# Obstruction theories

Or: an ode to the homological t-structure (got 21 papers out of it)

$E$ Adams type (next week: let go of this)

**Problem**: Let $M \in \mathrm{Comod}_{E_*E}$. Is there an $X \in \mathrm{Sp}$ with $E_*(X) \cong M$ ?

or CAlg, etc.

Used a lot: Morava $E$-thy as $\mathbb{E}_\infty$-ring constructed like this, BP originally constructed like this, $A_\infty$-str's on $K(n)$'s, algebraicity results for spectra, rigidity results in higher algebra, ...

**Solution**: there are inductively defined obstructions

$$\varkappa_k \in \mathrm{Ext}^{k+2,k}_{E_*E}(M,M) \quad \text{or} \quad \mathrm{Ext}(\mathbb{L}_M, M) \text{ for } \mathbb{E}_\infty\text{-version}$$

which vanish $\iff$ such $X$ exists.

"linear" version: Toda obstruction theory

$\mathbb{E}_\infty$-version: Goerss–Hopkins obstruction theory

Why Ext groups? Why synthetic spectra?

A sseq is an obstruction theory for maps: $f \in \mathrm{Hom}_{E_*E}(E_*X, E_*Y)$

$$d_r(f) \in \mathrm{Ext}^{r,r-1}(E_*X, E_*Y) \text{ are obstructions to lifting}$$

**Clue**: sseq comes about by saying $\nu Y = (\nu Y)^\wedge_\tau = \lim_n \nu Y/\tau^n$

$$\mathrm{Map}(\nu X, \nu Y) \cong \lim_n \mathrm{Map}(\nu X, \nu Y/\tau^n).$$

This tells us how we should proceed:

$$M \in \mathrm{Comod}_{E_*E} \subseteq \mathrm{Mod}_{C\tau}(\mathrm{Syn}_E).$$

We have "$\nu X/\tau$"; our task is to build the rest.

**Rmk**: These things can be done much more generally: [Pstr.–VK], or [Barkan] for a formulation in terms of deformations with a good enough t-structure.

This cannot be done in FilSp: this is truly synthetic.

## §1. t-str

Recall: The homological t-str. on $\mathrm{Syn}_E$ has:

- $X \in \mathrm{Syn}_{\geq 0} \iff \nu E_{*,s}(X) = 0 \quad \forall s > 0.$
- $X \in \mathrm{Syn}_{\leq 0} \iff [\nu E_{*,s}(X) = 0 \ \forall s < 0$ and $X$ is $\nu E$-local$]$
- right complete (but in general not even left separated). ⚠
- $\nu X$ connective. Specifically:

$$\nu E_{*,*}(\nu X) \cong E_*(X)[\tau].$$

- $(\nu X)/\tau \cong \tau_{\leq 0}(\nu X)$.
- $\mathrm{Syn}^{\heartsuit} \simeq \mathrm{Comod}_{E_*E}$, and this sends

$$\tau_{\leq 0}(\nu X) \longmapsto E_*(X)$$

- $\mathrm{Mod}_{C\tau}(\mathrm{Syn}) \simeq \mathrm{Stable}_{E_*E}$ $\quad$ ($L_{\nu E}\,\mathrm{Mod}_{C\tau}(\mathrm{Syn}) \simeq D(\mathrm{Comod}_{E_*E})$).

In fact, Postnikov tower of $\nu X$ is its $\tau$-adic tower:

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\leq 1}(\nu X) \arrow[r] \arrow[d, phantom, "\parallel"] & \tau_{\leq 0}(\nu X) \arrow[d, phantom, "\parallel"] \\
\cdots \arrow[r] & \nu X/\tau^2 \arrow[r] & \nu X/\tau
\end{tikzcd} \quad (\circledast)$$

⚠ This is wrong for general synthetic spectra.

## §2. Potential k-stages

Need to characterise when an object could be called "$\nu X/\tau^k$".

Clue: $\nu E_{*,*}(\nu X/\tau^k) \cong E_*(X)[\tau]/\tau^k$.

