we work with ∞-category *cellular* synthetic spectra. This, by definition, is the smallest stable subcategory $\mathrm{Syn}_E^{\mathrm{cell}}$ of Pstrągowski's Syn_E that is closed under colimits and contains the synthetic analogues of the spheres $\nu(\mathbf{S}^n)$. We do this to make Proposition 2.5 (4) true; in general, one has to work with the synthetic analogues of all finite E-projectives in the place of the spheres. We do this for two reasons: first, none of the applications for spectral sequences change under this modification, and second, a number of desirable properties are only true for the cellular subcategory. For many E, the cellular subcategory is simply equal to Syn_E. The author does not know if it holds in general, but also does not know of a counterexample. We discuss this more in. For the rest of the lecture, we write $\mathrm{Syn}_E^{\mathrm{cell}}$ for Syn_E. ref

Next, we turn to properties of the functor ν.

Proposition 2.7.

(1) *The functor $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$ is fully faithful, additive, and preserves filtered colimits. In particular, ν preserves arbitrary coproducts.*

[1] This should not be confused with what one might call an *E-finite projective* spectrum, meaning a spectrum P such that E_*P is a finite projective E_*-module. This need not imply that the spectrum P is itself finite, an assumption we very intentionally require on P.

(2) *Consider a cofibre sequence of spectra*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Then the sequence

$$\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z$$

is a cofibre sequence of synthetic spectra if and only if

$$0 \longrightarrow E_*X \xrightarrow{f_*} E_*Y \xrightarrow{g_*} E_*Z \longrightarrow 0$$

is short exact, or in other words, if the boundary map $Z \to \Sigma X$ is zero on E_*-homology.

(3) *The comparison map $\nu X \otimes \nu Y \to \nu(X \otimes Y)$ coming from the lax symmetric monoidal structure on ν is an isomorphism whenever X or Y is a filtered colimit of finite E-projective spectra.*

More generally, if the E_*-homology of X or Y is flat as an E_*-module, then the map $\nu X \otimes \nu Y \to \nu(X \otimes Y)$ is a νE-equivalence.

Proof. ■ references

Both conditions of Proposition 2.7 (3) are a type of flatness condition. This is obvious for the second one. For the first, compare this with the algebraic result that a module over a ring is flat if and only if it can be written as a filtered colimit of finite free modules; see [Stacks, Tag 058G].

Example 2.8. The definition of Adams type directly implies that $\nu E \otimes X \to \nu(E \otimes X)$ is an isomorphism for all spectra X. ▲

Example 2.9. Suppose $E = \mathbf{F}_p$, or more generally a spectral field. Then every finite spectrum is E-projective. The smallest subcategory of Sp that contains all finite spectra and is closed under filtered colimits is equal to all of Sp. As a result, we learn from Proposition 2.7 (3) that the E-synthetic analogue is a strong symmetric monoidal functor if E is a spectral field. ▲

Take particular note that ν is *not* an exact functor, even though it is a functor between stable ∞-categories. For example, Proposition 2.7 (2) implies that $\Sigma(\nu X) \cong \nu(\Sigma X)$ if and only if $E_*X = 0$. In terms of the E-Adams spectral sequences, asking to have zero E-homology is a very degenerate case, so the functor ν practically never preserves suspensions.

The difference between suspending in spectra and in synthetic spectra has a conceptual meaning as well: the former has the effect of shifting its Adams spectral sequence one to the right, while suspending its synthetic analogue also shifts it down by one filtration. This is made precise by the following definition of the *synthetic bigraded spheres*. The indexing convention we use here turns out to be the most practical.

Definition 2.10 (Synthetic bigraded spheres). Let n and s be integers.

(1) The **synthetic (n, s)-sphere** is

$$\mathbf{S}^{n,s} := \Sigma^{-s}\, \nu(\mathbf{S}^{n+s}).$$

We refer to n as the **stem**, and to s as the **filtration**.

(2) We write $\Sigma^{n,s} \colon \mathrm{Syn}_E \to \mathrm{Syn}_E$ for the functor given by tensoring with $\mathbf{S}^{n,s}$ on the left.

(3) We write $\pi_{n,s} \colon \mathrm{Syn}_E \to \mathrm{Ab}$ for the functor

$$\pi_{n,s}(-) := [\mathbf{S}^{n,s},\ -].$$

(4) The map $\tau \colon \mathbf{S}^{0,-1} \to \mathbf{S}^{0,0}$ is the colimit-comparison map

$$\tau \colon \mathbf{S}^{0,-1} = \Sigma(\nu\mathbf{S}^{-1}) \longrightarrow \nu\mathbf{S} = \mathbf{S}^{0,0}.$$

If now X is a synthetic spectrum, then tensoring it with the map $\tau \colon \mathbf{S}^{0,-1} \to \mathbf{S}$ results in a map $\Sigma^{0,-1}X \to X$. This turns $\pi_{*,*}$ into a functor $\mathrm{Syn}_E \to \mathrm{Mod}_{\mathbf{Z}[\tau]}(\mathrm{bigrAb})$.

Note that Proposition 2.5 (4) says that the collection of bigraded spheres form a set of compact generators for Syn_E. As a result, bigraded homotopy groups detect isomorphisms.

cite yanovski's paper

Remark 2.11. If X is a spectrum, then we also have the natural colimit-comparison map $\Sigma(\nu X) \to \nu(\Sigma X)$. This coincides with the map $\tau \otimes \nu X$: see [Pst22, Proposition 4.28].

Example 2.12.

