§1. Computation of stable stems

Goal: $\pi_{\leq 14, *}(\sim_{\mathbb{F}_2} \mathbb{S}_2^\wedge)$. Abbreviate by $\pi_{*,*}(\mathbb{S})$.

Input: $\pi_{\leq 15, *}(C\tau)$. This is algebraic data: Ext groups of $\mathcal{A}_*$-comodules.

Prop: $\pi_{0,*}(\mathbb{S}) \cong \mathbb{Z}_2[\tau][\tilde{2}]/(\tau\tilde{2} = 2)$, and $\tilde{2} \mapsto h_0$ under $\mathbb{S} \to C\tau$

Proof: First check that there are no differentials into zero-stem.

Suppose $d_r(h_1) = h_0^{r+1}$. $\overset{\text{Leibniz}}{\rightsquigarrow}$ $\underset{\substack{\| \\ d_r(0)=0}}{d_r(h_0 h_1)} = h_0^{r+2}$ $\lightning$

Leibniz rule is: $\partial : C\tau \longrightarrow \Sigma^{1,-2} C\tau$ is a derivation

By omnibus: $\pi_{0,*}(\mathbb{S})$ is τ-power torsion-free.

$$\Longrightarrow \quad \begin{array}{ccc} \pi_{0,*}\mathbb{S} & \hookrightarrow & \pi_{0,*}\mathbb{S}[\tau^{-1}] \cong \mathbb{Z}_2[\tau^{\pm}]. \\ \downarrow & & \\ \pi_{0,*}C\tau \cong \mathbb{F}_2[h_0] & & \end{array}$$

is precisely the quotient by the τ-multiples ⊛

Can now determine $\pi_{0,*}\mathbb{S}$ as a $\mathbb{Z}[\tau]$-submodule of $\mathbb{Z}_2[\tau^{\pm}]$.

We know $\pi_{0,\leq 0}(\mathbb{S}) \cong \mathbb{Z}_2[\tau]$. Only need to determine positive filtrations

⊛ implies:

$$\boxed{\text{something in } \pi_{0,*}\mathbb{S} \text{ is } \tau\text{-div.} \Longleftrightarrow \text{maps to zero in } C\tau.}$$

Now we use: $\pi_{0,*}C\tau \cong \mathbb{F}_2[h_0]$ $\quad |h_0| = (0,1)$.

$1 \mapsto 1 \neq 0$, $\quad 2 \mapsto 2 = 0$, so $\boxed{\frac{2}{\tau} \in \pi_{0,*}\mathbb{S}.} \longleftarrow =: \tilde{2}$

Likewise $\tilde{4} := \frac{4}{\tau^2}$, $\tilde{6} := \frac{6}{\tau}$, ... in general, $\tilde{n} = \frac{n}{\tau^{\nu_2(n)}}$.

Then must have $\tilde{2} \mapsto h_0$ by surjectivity.

Thus determines everything: $\pi_{0,*} \cong \mathbb{Z}_2[\tau, \tilde{2}]/(\tau\tilde{2} = 2)$. □

Observe: All $\pi_{n,*}$ are modules over $\pi_{0,*}\mathbb{S}$.

Prop: $\pi_{1,*} \cong \mathbb{Z}/2[\tau,\tilde{2}]\langle\eta\rangle/(\tilde{2}\eta=0)$. $|\eta|=(1,1)$, $\eta \mapsto h_1$.

Proof: Again τ-torsion free.

h_1 supports no differentials $\overset{\text{omnibus}}{\Longrightarrow}$ lifts to $\pi_{1,1}\mathbb{S}$.

Lift is unique: no elements of higher filtration. Call this η.

Remaining: $\tilde{2}\cdot\eta = ?$

$\tilde{2}\cdot\eta \in \pi_{1,2}$ but $\pi_{1,2}=0$ because $E_2^{1,s}=0$ for $s \geq 2$.

Lastly, $2\cdot\eta = \tau\cdot\tilde{2}\eta = 0$, so this is indeed a $\mathbb{Z}/2$-module. $\square$

Rmk: Write η instead of $\tilde{\eta}$: we are not reconstructing $\pi_{*,*}\mathbb{S}$ from $\pi_*\mathbb{S}$, but building $\pi_{*,*}\mathbb{S}$ from the ground up. (But $\tilde{2}$ is forced on us: there we can <u>only</u> think about it as lifting the classical element.)

