Goal: Motivate & explain some constructions. (Those who were in Bonn might recognise some ideas from William's talk: I would say William's ideas are generalisations of these)

§1. General Adams sseq's

Def: $\mathcal{C}$ stable ∞-cat, $\mathcal{A}$ ab. cat.
A **homological functor** is $h: \mathcal{C} \to \mathcal{A}$ s.t.

$$X \to Y \to Z \text{ cofib.} \Rightarrow h(X) \to h(Y) \to h(Z).$$

Let $\mathcal{A}$ be equipped with an auto-equiv. $[1]: \mathcal{A} \to \mathcal{A}$.
A **homology theory** $\mathcal{C} \to (\mathcal{A}, [1])$ is homological functor h together with "suspension iso's":

$$h(\Sigma X) \cong h(X)[1].$$

Eg. $\mathcal{A} = \mathrm{grAb}$, $[1]$ is grading shift: $(M[1])_n = M_{n-1}$.

Then $E \in \mathrm{Sp}$ yields hlgy thy $E_*(-): \mathrm{Sp} \to \mathrm{grAb}$.

Def: A homological functor $h: \mathcal{C} \to \mathcal{A}$ is **adapted** if

1) $\mathcal{A}$ has enough injectives.

2) $\forall I \in \mathcal{A}$ injective, $\exists X_I \in \mathcal{C}$ s.t.

$$[-, X_I]_{\mathcal{C}} \cong \mathrm{Hom}_{\mathcal{A}}(h(-), I).$$

if exists, then determined uniquely up to isom. by Yoneda

3) $h(X_I) \cong I$.

Rmk. (1) & (2) is usually formal (Brown representability).
(3) is very not formal: that's where the content is.

Idea: h adapted means: $\mathcal{C}$ has an h-ASS whose E_2-page is computed by $\mathrm{Ext}_{\mathcal{A}}$.

Eg. $A \in \mathrm{CAlg}(\mathrm{Sp})$. Then $\pi_*: \mathrm{Mod}_A \to \mathrm{Mod}_{A_*}$ is adapted.

If I injective, then by Brown $\exists X_I$ as in (2). But:

$$\pi_n X_I = [\Sigma^n A, X_I]_A \cong \mathrm{Hom}_{A_*}(A_*[n], I) \cong I_n.$$

So $\pi_* X_I \cong I$. Resulting ASS is the **Ext-sseq**.

⚠ Adaptedness can be destroyed by precomposing with other functors.

Eg. $H_*(-;\mathbb{F}_p)$ is $Sp \longrightarrow Mod_{\mathbb{F}_p}(Sp) \xrightarrow{\pi_*} grVect_{\mathbb{F}_p}$.

Then $\mathbb{F}_p \in grVect_{\mathbb{F}_p}$ is injective, and $\mathbb{F}_p$ is injective lift:

$$Hom_{\mathbb{F}_p}(H_*(X;\mathbb{F}_p), \mathbb{F}_p) \cong H^*(X;\mathbb{F}_p) \cong [X, \mathbb{F}_p].$$

But: $H_*(\mathbb{F}_p;\mathbb{F}_p) = \mathcal{A}_*$ is very different from $\pi_*(\mathbb{F}_p) = \mathbb{F}_p$.

Reason this fails: $H_*(X;\mathbb{F}_p)$ as $\mathbb{F}_p$-v.s. does not remember enough about X. Need to "add information" ⇝ $\mathcal{A}_*$-comodules.

$$\begin{tikzcd}
Sp \arrow[r] \arrow[d, dashed, "H_*(-;\mathbb{F}_p)"'] & Mod_{\mathbb{F}_p}(Sp) \arrow[d, "\pi_*"] \\
Comod_{\mathcal{A}_*} \arrow[r] & grVect_{\mathbb{F}_p}
\end{tikzcd}$$

Thm: E Adams type. Then $E_*(-): Sp \to Comod_{E_*E}$ is adapted.