(1) For every n, the synthetic spectrum $\nu(\mathbf{S}^n)$ is the bigraded sphere $\mathbf{S}^{n,0}$. Later we will see that these are the only synthetic spheres that are in the essential image of ν: see Example 4.5. This the first instance where we see that ν places everything in Adams filtration zero.

more about this later!!

We will abuse notation and abbreviate $\mathbf{S}^{0,0}$ simply by $\mathbf{S}$, and refer to it as *the synthetic sphere*. It is the unit for the monoidal structure on Syn_E (because ν is unital). Most of the time, the context will allow one to see whether the sphere spectrum or the synthetic sphere is meant by this notation.

(2) If X is a spectrum, then we have a natural isomorphism $\Sigma^{n,0}\,\nu X \cong \nu(\Sigma^n X)$.

(3) Categorical suspension is given by the bigraded suspension $\Sigma^{1,-1}$. ▲

While the grading convention of Definition 2.10 has become more standard, it is not the only one in the literature. We refer to the above indexing as **Adams grading** of synthetic spectra. This is not the convention used in [Pst22], which instead follows the *motivic grading*. Unless explicitly said otherwise, we will not use motivic grading in these notes.

Remark 2.13 (Motivic grading). The **motivic grading** on synthetic spectra is to define

$$\mathbf{S}^{t,w} := \Sigma^{w-t}\,\nu(\mathbf{S}^{w}).$$

Conversion from Adams to motivic grading is given by

$$(n,s) \longmapsto (n,\, n+s),$$

and conversion from motivic to Adams grading is given by

$$(t,w) \longmapsto (t,\, w-t).$$

The only cases in which motivic grading agrees with Adams grading is the case where $t = 0$. (In particular, the map τ from Definition 2.10 (4) has bidegree $(0,-1)$ in both conventions.) In [Pst22], the degree w is called the **weight**, and the difference $t-w$ is called the **Chow degree**. In Adams grading, the weight of $\mathbf{S}^{n,s}$ is given by $n+s$, while the Chow degree is given by $-s$. The motivic grading is designed to match with the standard indexing conventions of motivic homotopy theory; see for more information.

rewrite sentence

ref

Remark 2.14. In the specific case of $\mathbf{F}_p$-synthetic spectra, it is becoming more and more common to use the letter λ to denote the map otherwise denoted by τ. This is done to accommodate for computations that involve both BP-synthetic and $\mathbf{F}_p$-synthetic arguments at the same time. Because this document is not aimed at these computations, we will still use the letter τ even in the $\mathbf{F}_p$-synthetic case.

add citations

3 Synthetic spectra as a deformation

need to somehow shoe-horn in the overarching story: we want to understand the structure of this category (that includes knowing what the morphisms between the objects are; this is a segue for the geometric Adams filtration)

At this point, it is natural to ask what the relationship is between the synthetic bigraded spheres Definition 2.10 and the filtered bigraded spheres. The map τ from Definition 2.10 realises Syn_E as a deformation in the sense of **????**.

Lemma 3.1. *There is a natural symmetric monoidal structure on the functor* $\mathbf{Z} \to \mathrm{Syn}_E$ *given by the multiplication-by-*τ *tower on the unit:*

$$\cdots \xrightarrow{\tau} \mathbf{S}^{0,-1} \xrightarrow{\tau} \mathbf{S} \xrightarrow{\tau} \mathbf{S}^{0,1} \xrightarrow{\tau} \cdots.$$

Proof. We will prove this later as. ∎

ref to later section on constructions

Notation 3.2. By the universal property of FilSp from **??**, the symmetric monoidal functor $\mathbf{Z} \to \mathrm{Syn}_E$ from Lemma 3.1 induces an adjunction

$$\mathrm{FilSp} \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \mathrm{Syn}_E$$

where the left adjoint ρ is a symmetric monoidal functor. As a result, the functor σ is naturally lax symmetric monoidal. We refer to σ as the **signature functor**.

cite Lurie/that other paper

Remark 3.3. As a special case of **??**, the functor σ can be described as follows. Write $\operatorname{map}(-,-)$ for the mapping spectrum functor of the stable ∞-category Syn_E. Then σ is given by levelwise applying $\operatorname{map}(\mathbf{S},-)$ to the multiplication-by-τ tower functor. In diagrams: for $X \in \mathrm{Syn}_E$, the filtered spectrum σX is given by

$$\cdots \xrightarrow{\tau} \operatorname{map}(\mathbf{S}, \Sigma^{0,-1}X) \xrightarrow{\tau} \operatorname{map}(\mathbf{S}, X) \xrightarrow{\tau} \operatorname{map}(\mathbf{S}, \Sigma^{0,1}X) \xrightarrow{\tau} \cdots .$$

Since τ is an endomorphism of the synthetic sphere, and every synthetic spectrum is canonically a module over the synthetic sphere, it does not matter whether we view the connecting maps as coming from multiplication by τ on the first or the second argument.

Remark 3.4. The adjunction $\rho \dashv \sigma$ is very close to a monadic adjunction. More precisely, it is a monadic adjunction if and only if Syn_E is cellular in the sense of. What requires more assumptions is to then identify the monad on filtered spectra without making reference to the synthetic category. We will discuss this more in.

add ref to remark and section

add ref

By definition, the functor ρ sends τ to the synthetic map τ as defined in Definition 2.10 (4). We can now import everything we did in filtered spectra. For example, as ρ is symmetric monoidal and exact, the $\mathbf{E}_\infty$-ring structure on $C\tau$ in filtered spectra pushes forward to an $\mathbf{E}_\infty$-ring structure on the cofibre $C\tau$ formed in synthetic spectra.

edit this paragraph (is out of place)

edit

segue

3.1 Synthetic homotopy groups and the signature spectral sequence

The deformation picture tells us how to understand of the bigraded homotopy groups of a synthetic spectrum. This comes from a comparison of the filtered and synthetic bigraded spheres. To avoid confusion, we will for the moment distinguish these by writing

$$\mathbf{S}^{n,s}_{\mathrm{fil}} \quad \text{and} \quad \mathbf{S}^{n,s}_{\mathrm{syn}}$$

for the filtered and synthetic spheres, respectively, and similarly $\pi^{\mathrm{fil}}_{*,*}$ and $\pi^{\mathrm{syn}}_{*,*}$ for the homotopy groups.