Prop: $\pi_{3,*} \cong \mathbb{Z}/8[\tau,\tilde{2}]\langle\nu\rangle/(\tilde{2}^3\cdot\nu=\eta^3,\ \tilde{2}^3\cdot\nu=0)$.

Proof: τ-torsion-free, and h_2 lifts to $\pi_{3,1}$.

Choose a lift ν of h_2. Or: you can lift your preferred $\nu \in \pi_3\mathbb{S}$.

$\tilde{2}^3\cdot\nu - \eta^3 \longmapsto h_0^3 h_2 - h_1^3 = 0$ under $\mathbb{S}\to C\tau$.

$$\begin{array}{ccccc}
\pi_{3,4}\mathbb{S} & \xrightarrow{\tau} & \pi_{3,3}\mathbb{S} & \longrightarrow & \pi_{3,3}C\tau \\
\times & \longmapsto & \tilde{2}^3\cdot\nu-\eta^3 & \longmapsto & 0
\end{array}$$

But $\pi_{3,4}\mathbb{S}=0$.

As before, $8\nu = \tau^3\cdot\tilde{2}^3\cdot\nu = 0$. $\square$

One continues in the same way, until we get to stem 14, where differentials could (and do indeed) appear.

Prop: $\pi_{7,*} \cong \mathbb{Z}/16[\tau,\tilde{2}][\sigma]/(\tilde{2}^4\cdot\sigma=0)$ $|\sigma|=(7,7)$.

Then $\sigma\cdot\sigma = -\sigma\cdot\sigma$, so $2\sigma^2=0$.

$\Rightarrow$ $0 = 2\sigma^2 = \tau\cdot\tilde{2}\sigma^2$.

Omnibus: $\tilde{2}\sigma^2$ is hit by a diff of length at most 2^n.
$h_0h_3^2$.

Only one option:

Prop: $d_2(h_4) = h_0 h_3^2$ first Hopf invariant 1 differential.

Rmk: While this time we didn't need it, it's a very general & powerful method to have equations put bounds on differentials.
Also, we got more than the differential: the specific lift $\tilde{2}\sigma^2$ is τ-torsion. (In general, only have: $d_r \Rightarrow \exists\ \tau^{r-1}$- torsion lift)

Next goal: $d_3(h_0 h_4) = h_0 d_0$.

Cannot follow from Leibniz rule ☹
We'll use a beefed-up version.

Recall: ∂ is boundary map $S^{0,-1} \xrightarrow{\tau} S \longrightarrow C\tau \xrightarrow{\partial} \Sigma^{1,-2} S$.

Prop: $\partial(h_4) = \tilde{2}\sigma^2$.

$$\begin{tikzcd} \pi_{n,s} C\tau \arrow[r, "\partial"] \arrow[dr, "d_2"'] & \pi_{n-1,w+2} S \arrow[d] \arrow[dl, phantom, "\#"] \\ & \pi_{n-1,w+2} C\tau = E_2^{n-1,w+2} \end{tikzcd}$$

Proof: Know that $\tau\cdot\tilde{2}\sigma^2 = 0$. LES:

$$\mathbb{Z}/2\langle h_4\rangle \cong \pi_{15,1}(C\tau) \longrightarrow \pi_{14,3} S \xrightarrow{\tau} \pi_{14,2} S \qquad \square$$

$$\tilde{2}\sigma^2 \longmapsto 0$$

only option

Prop: $d_3(h_0 h_4) = h_0 d_0$.

Proof:

$$\partial(h_0 h_4) = \partial(\tilde{2}\cdot h_4) = \tilde{2}\cdot\partial(h_4)$$
$$= \tilde{2}\cdot\tilde{2}\cdot\sigma^2 \overset{?}{=} \tau\cdot\tilde{2}\kappa.$$

$\boxed{\text{In part, } \tau^2\tilde{2}\kappa = 0}$

This implies $d_3(h_0 h_4) = \overline{\tilde{2}\kappa} = h_0 d_0$. $\square$

$\tilde{2}\sigma^2$ contributes to $\pi_{14,3}$ not to $\pi_{14,2}$

Prop: $\pi_{14,*} = \mathbb{Z}/8[\tilde{2},\tau]\langle\kappa,\sigma^2\rangle / (\tilde{2}\sigma^2 = \tau\tilde{2}\kappa,\ \tau\cdot(\tilde{2}\sigma^2) = 0,\ \tilde{8}\kappa = 0)$.

The relation $\tilde{4}\sigma^2 = \tau\cdot\tilde{2}\kappa$ has one τ in it, causing a differential stretch by one.
In a precise way, these "hidden extensions" are equivalent to differential stretching.