The k-th term in the tower $\circledast$ has more structure than a synthetic spectrum: it's a $C\tau^{k+1}$-module, and the reduction map is a $C\tau^{k+1}$-module map. (Having both of these descriptions is useful !)

To abstract this notion, we need to take this info account.

Rmk: Let $M \in \mathrm{Syn}^{\heartsuit}$. Then $M$ gets a canonical module str. over $\tau_{\leq 0}\mathbb{S} = C\tau$.
More generally: $M \in \mathrm{Syn}_{\leq k}$ canonically a $\tau_{\leq k}\mathbb{S} = C\tau^{k+1}$-module.

**Def**: A potential $k$-stage ($k \geq 0$) is $M \in \mathrm{Mod}_{C\tau^{k+1}}(\mathrm{Syn})$ s.t.

(1) $M \in \mathrm{Syn}_{\geq 0}$

(2) $C\tau \otimes_{C\tau^{k+1}} M \xrightarrow{\simeq} \tau_{\leq 0} M$.

(nat. map is obtained via: $\tau_{\leq 0} M \in \mathrm{Syn}^{\heartsuit}$, so canonically a $C\tau$-module, then ext$^n$ of scalars)

**Rmk**: If (1) holds, then (2) is equiv. to: $C\tau \otimes_{C\tau^{k+1}} M$ is 0-truncated.

Write $\mathcal{M}_k \subseteq \mathrm{Mod}_{C\tau^{k+1}}(\mathrm{Syn})$ for full subcat. on potential $k$-stages.

**Eg.**
- Potential 0-stage: a discrete obj. of Syn $\overset{\text{earlier rmk}}{\Rightarrow}$ $\mathcal{M}_0 \simeq \mathrm{Syn}^{\heartsuit} \simeq \mathrm{Comod}$.
- Computation of $\nu E_{*,*}(\nu X)$ shows $\nu X/\tau^{k+1}$ is potential $k$-stage.

⚠ • $\mathcal{M}_k$ is not stable under $\Sigma$ or $\Omega$ & $\mathcal{M}_k$ has underlying obj. in $\mathrm{Syn}_{\leq k-1}$.

- Also not closed under $\otimes$ & Eg, $X \otimes_{C\tau} Y$ for $X, Y \in \mathrm{Syn}^{\heartsuit}$ is the derived tensor prod. $X \otimes^{\mathbb{L}} Y \in \mathcal{D}(\mathrm{Comod}) \subseteq \mathrm{Mod}_{C\tau}(\mathrm{Syn})$.

$\rightsquigarrow$ So $\mathcal{M}_0$ not closed under $\otimes$; similar phenomena for $\mathcal{M}_k$.

- $\mathcal{M}_k$ for $0 < k < \infty$ generally does not have finite limits (in part, not presentable)

**Rmk**: Most papers restrict to $\mathrm{Syn}_{\geq 0}$ in all of their discussion. For clarity, we keep this explicit. (Makes later things like "periodic obj's" less confusing).

By def., $C\tau^k \otimes_{C\tau^{k+1}} -$ restricts to $\mathcal{M}_k \to \mathcal{M}_{k-1}$.

$\Rightarrow$ $\cdots \to \mathcal{M}_k \to \cdots \to \mathcal{M}_1 \to \mathcal{M}_0 \simeq \mathrm{Syn}^{\heartsuit} \simeq \mathrm{Comod}_{E_*E}$.

For fixed $M \in \mathrm{Comod}_{E_*E}$, the pullback

$$\begin{tikzcd} \mathrm{Lift}^k(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_k \arrow[d, "C\tau \otimes_{C\tau^{k+1}} -"] \\ \{M\} \arrow[r] & \mathrm{Comod}_{E_*E} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Lift}^k_{k-1}(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_k \arrow[d] \\ \{M\} \arrow[r] & \mathcal{M}_{k-1} \end{tikzcd}$$

turns out to be a ($k$-truncated) space; the space of $k$-lifts. Our first goal is to understand this space, in part, when it's non-empty.