This leads to the E-ASS as before. (This is not nec. the one from $E^{\otimes \bullet+1}$)

Thm: Let $h: \mathcal{C} \to \mathcal{A}$ be homological functor. Then $\exists$ adapted $H: \mathcal{C} \to \mathcal{B}$ together with an exact functor $\mathcal{B} \to \mathcal{A}$ and

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[dr, "h"'] & \mathcal{B} \arrow[d, "\#"'] \\
& \mathcal{A}
\end{tikzcd}$$

Moreover, $\mathcal{B}$ is comonadic over $\mathcal{A}$ for a left exact comonad.

Eg. $Comod_{\mathcal{A}_*} \cong Comod_C(grVect_{\mathbb{F}_p})$ for $C: V \mapsto \mathcal{A}_* \otimes_{\mathbb{F}_p} V$.

Rmk:
- This $(\mathcal{B}, H)$ is unique if it exists: adapted hlgy thy's form a poset
- The proof does not help in finding an explicit $\mathcal{B}$ in practise.

Later: can build a "synthetic category" $Syn_h(\mathcal{C})$ when h adapted.

§2. Synthetic A-modules (Take it slow: split it up into two steps.)

$A \in CAlg$. Define synthetic cat for the Ext sseq for Mod_A:

Def. $$Syn_A(Mod_A) := \underbrace{PSh_\Sigma}_{\text{presheaves preserving finite products}}(\underbrace{Mod_A^{ff}}_{\text{finite \& free}}; Sp).$$

← closed under ⊗ in Mod_A ⇒ get Day convolution

Think: $\mathrm{Syn}_A(\mathrm{Mod}_A)$ is associated to adapted $\pi_*: \mathrm{Mod}_A \to \mathrm{Mod}_{A_*}(\mathrm{grAb})$

Gets t-structure: use levelwise t-str. using standard one on Sp.
Then

$$\begin{aligned}\mathrm{Syn}_A(\mathrm{Mod}_A)^\heartsuit &\simeq \mathrm{PSh}_\Sigma(\mathrm{Mod}_A^{ff}; \mathrm{Set})\\ &\simeq \mathrm{PSh}_\Sigma(\mathrm{Mod}_A^{ff}; \mathrm{Ab})\\ &\simeq \mathrm{PSh}_\Sigma(h\mathrm{Mod}_A^{ff}; \mathrm{Ab})\\ &\simeq \mathrm{Mod}_{A_*}(\mathrm{grAb})\end{aligned}$$

because we're looking at prod.-pres. presheaves,

(this sounds fancy but it's not)

$h\mathrm{Mod}_A^{ff} \simeq \mathrm{Mod}_{A_*}^{ff}(\mathrm{grAb})$ is even an equivalence of Lawvere theories

Def: $\nu: \mathrm{Mod}_A \to \mathrm{Syn}_A(\mathrm{Mod}_A)$ is $M \mapsto \mathrm{Map}(-, M)$.

Here uses: $\mathrm{PSh}_\Sigma(\mathrm{Mod}_A^{ff}; \mathcal{S}) \simeq \mathrm{PSh}_\Sigma(\mathrm{Mod}_A^{ff}; \mathrm{Sp}_{\geq 0}) \subseteq \mathrm{PSh}_\Sigma(\mathrm{Mod}_A^{ff}; \mathrm{Sp})$

Then ν lands in $\mathrm{Syn}^{\geq 0}$. Have:

$$\mathrm{Mod}_A \xrightarrow{\nu} \mathrm{Syn}^{\geq 0} \xrightarrow{\tau_{\leq 0}} \mathrm{Syn}^\heartsuit \simeq \mathrm{Mod}_{A_*}(\mathrm{grAb}) \text{ is } \pi_*$$

Could say: I'm not gonna explain everything in terms of this construction: I already showed how everything can be done in terms of the abstract properties.

But: $\mathrm{PSh}_\Sigma(\mathrm{Mod}_{A_*}^{ff}; \mathrm{Sp}) \simeq \mathcal{D}(A)$. ⚠

Prop: ν is fully faithful.

Proof: fully faithful on Mod_A^{ff} by Yoneda. But also preserves filtered colimits. □

§3. Synthetic spectra

We essentially have to modify the previous setting to "make it adapted" again.

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "E\otimes -"] \arrow[d] & \mathrm{Mod}_E \arrow[d, "\pi_*"] \\ \mathrm{Comod}_{E_*E} \arrow[r] & \mathrm{Mod}_{E_*} \end{tikzcd}$$

this is adapted if E Adams type

this picture is only true if E is highly structured.