Proposition 3.5. *Let n and s be integers.*

(1) *We have isomorphisms*

$$\rho(\mathbf{S}^{n,s}_{\mathrm{fil}}) = \mathbf{S}^{n,\,s-n}_{\mathrm{syn}},$$

and the functor ρ sends the filtered map τ_{fil} to the synthetic map τ_{syn}.

(2) *We have natural isomorphisms, where $X \in \mathrm{Syn}_E$,*

$$\pi^{\mathrm{syn}}_{n,s}(X) = \pi^{\mathrm{fil}}_{n,\,s+n}(\sigma X),$$

and the map $\tau_{\mathrm{fil}} \colon \Sigma^{0,-1}\,\sigma X \to \sigma X$ is given by σ applied to $\tau_{\mathrm{syn}} \colon \Sigma^{0,\,-1}X \to X$.

(3) *The functor σ is conservative and preserves filtered colimits.*

Proof.

write the rest

From the formula for σ of Remark 3.3, we learn that $\sigma X = 0$ if and only if $\pi^{\mathrm{syn}}_{*,*} X = 0$. The bigraded homotopy groups detect isomorphisms of synthetic spectra, so this implies σ is conservative.

refer back. maybe make that statement a proposition

The fact that σ preserves filtered colimits follows from the fact that ρ sends a collection of compact generators of FilSp (namely, the filtered spheres) to compact objects in Syn_E. In more detail: fix a filtered diagram I in Syn_E. We have a chain of isomorphisms

$$\begin{aligned} \mathrm{Map}(\mathbf{S}^{n,s}_{\mathrm{fil}}, \operatorname{colim} \sigma I) &\cong \operatorname{colim} \mathrm{Map}(\mathbf{S}^{n,s}_{\mathrm{fil}}, \sigma I) \\ &\cong \operatorname{colim} \mathrm{Map}(\rho(\mathbf{S}^{n,s}_{\mathrm{fil}}), I) \\ &\cong \mathrm{Map}(\rho(\mathbf{S}^{n,s}_{\mathrm{fil}}), \operatorname{colim} I) \\ &\cong \mathrm{Map}(\mathbf{S}^{n,s}_{\mathrm{fil}}, \sigma \operatorname{colim} I), \end{aligned}$$

where for the first isomorphism we used that $\mathbf{S}^{n,s}_{\mathrm{fil}}$ is compact in FilSp, and for the third we used that $\rho(\mathbf{S}^{n,s}_{\mathrm{fil}}) = \mathbf{S}^{n,s-n}_{\mathrm{syn}}$ is compact in Syn_E. Because bigraded filtered homotopy groups detect equivalences of filtered spectra, we learn from this that $\operatorname{colim} \sigma I \to \sigma(\operatorname{colim} I)$ is an isomorphism of filtered spectra. ∎

I think this argument is actually showing ρ preserves compactness in general

Corollary 3.6. *Let X be a synthetic spectrum.*

(1) *We have a natural isomorphism of graded $\mathbf{Z}[\tau]$-modules*

$$\pi_{n,*}(X) \cong \pi_{n,*+n}(\sigma X).$$

(2) *We have natural isomorphisms of filtered spectra*

$$\sigma(C\tau \otimes X) \cong C\tau \otimes \sigma(X) \qquad \text{and} \qquad \sigma(X[\tau^{-1}]) \cong \sigma(X)[\tau^{-1}].$$

Remark 3.7. The above reindexing formula is merely a consequence of the way we usually index Adams spectral sequences. It is customary to start Adams spectral sequences as beginning on the second page, while in previous chapters, we indexed spectral sequences associated to filtered spectra to start on the *first* page. The indexing conventions for synthetic and filtered bigraded spheres reflect these choices, and as a result are incompatible. The reindexing $(n,s) \mapsto (n, s-n)$ of Proposition 3.5 is precisely the algebraic page-turning of.

need better name

add ref

Warning 3.8. Even though $\rho(\mathbf{S}^{n,s}_{\mathrm{fil}})$ is a bigraded synthetic sphere, the filtered spectrum $\sigma(\mathbf{S}^{n,s}_{\mathrm{syn}})$ is *very* different from a bigraded filtered sphere. Indeed, the spectral sequence associated to a filtered sphere is uninteresting, while the spectral sequence associated

earlier remark

to $\sigma(\mathbf{S}^{n,s}_{\mathrm{syn}})$ is (a shift of) the E-Adams spectral sequence for the sphere spectrum, which is very interesting and highly nontrivial. In particular, ρ is very far from preserving bigraded homotopy groups.

Combining Corollary 3.6 with the filtered Omnibus **??**, we obtain an interpretation of the bigraded homotopy groups of a synthetic spectrum X: it captures the spectral sequence associated to $\sigma(X)$. Accordingly, we will refer to this as the **signature spectral sequence** of X. In keeping with synthetic conventions, we will reindex it to start on the second page. In this form, it looks like

$$\mathrm{E}_2^{n,s} = \pi_{n,s}(C\tau \otimes X) \implies \pi_n(X[\tau^{-1}]).$$

Here we drop the second index on the homotopy of $X[\tau^{-1}]$: any choice of index yields an isomorphic group (or, more functorially, we take the colimit over the second index).