**Problem**: It's hard to understand $C\tau^k$-modules. We need to reduce it to algebra, ie, to $C\tau$-modules &

## §3. Defining the obstructions.

Have fibre seq. $\Sigma^{0,-k} C\tau \xrightarrow{\tau^k} C\tau^{k+1} \to C\tau^k$. (think $\mathbb{Z}/p \xrightarrow{p^k} \mathbb{Z}/p^{k+1} \to \mathbb{Z}/p^k$.)

**Thm:** Have a square-zero structure

$$\begin{tikzcd} C\tau^{k+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C\tau^k \arrow[d, "d"] \\ C\tau^k \arrow[r, "d_0"'] & C\tau^k \oplus \Sigma^{1,-k-1} C\tau \end{tikzcd}$$

$d_0$ inclusion first comp. ("trivial derivation")

$d$ derivation classifying $C\tau^{k+1}$.

$C\tau^k \oplus \Sigma^{1,-k-1} C\tau$ ⇝ trivial square-zero extension

**Prop:**

$$\begin{tikzcd} \mathrm{Mod}_{C\tau^{k+1}}(\mathrm{Syn}_{\geq 0}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Mod}_{C\tau^k}(\mathrm{Syn}_{\geq 0}) \arrow[d] \\ \mathrm{Mod}_{C\tau^k}(\mathrm{Syn}_{\geq 0}) \arrow[r] & \mathrm{Mod}_{C\tau^k \oplus \Sigma^{1,-k-1} C\tau}(\mathrm{Syn}_{\geq 0}) \end{tikzcd}$$

So $M \in \mathrm{Mod}_{C\tau^{k+1}}$ with $M$ connective $\iff (N, P, \alpha : d^* P \cong d_0^* N)$

$\iff (N, P, \alpha' : P \to \underbrace{d_* d_0^* N}_{=: \Theta}$ s.t. $\alpha$ is isom.$)$

For $p : C\tau^k \oplus \Sigma^{1,-k-1} C\tau \to C\tau^k$ projection have $d_0 \circ p = d \circ p = \mathrm{id}$.

$\Rightarrow$ nat. transf. $\pi : \Theta N = d_* d_0^* N \longrightarrow d_* p_* p^* d_0^* N \cong N$.

**Observe:** $\alpha$ is an iso $\iff \pi \circ \alpha' : P \to \Theta N \to N$ is an iso.

This reformulation results in:

**Prop:** $\mathrm{Mod}_{C\tau^{k+1}} \simeq \{ (N \in \mathrm{Mod}_{C\tau^k},\ s : N \to \Theta N,\ h : \pi \circ s \simeq \mathrm{id}) \}$.

"$\infty$-category of sections of $\Theta \to \mathrm{id}$".

**Key last step:** $\Theta N \xrightarrow{\pi} N$ has a section $\iff N \to \mathrm{cofib}\,\pi$ is null.

Can compute: $\mathrm{cofib}\,\pi \simeq \Sigma^{2,-k-2} \underbrace{C\tau \otimes_{C\tau^k} N}$. DOUBLE CHECK !

discrete $\Rightarrow$ canonically a $C\tau$-module ?

So $N \to \mathrm{cofib}\,\tau$ is null $\iff$ $C\tau \otimes_{C\tau^k} N \to \mathrm{cofib}\,\tau$ is null as a map of $C\tau$-modules.

This last map defines an elt of

$$[C\tau \otimes_{C\tau^k} N, \mathrm{cofib}\,\tau]_{C\tau} = [C\tau \otimes_{C\tau^k} N, \Sigma^{2,-k-2} C\tau \otimes_{C\tau^k} N]_{C\tau}.$$
$$= \mathrm{Ext}^{k+2,k}_{E_*E}(C\tau \otimes_{C\tau^k} N, C\tau \otimes_{C\tau^k} N).$$

Cor: Let $M \in \mathrm{Comod}_{E_*E}$. Then $M$ lifts to $m_k$ $\iff$

$$x_i \in \mathrm{Ext}^{i+2,i}_{E_*E}(M, M).$$

vanishes for $i = 0, \ldots, k$.