Let E be htpy assoc. ring sp.

Def: $\mathrm{Sp}_E^{fp} \subseteq \mathrm{Sp}^\omega$ those P s.t. E_*P projective ($\Leftrightarrow$ dualisable).
Equip Sp_E^{fp} with a designated class of "epimorphisms": the E_*-surjections.

Def: $\mathrm{Syn}_E \subseteq \mathrm{PSh}_\Sigma(\mathrm{Sp}_E^{fp}, \mathrm{Sp})$ full subcat. on those F s.t.

⊛ $X \to Y \to Z$ cofib. seq. of spectra in Sp_E^{fp} with $\underset{\text{surj}}{E_*Y \to E_*Z}$ $\Rightarrow$ $F(Z) \to F(Y) \to F(X)$ cofib. seq.

It's very convenient to be able to phrase this in the language of sheaves.

Make Sp_E^{fp} a site: a cover of P is $\{Q \to P\}$ single map with $E_*Q \twoheadrightarrow E_*P$ surj.

sheaf condition is similar to comonad in the adapted factorisation

Thm [Pstrągowski]: $\circledast \Leftrightarrow$ F a sheaf for this topology.

Rmk. We really have a functor $\mathrm{Alg}(h\mathrm{Sp}) \longrightarrow \{\text{sites}\}$, so also get a functor $\mathrm{Syn}_{(-)}(\mathrm{Sp}) : \mathrm{Alg}(h\mathrm{Sp}) \longrightarrow \mathrm{Pr}^L_{st}$. (in fact, have both adjoints)

Thm [Goerss–Hopkins]: (A,Γ) Hopf algebroid of Adams type. Then

$$\mathrm{Comod}_{(A,\Gamma)} \simeq \mathrm{Sh}_\Sigma(\mathrm{Comod}^{fp}_{(A,\Gamma)}; \mathrm{Set}).$$

coverings are surjections

Thm [Pstrągowski]: Let E be Adams type. Then

$$\mathrm{Sh}_\Sigma(\mathrm{Comod}^{fp}_{E_*E}; \mathrm{Set}) \simeq \mathrm{Sh}_\Sigma(\mathrm{Sp}^{fp}_E; \mathrm{Set}).$$

induced by $E_*(-) : \mathrm{Sp}^{fp}_E \to \mathrm{Comod}^{fp}_{E_*E}$.

Also: $\mathrm{Sp} \xrightarrow{\sim} \mathrm{Syn}_E^{\geq 0} \to \mathrm{Syn}_E^{\heartsuit}$ is $E_*(-)$.

$\Rightarrow$ only for those E do we understand Syn_E as defined like this

But: sheaves of spectra are different:

$$\mathrm{Sh}_\Sigma(\mathrm{Comod}^{fp}_{E_*E}; \mathrm{Sp}) \simeq \mathrm{Stable}_{E_*E}. \rightsquigarrow \simeq \mathrm{Mod}_{C\tau}(\mathrm{Syn}_E\mathrm{Sp}).$$

"Algebra can only reconstruct the E_2-page".

(but this inclusion is subtle)

Still have $\mathrm{Sh}_\Sigma(-;\mathcal{S}) \simeq \mathrm{Sh}_\Sigma(-;\mathrm{Sp}_{\geq 0}) \subseteq \mathrm{Sh}_\Sigma(-;\mathrm{Sp})$, so can define

Def: $\nu : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E(\mathrm{Sp})$, $X \mapsto \mathrm{Map}(-,X)$.

This is fully faithful by the same argument.

Turns out: hypercomplete $\Leftrightarrow$ νE-local (hypercomplete: receiving no maps from ∞-conn.)

§4. Filtered models

(Essentially in [BHS, App. C], but this slightly different perspective will be in [Barkan–vN.])

Recall: $\mathrm{FilSp} \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \mathrm{Syn}_E$.

Thm: This is monadic $\Leftrightarrow$ Syn_E is generated under colimits by $\{\mathbb{S}^{n,s}\}_{n,s}$.
If this happens, we say Syn_E is cellular. $\rightsquigarrow$ [Lawson], [Barkan–vN.].