The filtered Omnibus Theorem translates directly into a synthetic Omnibus Theorem, where it is more convenient to reindex everything to make the differentials one longer. For example, if a class is hit by a d_r-differential, this class lifts to an element α in the synthetic homotopy groups such that $\tau^{r-1} \cdot \alpha = 0$; cf. **?? ??**. For the sake of brevity, we will not restate the resulting Omnibus Theorem here, but leave it to the reader to reindex the previous one.

Of course, all of this raises the question what spectral sequence this is concretely. Accordingly, we should understand what τ-invertible synthetic spectra and $C\tau$-modules in synthetic spectra are. The first ends up being the same as in filtered spectra, but the second is very different, and is what makes synthetic spectra what they are. After we have done this, we will analyse the signature of a synthetic analogue, and identify it with the Adams spectral sequence in the usual sense. Combining this with the synthetic Omnibus Theorem recovers the version of Burklund–Hahn–Senger.

cite again? or refer back to earlier discussion?

3.2 Inverting τ

Recall that the functor $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$ is fully faithful. However, since it is not an exact functor, we should not think too strongly of the image of ν as an embedding of spectra into synthetic spectra. If we make ν exact in a universal way, this does result in an embedding of spectra into synthetic spectra, and these happen to be exactly the τ-invertible synthetic spectra.

The exact same discussion as in **??** applies here. A synthetic spectrum is called **τ-invertible** if the endomorphism τ on it is an isomorphism. This is again a smashing localisation, being given by tensoring with the τ-inversion of the synthetic sphere. Since ρ in particular makes Syn_E tensored over FilSp, one can equivalently describe this as being the modules over $\rho(\mathbf{S}[\tau^{-1}])$. This is indeed the same because ρ preserves colimits and sends τ to τ by Proposition 3.5.

What is not automatic is that $\mathrm{Syn}_E[\tau^{-1}]$ is equivalent to spectra. This is true, but requires additional analysis.

Definition 3.9. Write $\text{よ} \colon \mathrm{Sp} \to \mathrm{Syn}_E$ for the functor $\nu(-)[\tau^{-1}]$.

By definition, よ lands in τ-invertible synthetic spectra.

Theorem 3.10 ([Pst22], Theorem 4.37). *The functor* よ *is fully faithful, exact, and symmetric monoidal, and restricts to a symmetric monoidal equivalence*

$$\text{よ} \colon \mathrm{Sp} \xrightarrow{\simeq} \mathrm{Syn}_E[\tau^{-1}].$$

As before, we will write $(-)^{\tau=1}$ for the composite

$$\mathrm{Syn}_E \xrightarrow{\tau^{-1}} \mathrm{Syn}_E[\tau^{-1}] \simeq \mathrm{Sp}.$$

Because τ^{-1} is left adjoint to the inclusion, it follows that we have an adjunction

$$\mathrm{Syn}_E \underset{\text{よ}}{\overset{(-)^{\tau=1}}{\rightleftarrows}} \mathrm{Sp}.$$

Example 3.11. Recall the definition $\mathbf{S}^{n,s} = \Sigma^{-s}\nu(\mathbf{S}^{n+s})$ from Definition 2.10. As τ-inversion is an exact functor on synthetic spectra, it preserves suspensions, so we find that

$$\mathbf{S}^{n,s}[\tau^{-1}] = \Sigma^{-s}\nu(\mathbf{S}^{n+s})[\tau^{-1}] = \Sigma^{-s}\,\text{よ}(\mathbf{S}^{n+s}) \cong \text{よ}(\mathbf{S}^{n}).$$

In other words, $(\mathbf{S}^{n,s})^{\tau=1} = \mathbf{S}^n$. We can think of this as saying that inverting τ forgets the Adams filtration. ▲

Said differently: the functor $(-)^{\tau=1}$ is a right inverse to ν. In this way, one can think of Syn_E as a type of bundle over Sp, of which ν is a section.

in the lectures as a whole, need to bring the use of synthetic spectra as "remembering the Adams filtration" more!

Y is the zero section, ν is an interesting section

make this more centerstage

3.3 Modding out by τ and the homological t-structure

While τ-invertible spectra do not look different from τ-invertible filtered spectra, things look very different when we look at the cofibre of τ in synthetic spectra. Understanding modules over it relies on one of the most important categorical tools for interacting with synthetic spectra: the *homological t-structure*.

As the name suggests, the key feature of the homological t-structure is that it looks at *νE-homology* of E-synthetic spectra to measure (co)connectivity, *not* at the bigraded homotopy groups. This is both an upside and a downside: due to the special role that E plays for E-synthetic spectra, this homology tends to be a lot simpler than the homotopy. On the other hand, this means that taking truncations or connective covers can have very unpredictable effects on bigraded homotopy groups.