## §4. The limit

If we could build a tower

$$\cdots \longrightarrow \text{"}\nu X/\tau^{2}\text{"} \longrightarrow \text{"}\nu X/\tau\text{"}$$

then the limit will be the synthetic analogue of a spectrum with the desired E-homology. However, we want more: we want to know a description of maps between such (synthetic) spectra too. (That would in particular tell us about uniqueness.)

$\nu : Sp \to Syn_E$ is fully faithful (it's "normalised correctly" to answer such a question.)

But: for $X \in Sp$, in general: $\nu X \not\simeq \lim \nu X/\tau^k$.
This is a equiv. $\iff$ ASS for $X$ converges.
So can only hope to work with $Sp_E$.

(probably the same as asking: $X/\tau \in Syn_{\leq 0}$)

Lemma: $X \in Syn$ in image of $\nu$ $\iff$ $[X \in Syn_{\geq 0}$ & $C\tau \otimes X \xrightarrow{\simeq} \tau_{\leq 0} X]$.

Proof "$\Leftarrow$": Show $X = \tau_{\geq 0}(X[\tau^{-1}])$. $\square$

Prop: $L_{\nu E} Syn$ is left complete $\implies$ $L_{\nu E} Syn \longrightarrow \lim_k \mathrm{Mod}_{C\tau^k}(Syn)$ is equiv.

this in part. requires all Postnikov towers to converge in Syn.
This implies $\nu X$ to be $\tau$-complete for all E-local $X \in Sp$.

This is very strong ASS convergence. Requires things like E = Morava E-thy

Some fixes: can pull back to specific $M \in Syn$?

**Thm:** Suppose $L_{\nu E}\mathrm{Syn}$ is left complete. Then

$$\{E\text{-local spectra}\} \simeq \lim_k \mathcal{M}_k.$$

Requiring left completeness of $L_{\nu E}\mathrm{Syn}_E$ is very strong, and not needed if you're only interested in one lifting problem at a time.

**Eg** $L_{\nu\mathbb{F}_2}\mathrm{Syn}_{\mathbb{F}_2}$ is not left complete. But if $M \in \mathrm{Comod}_{A_*}$ is bounded below (in grading), then

$$\mathcal{M}_\infty \times_{\mathcal{M}_0} \{M\} \simeq \lim_k \left(\mathcal{M}_k \times_{\mathcal{M}_0} \{M\}\right).$$

So we still get an obstruction theory for $A_*$-comodules. Toda obstr. theory.

## §5. Applications

**Morava E-theory:** Have $E(k, \mathbb{G}) \in \mathrm{CAlg}(h\mathrm{Sp})$.

Show: all obstructions to lifting to $\mathbb{E}_\infty$-ring vanish

$\rightsquigarrow$ functor $\{\text{formal groups}\} \longrightarrow \mathrm{CAlg}(\mathrm{Sp})$.

**Chromatic algebraicity:** $\exists$ ab. cat. $\mathcal{A}$ s.t.

$$h\,\mathrm{Sp}_{E(h)} \simeq hD(\mathcal{A}).$$

Proof shows that $\mathcal{M}_k(\mathrm{Syn}_{E(h)})$ and $\mathcal{M}_k(\mathrm{Syn}(D(\mathcal{A})))$ have equiv. htpy cats. for certain $k$, and that the original htpy cats can be recovered from the $\mathcal{M}_k$ for $k$ in the same range.

**Separability:** An algebra is separable if $\mu: A\otimes A \longrightarrow A$ admits a section of $(A,A)$-bimodules.

Then $\mathrm{CAlg}^{sep}(\mathrm{Sp}) \xrightarrow{\simeq} \mathrm{CAlg}^{sep}(h\mathrm{Sp})$. (!!!)