Remains to identify the monad on FilSp. Can identify $\sigma(\nu\mathbb{S})$, but we need to identify this as an $\mathbb{E}_\infty$-alg. For that, need lots of structure on E:

Prop: If E is $\mathbb{E}_\infty$ and E-ASS($\mathbb{S}$) conv. cond., then $\mathrm{Syn}_E \simeq \mathrm{Mod}_{\mathrm{ASS}_E(\mathbb{S})}(\mathrm{Fil}\,\mathrm{Sp})$. Eg. MU.

§4. Recollection on derived ∞-cats.

Def: A prestable ∞-cat. is an ∞-cat. $\mathcal{C}$ s.t. $\exists$ stable $\mathcal{D}$ with t-str. s.t. $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$.

A Grothendieck prestable ∞-cat. is a prestable one that's presentable and where filtered colimits are left exact.

If $\mathcal{C}$ is prestable and $\mathcal{C} = \mathcal{D}_{\geq 0}$, then notion of "n-truncated obj." in $\mathcal{C}$ agrees with the t-structure one on $\mathcal{D}$. $\mathcal{C}_{\leq n} \simeq (\mathcal{D}_{\geq 0})_{\leq n}$.
Write $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\leq 0}$. ("Right complete t-str" $\iff$ $\mathcal{D} = \mathrm{Sp}(\mathcal{D}_{\geq 0}) = \mathrm{Sp}(\mathcal{C})$)

Proto-theorem: if $\mathcal{A}$ ab. cat, $\mathcal{E}$ prestable, then

$$\mathrm{Fun}^{ex}(\mathcal{D}_{\geq 0}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{Fun}^{ex}(\mathcal{A}, \mathcal{E}^{\heartsuit}).$$

To make precise, need some hypotheses:

$\mathrm{Fun}^{ex}(\mathcal{D}^{b}_{\geq 0}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{Fun}^{ex}(\mathcal{A}, \mathcal{E}^{\heartsuit})$. $\mathcal{E}$ finite limits [PP, Thm. 5.18]

$\mathrm{LFun}^{ex}(\check{\mathcal{D}}_{\geq 0}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{LFun}^{ex}(\mathcal{A}, \mathcal{E}^{\heartsuit})$ $\mathcal{E}$ Grothendieck

$\mathrm{LFun}^{ex}(\mathcal{D}_{\geq 0}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{LFun}^{ex}(\mathcal{A}, \mathcal{E}^{\heartsuit})$ $\mathcal{E}$ Grothendieck & separated [SAG, Thm. C.5.4.9]

$\mathcal{D}(\mathcal{A}) = \check{\mathcal{D}}(\mathcal{A}) / \{\infty\text{-connective objects}\}$. "hypercompletion"

Construction [Patchkoria–Pstrągowski]: For $\mathcal{C}$ stable, compactly gen., $h: \mathcal{C} \to \mathcal{A}$ adapted, $\mathcal{A}$ Grothendieck abelian, h pres. arbitrary sums

$\rightsquigarrow$ $\check{\mathcal{D}}(\mathcal{C}; h)$ stable ∞-cat. with right complete t-str.
$\check{\nu}: \mathcal{C} \to \check{\mathcal{D}}_{\geq 0}(\mathcal{C}; h)$.

$\rightsquigarrow$ $\mathcal{D}(\mathcal{C}; h) := \check{\mathcal{D}}(\mathcal{C}; h) / \{\infty\text{-conn.}\}$

Similar properties as Syn_E, but notably:

1) no monoidal str. in general.

2) t-str. slightly less structured: in general, there's no object X s.t. (co)connectivity is measured by $\pi_{*,*}(X \otimes -)$.

maybe should emphasise this more in other notes ?

Still, $\check{\mathcal{D}}^{\heartsuit}(\mathcal{C}; h) \simeq \mathcal{A}$, and even:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\check{\nu}"] \arrow[rr, bend right, "h"'] & \mathcal{D}_{\geq 0}(\mathcal{C}; h) \arrow[r, "\tau_{\leq 0}"] & \mathcal{D}^{\heartsuit}(\mathcal{C}; h) \simeq \mathcal{A} \end{tikzcd}$$

So t-str. htpy groups is homology with values in $\mathcal{A}$.