Theorem 3.12. *There exists a unique accessible t-structure on* Syn_E *that satisifes the following.*

(a) *A synthetic spectrum* X *is connective if and only if*

$$\nu E_{n,s}(X) = 0 \qquad \text{whenever } s > 0.$$

(b) *A synthetic spectrum* X *is* 0*-truncated if and only if* X *is* νE*-local and*

$$\nu E_{n,s}(X) = 0 \qquad \text{whenever } s < 0.$$

(c) *The connective cover* $\tau_{\geqslant 0} X \to X$ *induces an isomorphism*

$$\nu E_{n,s}(\tau_{\geqslant 0} X) \xrightarrow{\cong} \nu E_{n,s}(X) \qquad \text{whenever } s \leqslant 0.$$

Likewise, the 0*-truncation* $X \to \tau_{\leqslant 0} X$ *induces an isomorphism*

$$\nu E_{n,s}(X) \xrightarrow{\cong} \nu E_{n,s}(\tau_{\leqslant 0} X) \qquad \text{whenever } s \geqslant 0.$$

(d) *The t-structure is right complete* (*but in general, not even left separated*).

(e) *The t-structure is compatible with the monoidal structure.*

(f) *There exists a monoidal equivalence of categories*

$$\mathrm{Syn}_E^{\heartsuit} \simeq \mathrm{Comod}_{E_*E}$$

such that there exists a commutative diagram of lax monoidal functors

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{\tau_{\leqslant 0}\tau_{\geqslant 0}\nu} & \mathrm{Syn}_E^{\heartsuit} \\ & \searrow^{E_*(-)} & \downarrow \simeq \\ & & \mathrm{Comod}_{E_*E}. \end{array}$$

Moreover, if E is homotopy commutative, then these equivalences are naturally symmetric monoidal.

We refer to this t-structure as the ***homological t-structure*** *on* Syn_E.

Proof. Property (a) determines this t-structure uniquely, and is formal. Property (b) follows from this using the Adams type property of E, and (c) follows directly from it. The difficult part is the identification of the heart; see. ■

Remark 3.13. The name *homological t-structure* is taken from [BHS23, Appendix A]. In [Pst22, Section 4.2], it called the *natural t-structure*.

suppose all spectra have nonvanishing E homology. is Syn left separated then?

need a comment somewhere that these symmetric monoidality issues only arise when comparing a previously constructed abelian category with a topologically defined one. (your thought of "if you do not cross these realms, then you do not have any symmetry issues"). though I think that this would be the first instance where we see this in action?

Lurie

do we want to spell this out in these notes?

write the rest

Remark 3.14. The description of the connective objects is somewhat confusing, in that an object is connective when certain groups in a *positive* degree vanish. This clash is because the filtration in Adams spectral sequences is indexed cohomologically, while we usually index t-structures homologically (at least in homotopy theory). Arguably, it would be less confusing to index this t-structure cohomologically instead (as is more common in, e.g., algebraic geometry), writing $\tau^{\leqslant 0}$ for what we normally write as $\tau_{\geqslant 0}$, and $\tau^{\geqslant 0}$ for $\tau_{\leqslant 0}$.

should we do that?

Warning 3.15. Often with t-structures, one writes $\pi_n^{\heartsuit}$ for the functor $\Sigma^{-n}\tau_{\leqslant n}\tau_{\geqslant n}$ considered as landing in the heart of the t-structure. Because this t-structure is more concerned with homology than homotopy, this can get confusing: the functor $\pi_n^{\heartsuit}$ is *not* related to bigraded homotopy groups. Instead, we have an isomorphism

$$\pi_n^{\heartsuit}(X) \cong \nu E_{*+n,-n}(X).$$

Note also the minus sign in the filtration on the right-hand side; this is again due to the difference between homological and cohomological grading of Remark 3.14. To avoid the potential confusion with the bigraded homotopy groups, we will generally not use the notation $\pi_n^{\heartsuit}$ and work directly with the νE-homology.

As we have not yet computed anything in synthetic spectra, it is at this point not clear if we have any examples of connective or truncated synthetic spectra. Shortly, we will compute the νE-homology of synthetic analogues, and in particular find that *all* synthetic analogues are connective (even if the original spectrum is not); see Example 4.5. In fact, we will see that for all X, we have

$$\nu X \cong \tau_{\geqslant 0}\, \text{よ}(X).$$

also include statement about whitehead tower? at that point, phrase things as a proposition. emphase the two roles that $\nu \to$ よ plays: it gives the source a univ prop by saying it's a conn cover, while it gives the target a univ prop by saying it's tau inversion

Remark 3.16. The functor $\nu\colon \mathrm{Sp} \to \mathrm{Syn}_E$ does not have a left or right adjoint, as it is not even an exact functor. When considered as landing in connective synthetic spectra however, its categorical properties improve: it is then right adjoint to inverting τ. This follows from the relation $\nu = \tau_{\geqslant 0} \circ$ よ and by pasting adjunctions, using that よ is right (and left) adjoint to τ^{-1} (being its inverse) as a functor to τ-invertible synthetic spectra: the horizontal composites in

$$\mathrm{Sp} \underset{\text{よ}}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Syn}_E[\tau^{-1}] \underset{}{\overset{\tau^{-1}}{\longleftrightarrow}} \mathrm{Syn}_E \underset{\tau_{\geqslant 0}}{\overset{\longleftarrow}{\longrightarrow}} (\mathrm{Syn}_E)_{\geqslant 0}$$

form the adjunction

$$\mathrm{Sp} \underset{\nu}{\overset{(-)^{\tau=1}}{\longleftrightarrow}} (\mathrm{Syn}_E)_{\geqslant 0}.$$

Using the homological t-structure, one can show the following.

Theorem 3.17. *There is an equivalence of monoidal ∞-categories*

only fully faithful in general

$$\mathrm{Mod}_{C\tau}(\mathrm{Syn}_E) \simeq \mathrm{Stable}_{E_*E}.$$

If E is homotopy commutative, then this equivalence is naturally symmetric monoidal.

explain Stable

4 The signature of a synthetic analogue

Definition 4.1. The ***E*-based Adams filtration** is the functor $\sigma \circ \nu_E \colon \mathrm{Sp} \to \mathrm{FilSp}$.