Eg. Suppose we know $\partial(h_4) = \tilde{2}\sigma^2$ & $\partial(h_0 h_4) = \tau\tilde{2}\kappa$.

Observe: $h_0 h_4 = \tilde{2}\cdot h_4$. So

$$\tilde{2}\cdot(\tilde{2}\sigma^2) = \tilde{2}\,\partial(h_4) = \partial(\tilde{2}h_4) = \partial(h_0 h_4) = \tau\tilde{2}\kappa. \qquad \square$$

§2. Keeping track of filtration

In Sp, it is not always clear what a construction will do to the Adams filtration of a map.
In Syn, the Ad fil manifests geometrically, which makes studying this easier.

$$f, g, h \in \pi_* S \quad \text{with} \quad fg = 0 \ \& \ gh = 0 \ \longrightarrow \ \langle f, g, h \rangle \subseteq \pi_* S$$

in degree $|f| + |g| + |h| \underbrace{-1}$.

due to suspension

But there's indeterminacy, and we have no control over Adams filtration.

Eg. $\langle \overset{1}{\eta}, \overset{0}{2}, \overset{6}{\nu^2} \rangle = \{ \varepsilon, \varepsilon + \eta\sigma \} \subseteq \pi_8 S$.

If we have synthetic lifts $\tilde{f}, \tilde{g}, \tilde{h}$ and $\tilde{f}\tilde{g} = 0$ & $\tilde{g}\tilde{h} = 0$ still, then get

$$\langle \tilde{f}, \tilde{g}, \tilde{h} \rangle \subseteq \pi_{*,*} S \quad \text{in stem } |f| + |g| + |h| - 1$$

and filtr. $\mathrm{fil}(f) + \mathrm{fil}(g) + \mathrm{fil}(h) + 1$.

Eg. $\langle \underset{(1,1)}{\eta}, \underset{(0,1)}{\tilde{2}}, \underset{(6,2)}{\nu^2} \rangle = \{ \varepsilon \} \subseteq \pi_{8,3} S$.

So one can now define ε as $\langle \eta, \tilde{2}, \nu \rangle$.

Sometimes we need to insert τ's to make synthetic brackets be zero.

Eg. $\langle \sigma^2, \underset{= \tau\tilde{2}}{2}, \eta \rangle \subseteq \pi_{16,2} S$. This detects $[h_1 h_4]$.

$$\tilde{2}\sigma^2 \neq 0 \quad \text{while} \quad 2\sigma^2 = 0.$$

Have a procedure for evaluating Toda brackets with τ's in them: Moss' theorem.

"Keeping track of filtration" (also in more involved ways) is at the heart of Burklund–Hahn–Senger.

§4. Importing algebraic differentials

Warning: anachronistic perspective!

So far, we only looked internally at the sseq coming from a synthetic spectrum, and then used beefed up methods to compute this.
Now, we'll take a different approach: use Syn_{MU} to create a zig zag

$$\mathrm{ANSS}(\mathbb{S}) \leftarrow \{E_r^{*,*}\} \longrightarrow \text{"algebraic Novikov sseq"}$$

at which point you can import differentials.

$\Rightarrow$ Isaksen – Wang – Xu stem 90.

Important: previous things (total diff, $\mathbb{S}/\tilde{\delta}$, etc) could, with some extra work, have been done with filtered spectra instead.
This application is truly synthetic.

Idea: You can run the $\mathbb{F}_p$-Adams sseq in MU-synthetic spectra.

This will result in an E_2-term with an additional grading:

$$E_2^{n,w,s} = \mathrm{Ext}^{s,n+s,w-s}_{\mathcal{A}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \Rightarrow \pi_{n,w}\mathbb{S}_p^{\wedge}.$$

The w is new: it measures Adams–Novikov filtration.
$\rightsquigarrow$ Another case of "synthetic spectra help you keep track of the Adams filtration.

Observe: when invert τ, we obtain ordinary $\mathbb{F}_p$-ASS. So τ^{-1} gives a map

$$\nu\mathbb{F}_p\text{-ASS}(\nu\mathbb{S}) \longrightarrow \mathbb{F}_p\text{-ASS}.$$

Can compute E_2-page of sASS: looks similar, but with τ's adjoined. The additional grading is new information however.