In fact, it's the universal such pair:

Thm: If $\mathcal{E}$ is Groth. prestable, then

$$G : \check{D}_{\geq 0}(\mathcal{C};h) \to \mathcal{E} \quad \Longleftrightarrow \quad G_0 : \mathcal{A} \to \mathcal{E}^\heartsuit \quad + \quad \begin{tikzcd} & & \mathcal{E} \arrow[d, "\tau_{\leq 0}"] \\ \mathcal{C} \arrow[r, "h"] \arrow[urr, bend left, "\tilde{G}_0"] & \mathcal{A} \arrow[r, "G_0"] & \mathcal{E}^\heartsuit \end{tikzcd}$$

exact, colim-pres. ⟺ exact, colim-pres.

"prestable enhancement"

Construction: idea is: for Syn, the equiv. $\mathrm{Sh}_\Sigma(\mathrm{Sp}_E^{fp};\mathrm{Set}) \simeq \mathrm{Comod}_{E_*E}$ was key. To generalise this, would need to find a suitable version of this more gen.

Universal case: $\mathcal{A} = \mathrm{PSh}_\Sigma(\mathcal{C}^\omega; \overset{\text{or Ab}}{\mathrm{Set}})$ (think: ridiculous enlargement of π_*).

and $h : X \mapsto [-,X]_\mathcal{C}$. (preserves ⊕ by compactness).

Then define $\check{D}(\mathcal{C};h) := \mathrm{PSh}_\Sigma(\mathcal{C}^\omega; \mathrm{Sp})$

Compactly gen. case: $h : \mathcal{C} \to \mathcal{A}$ ⟺ quotient $\mathrm{PSh}_\Sigma(\mathcal{C}^\omega; \mathrm{Ab}) / \mathcal{K}$ localising subcat. $\mathcal{K}$.

Lift $\mathcal{K}$ to $\mathrm{PSh}_\Sigma(\mathcal{C}^\omega;\mathrm{Sp})$: take smallest loc. subcat. $\mathcal{L}$ containing $\mathcal{K} \subseteq \mathrm{PSh}_\Sigma(\mathcal{C}^\omega)^\heartsuit$.

Then define $D(\mathcal{C};h) := \mathrm{PSh}_\Sigma(\mathcal{C}^\omega;\mathrm{Sp}) / \mathcal{L}$.

§ Synthetic spectra as a derived ∞-cat.

Consider $E_*(-) : \mathrm{Sp} \to \mathrm{Comod}_{E_*E}$ ⇝ $\check{D}(\mathrm{Sp}; E_*(-))$.

⚠ $\mathrm{Syn} \not\simeq D$, but rather $\widehat{\mathrm{Syn}} \simeq D$! (but $\widehat{\mathrm{Syn}} \not\simeq \hat{D}$!!)

Prop: $\mathrm{Syn}_E(\mathrm{Sp}) \simeq \mathrm{Ind}\,\mathrm{Thick}(\check{\nu}P \mid P \in \mathrm{Sp}_E^{fp})$.

Now, Sp_E^{fp} has monoidal str. Give $\mathrm{Thick}(\check{\nu}P)$ a mon. str. by declaring $\check{\nu}$ to be symmetric monoidal on $P \in \mathrm{Sp}_E^{fp}$. Then Ind cat also gets a mon. str, and this is equiv. to $(\mathrm{Syn}_E, \otimes)$.

Sometimes this becomes simpler: if E is a field, then simply

$$\mathrm{Syn}_E \simeq \check{D}(\mathrm{Sp}; E_*(-)).$$

In part., such equiv's give Syn_E a new univ. property.

Variant: version of E-ASS defined via $E^{\otimes \bullet+1}$. By categorical constr., there is ab. cat. $\mathcal{A}$ with $F : \mathrm{Sp} \to \mathcal{A}$ adapted giving rise to this. But not clear how to describe this in general.

If $P \in \mathrm{Sp}_E^{fp}$, then $\mathrm{Ext}_\mathcal{A}(F(P), F(X)) \cong \mathrm{Ext}_{E_*E}(E_*P, E_*X)$.