The first thing we will prove is that, on suitable spectra, this coincides with the E-Adams spectral sequence as it is usually defined. However, the point of this is not to let go of its synthetic origins, but it serves as an illustration of how to work with synthetic spectra.[2] One can interact with the Adams spectral sequence entirely through the category theory of synthetic spectra.

edit (also footnote)

Remark 4.2. The functor $\sigma \circ \nu_E$ is a composite of two lax symmetric monoidal functors, making it into a symmetric monoidal functor. For general E, it is not known how to directly construct this structure on the Adams filtration as it is usually defined. This is one of the great benefits of synthetic spectra: to define the symmetric monoidal ∞-category Syn_E, we merely require E to be a homotopy associative ring spectrum. By contrast, to turn the standard definition of the E-Adams filtration into a lax symmetric monoidal functor, one would need an $\mathbf{E}_\infty$-structure on E. Such a structure does not always exist in cases of interest (e.g., BP or Morava K-theory), and the Adams spectral sequence does not depend on it, so this is not desirable. The version in Definition 4.1 does not have any of these defects.

bad sentence

only defect is that it requires E to be of Adams type. but we'll address that later

Irakli–Piotr, Remark 5.62 (and other places?)

Need to give a literature overview. the account here follows CDvN, but that in and of itself is a collection and rewording of many different things

talk about the LES

explanation of result: Adams sseq of a spectrum is zero in negative filtrations, so nothing changes, so tau is an iso there.

The following holds for homotopy classes of maps $Y \to X$ between two spectra, but for simplicity we record it only for homotopy groups.

Proposition 4.3 ([Pst22], Theorem 4.58). *Let X be a spectrum. Then for all $s \leqslant 0$ and all n, inverting τ induces a natural isomorphism*

$$\pi_{n,s}(\nu X) \xrightarrow{\cong} \pi_n X.$$

[2]In particular, those readers not familiar with the Adams spectral sequence should not despair.

Phrased differently, it induces a natural isomorphism of bigraded $\mathbf{Z}[\tau]$*-modules*

$$\pi_{*,\leqslant 0}(\nu X) \cong \pi_*(X)[\tau],$$

where $\pi_n X$ *is placed in bidegree* $(n, 0)$.

Proof. We note that $\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) = 0$ whenever $s < 0$. Part of the long exact sequence from **??** reads

$$\mathrm{Ext}^{s-2,n+s-1}_{E_*E}(E_*, E_*X) \longrightarrow \pi_{n,s}(\nu X) \xrightarrow{\tau} \pi_{n,s-1}(\nu X) \longrightarrow \mathrm{Ext}^{s-1,n+s-1}_{E_*E}(E_*, E_*X)$$

If $s \leqslant 0$, we therefore see that the two outer terms vanish, so that the map in the middle is an isomorphism. As a result, we only have to compute $\pi_{n,0}(\nu X)$. Because $\mathbf{S}^{n,0} = \nu\mathbf{S}^n$, the fact that ν is fully faithful implies that the map $\pi_{n,0}(\nu X) \to \pi_n X$ is an isomorphism. ■

For a particularly nice class of spectra, this computes the entirety of the synthetic homotopy groups.

Proposition 4.4 ([Pst22], Proposition 4.60). *Let* M *be a spectrum admitting a homotopy* E*-module structure. Then we have an isomorphism of bigraded* $\mathbf{Z}[\tau]$*-modules*

$$\pi_{*,*}(\nu M) \cong \pi_*(M)[\tau]$$

where $\pi_n M$ *is placed in bidegree* $(n, 0)$.

Proof. Using the previous result, we only have to show that $\pi_{n,s}(\nu M)$ vanishes when $s \geqslant 1$. We first show this for $s = 1$. Since M is a homotopy E-module, the Hurewicz homomorphism

$$E_*(-)\colon \pi_n M \longrightarrow \mathrm{Hom}_{E_*E}(E_*[n], E_*M)$$

is an isomorphism. Under the isomorphism $\pi_n M \cong \pi_{n,0}(\nu M)$, the Hurewicz homomorphism is exactly the right-most map in the exact sequence

Piotr remark 3.18, though there is a typo there (he flips the Hom term)

$$\mathrm{Ext}^{-1,n}_{E_*E}(E_*, E_*M) \longrightarrow \pi_{n,1}(\nu M) \xrightarrow{\tau} \pi_{n,0}(\nu M) \longrightarrow \mathrm{Ext}^{0,n}_{E_*E}(E_*, E_*M).$$

As the Ext group on the left vanishes, we learn that $\pi_{n,1}(\nu M) = 0$ for all n.

Next, we consider the case $s > 1$. Since M is a homotopy E-module, it is a retract of $E \otimes M$ (using the multiplication $E \otimes M \to M$ and the unit map $\mathbf{S} \to E$ tensored with M). As a result, E_*M is a retract of $E_*(E \otimes M) \cong E_*E \otimes_{E_*} E_*M$, so that E_*M is a retract of an extended comodule. In particular, it is an injective comodule, implying that the Ext groups

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*M)$$

vanish for all $s \geqslant 1$. By the long exact sequence, this means that multiplication by τ

$$\tau\colon \pi_{n,s+1}(\nu X) \xrightarrow{\cong} \pi_{n,s}(\nu X)$$

is an isomorphism for all $s \geqslant 1$. Since we previously showed that $\pi_{*,1}(\nu M) = 0$, this finishes the proof. ■

connectivity of synthetic analogues

Example 4.5. Recall from Example 2.8 that for all spectra X, we have an isomorphism $\nu E \otimes \nu X \cong \nu(E \otimes X)$. Because $E \otimes X$ is a homotopy E-module, we learn from Proposition 4.4 that

$$\nu E_{*,*}(\nu X) = \pi_{*,*}(\nu E \otimes \nu X) \cong \pi_{*,*}(\nu(E \otimes X)) \cong E_*(X)[\tau].$$

In particular, by Theorem 3.12 (a), this shows that νX is connective in the homological t-structure. We see why this is independent of the connectivity of X as a spectrum: this only affects the stem variable, while the homological t-structure is concerned with the filtration.