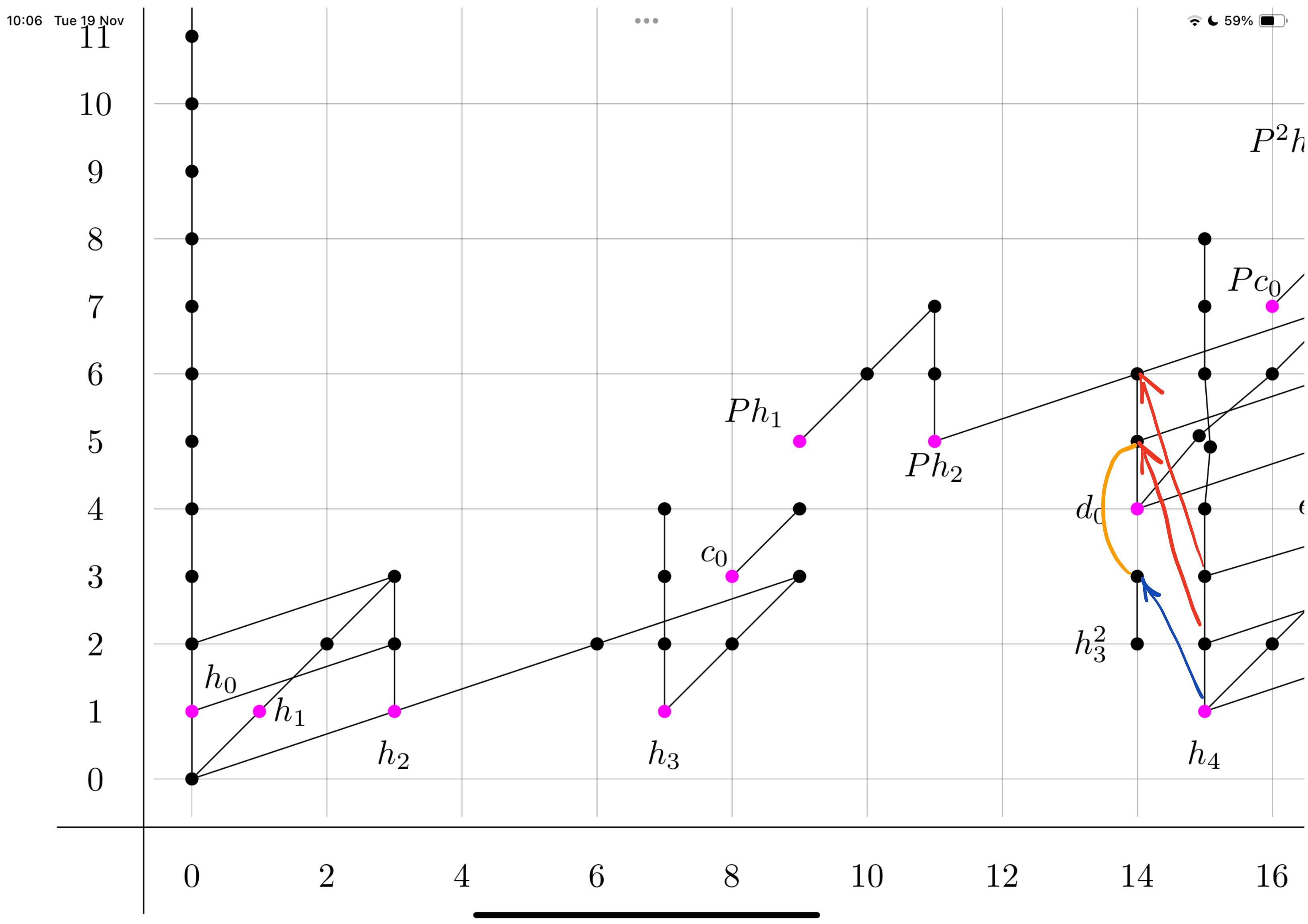
Isaksen: used the additional grading as a restraint on differentials.

Diff'l has to τ-invert to a classical one, and all classical differentials must arise in some way.
But sometimes, the third grading excludes differentials for degree reasons.

Deeper is to then also consider $C\tau \otimes -$. Then we get

$$(\nu\mathbb{F}_p)/\tau\text{-ASS}(C\tau) \text{ in } \mathrm{Stable}_{MU_*MU} \cong \text{algebraic Novikov sseq.}$$

Have two maps: $\nu\mathbb{S} \to C\tau$ and $C\tau \to \mathbb{S}^{1,-2}$. Can use both to obtain diff'ls in $\nu\mathbb{S}$. Then τ-invert.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