We can learn a number of things from this computation. First, we see that (unless $E_*(X)$ vanishes) the shift $\Sigma^{0,s}\,\nu X$ for $s \neq 0$ is not in the essential image of ν.

Another consequence is that $\nu E_{*,*}(\nu X)$ is τ-torsion free. As a result, we learn that

$$\nu E_{*,*}(C\tau \otimes X) \cong (\nu E_{*,*}(\nu X))/\tau \cong E_*(X),$$

where we mean the quotient by τ in the (non-derived) algebraic sense. This explains (apart from the νE-locality) why $C\tau \otimes \nu X$ is 0-truncated in the homological t-structure; cf. Theorem 3.12 (b).

Finally, inverting τ on $\nu E_{*,*}(\nu X)$ yields

$$\nu E_{*,*}(\text{よ}\, X) = \nu E_{*,*}(\nu X[\tau^{-1}]) \cong E_*(X)[\tau^{\pm}].$$

This explains why $\nu X \to \text{よ}\, X$ is a connective cover, and more generally, why the Whitehead tower of $\text{よ}\, X$

$$\cdots \longrightarrow \tau_{\geqslant 1}\, \text{よ}\, X \longrightarrow \tau_{\geqslant 0}\, \text{よ}\, X \longrightarrow \tau_{\geqslant -1}\, \text{よ}\, X \longrightarrow \cdots$$

can be identified with the multiplication-by-τ tower on νX

$$\cdots \xrightarrow{\ \tau\ } \Sigma^{0,-1}\,\nu X \xrightarrow{\ \tau\ } \nu X \xrightarrow{\ \tau\ } \Sigma^{0,1}\,\nu X \xrightarrow{\ \tau\ } \cdots. \qquad \blacktriangle$$

We can restate this result in terms of the signature of νM.

Corollary 4.6. *Let M be a spectrum admitting a homotopy E-module structure. Then there is a natural symmetric monoidal isomorphism of filtered spectra*

$$\sigma(\nu M) \cong \mathrm{Wh}(M).$$

Proof. For every s, inverting τ induces a natural map of spectra

should improve this: previously you showed σ pres filtered colimits and hence commutes with inverting τ. making precise what that means should immediately give you this

$$(\sigma(\nu M))^s = \mathrm{map}_{\mathrm{Syn}_E}(\mathbf{S}, \Sigma^{0,-s}\,\nu M) \xrightarrow{\tau^{-1}} \mathrm{map}_{\mathrm{Sp}}(\mathbf{S}, M) \cong M.$$

Because the transition maps in $\sigma(\nu M)$ are induced by τ, they become isomorphisms after inverting τ, so that this assembles to a natural transformation $\sigma(\nu M) \to \mathrm{Const}(M)$. This is even a symmetric monoidal natural transformation because τ-inversion is symmetric monoidal. Proposition 4.4 implies that $\sigma(\nu M)$ is connective in the diagonal t-structure on filtered spectra from **??**. Indeed, combining this with Proposition 3.5 (2), we see that

$$\pi_{n,s}(\sigma(\nu M)) \cong \pi_{n,s-n}(\nu M)$$

vanishes whenever $s - n > 0$, that is, whenever $n < s$. As a result, the natural map $\sigma(\nu M) \to \mathrm{Const}(M)$ factors through a natural map

$$\sigma(\nu M) \longrightarrow \tau^{\mathrm{diag}}_{\geqslant 0}(\mathrm{Const}(M)) = \mathrm{Wh}(M).$$

Moreover, this factorisation is through a symmetric monoidal transformation, because the diagonal t-structure on filtered spectra is monoidal (**?? ??**). To establish that it is an isomorphism, it suffices to show that $\sigma(\nu M)^s \to \tau_{\geqslant s}\, M$ is an isomorphism for all s. This is the other part of Proposition 4.4. ■

We can now describe the signature of a general synthetic analogue, at least if the Adams spectral sequence converges conditionally. Note that we will not use this formula directly to compute with synthetic spectra, but rather it serves a check that we are talking about the correct objects.

We require a brief digression on terminology. If E is a homotopy ring spectrum, then its unit map $\mathbf{S} \to E$ gives rise to a semicosimplicial spectrum $\Delta_{\mathrm{inj}} \to \mathrm{Sp}$ of the form

$$E^{\bullet+1} = \quad E \rightrightarrows E \otimes E \quad \substack{\to\\ \to\\ \to} \quad \cdots .$$

This receives a map from $\mathbf{S}$. Tensoring this with a spectrum X, we obtain a map

$$X \longrightarrow \mathrm{Tot}(E^{\bullet+1} \otimes X). \tag{4.7}$$

We say that X is ***E*-nilpotent complete** if this map is an isomorphism.

Theorem 4.8. *There is a natural map of filtered spectra*

CDvN Proposition 1.25

$$\sigma(\nu X) \longrightarrow \mathrm{Tot}(\mathrm{Wh}(E^{\bullet+1} \otimes X))$$

and it is an isomorphism if X is E-nilpotent complete.

Proof. Applying ν to the diagram $X \to E^{\bullet+1} \otimes X$ yields a map

$$\nu X \longrightarrow \mathrm{Tot}(\nu(E^{\bullet+1} \otimes X)). \tag{4.9}$$

Using **??**, we see that this map is canonically isomorphic to

$$\nu X \longrightarrow \mathrm{Tot}((\nu E)^{\bullet+1} \otimes \nu X).$$

One can check that this is an isomorphism if and only if (4.7) is: see [BHS23, Proposition A.13].

Because σ preserves limits (being a right adjoint), applying σ to (4.9) yields a map

$$\sigma\nu X \longrightarrow \mathrm{Tot}(\sigma\nu(E^{\bullet+1} \otimes X)).$$

We claim that this is of the claimed form. Indeed, for all $n \geqslant 1$, the synthetic spectrum $\nu(E^{\otimes n} \otimes X)$ is the synthetic analogue of a homotopy E-module, so Corollary 4.6 applies, yielding

$$\mathrm{Tot}(\sigma\nu(E^{\bullet+1} \otimes X)) \cong \mathrm{Tot}(\mathrm{Wh}(E^{\bullet+1} \otimes X)). \qquad \blacksquare$$

As a corollary, we get the Burklund–Hahn–Senger Omnibus.

Remark 4.10. It is natural to ask what happens when X is not E-nilpotent complete. In this case, the comparison map is *never* an isomorphism, and $\sigma(\nu X)$ is the preferable filtration. As a simple example: if $E_*(X) = 0$, then

$$\mathrm{Tot}(\mathrm{Wh}(E^{\bullet+1} \otimes X)) = 0, \qquad \text{while} \qquad \sigma(\nu X) = \mathrm{Const}(X).$$

This second claim is a rephrasing of the isomorphism $\pi_{*,*}(\nu X) \cong \pi_* X[\tau^{\pm}]$, which follows from Proposition 4.3. In general, the comparison map of Theorem 4.8 is the *τ-completion* of the filtered spectrum; i.e., it is an isomorphism on associated graded, and the target has a vanishing limit. The filtered $\sigma(\nu X)$ always has colimit X, while the colimit of the other is (by definition) isomorphic to X if and only if X is E-nilpotent complete. This being said, the filtration $\sigma(\nu X)$ is of course of very limited use if X is not E-nilpotent complete.

Remark 4.11. If E-synthetic spectra are cellular, then the above result can be strengthened to say that the comparison map is an isomorphism if *and only if* X is E-nilpotent complete. Indeed, the map (4.9) is an isomorphism if and only if X is E-nilpotent complete, so we need to exclude the possibility that σ kills its cofibre. Analysing the formula of σ, we see that for $A \in \mathrm{Syn}_E$, the filtered spectrum σA is zero if and only if $\pi_{*,*}A = 0$.

Remark 4.12. The left-hand side of the comparison map of Theorem 4.8 is naturally a lax symmetric monoidal functor. Under very heavy assumptions on E, the right-hand side also carries a natural lax symmetric monoidal structure, and the comparison map matches these up.

refer back to prev discussion

explain: limit of $\sigma\nu$ is zero iff nilpotent complete? make that more apparent in the proof above

ref

probably better to say somewhere: if cellular, then σ is conservative.

- If E carries an $\mathbf{E}_1$-structure, then the semicosimplicial spectrum $E^{\bullet+1}$ naturally extends to a cosimplicial spectrum $\Delta \to \mathrm{Sp}$.

  cite Mathew–Naumann–Noel. isn't this in some sense an iff?

- If E carries an $\mathbf{E}_\infty$-structure, then this naturally lifts to a cosimplicial $\mathbf{E}_\infty$-ring $\Delta \to \mathrm{CAlg}$. Because the Whitehead filtration is a symmetric monoidal functor, the filtered spectrum $\mathrm{Wh}(E^{n+1} \otimes X)$ is naturally a filtered $\mathbf{E}_\infty$-ring for every n. Taking limits is lax symmetric monoidal (being the right adjoint to a strong symmetric monoidal functor), so $\mathrm{Tot}(\mathrm{Wh}(E^{\bullet+1} \otimes X))$ is then naturally an $\mathbf{E}_\infty$-ring. In this case, the comparison map of Theorem 4.8 is a symmetric monoidal natural transformation.

  ref

  it's not a ring! also fix later points

Remark 4.13. One can do something similar for the Adams spectral sequence for maps $[Y, X]$ for a general spectrum Y; this is the generality of [PP21, Proposition 5.56, Theorem 5.60]. In this case, one would look at the filtered spectrum

$$\cdots \longrightarrow \mathrm{map}(\nu Y, \Sigma^{0,-1} \nu X) \longrightarrow \mathrm{map}(\nu Y, \nu X) \longrightarrow \mathrm{map}(\nu Y, \Sigma^{0,1} \nu X) \longrightarrow \cdots$$

with τ as transition maps. This defines a functor $\mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp} \to \mathrm{FilSp}$. The colimit of this filtered spectrum is $\mathrm{map}(Y, X)$, and the starting page of the spectral sequence (indexed to be page 2) is of the form

$$\mathrm{Ext}^{s,n+s}_{E_*E}(E_*(Y), E_*(X)).$$

In general however, this does not agree with the E-Adams spectral sequence as defined through tensor-powers of E; for this, we would require $E_*(Y)$ to be projective over E_*. We discuss this issue more in.

ref

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