Corrections to proofs in "An equilibrium closure result for discontinuous games"*

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G. Carmona has kindly pointed out to me that two proofs in my paper "An equilibrium closure result for discontinuous games" (Economic Theory **48** (2011), 47-65), a paper from now on referred to as ECR, contain errors. Here I correct these errors. As a consequence, all results in ECR, and in particular the main result in Theorem 1, remain valid as stated. Below notation, references, etc. are borrowed freely from ECR.

Proof of Lemma 5 in ECR. The proof must be corrected as follows. By compactness of S and Z the function $s \mapsto \phi_i(s) := \inf_{z \in L_s} z_i$ is not upper semicontinuous, as I claimed in ECR, but lower semicontinuous (as a direct consequence of Berge's theorem). That still causes $(s, z) \mapsto \phi_i(s) - z_i$ to be Borel measurable on $S \times Z$, so the set E, consisting of all $(s, z) \in L$ such that $\phi_i(s) - z_i \geq 0$, is Borel measurable. Now observe that for every $s \in S$ the section E_s, which is precisely equal to $\text{argmin}_{z \in L_s} z_i$, is compact. This allows me to apply Theorem 1 in reference [7] of ECR, and it yields that $s \mapsto \text{argmin}_{z \in L_s} z_i$ has a measurable selection $\hat{q}^i : S \to Z$, just as was stated in Lemma 5.

Proof of Case 2 on p. 64 of ECR. As stated, the proof of case 2 makes sense until its last line: "So (19) now follows from ... $\hat{q}^i_i \leq q^{**}_i$". This line must be replaced by an application of the following lemma, which implies that (19) of ECR is indeed valid in case 2.[1]

Lemma *For every fixed* $\bar{s}_i \in S_i$

$$\int_S q_i^{**} d\tilde{\beta} \geq \int_{S_{-i}} \hat{q}_i^i(\bar{s}_i, \cdot) d\tilde{\beta}_{-i}. \quad (1)$$

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[1]In addition, in lines 15-16 of p. 64 $\hat{q}^i$ must be read as $\hat{q}^i_i$.

Proof. By assumption (3) in ECR there exists a subsequence $\{S_i^{(n')}\}_{n'}$ (which is $\bar{s}_i$-dependent, of course) and associated elements $\bar{s}_{i,n'} \in S_i^{(n')}$ such that $\bar{s}_i = \lim_{n' \to \infty} \bar{s}_{i,n'}$. By the definition of the mixed Nash equilibrium profiles $(\alpha_i^{(n)})_i$ in ECR, we have

$$\int_{S^{(n')}} q_i^{(n')} d\tilde{\alpha}^{(n')} \geq \int_{S_{-i}^{(n')}} q_i^{(n')}(\bar{s}_{i,n'}, \cdot) d\tilde{\alpha}_{-i}^{(n')} \text{ for every } n'. \tag{2}$$

To handle the right side in (2), I define a sequence $\{\rho^{(n')}\}_n$ in $\text{Prob}(S_{-i} \times Z)$ by setting $\rho^{(n')}(A \times B) := \tilde{\alpha}_{-i}^{(n')}(A \cap (\bar{q}^{(n')})^{-1}(B))$, where $\bar{q}^{(n')} := q^{(n')}(\bar{s}_{i,n'}, \cdot) : S_{-i} \to Z$ defines a continuous function (use (4) in ECR). The definition of $\rho^{(n')}$ is motivated by the identity

$$\int_{S_{-i}^{(n')}} q_i^{(n')}(\bar{s}_{i,n'}, \cdot) d\tilde{\alpha}_{-i}^{(n')} = \int_{S_{-i} \times Z} z_i \rho^{(n')}(d(s_{-i}, z)), \tag{3}$$

and its definition mimicks the definition of $\pi^{(n)} \in \text{Prob}(S \times Z)$, which plays a major role in ECR. Recall here the following standard convention: any probability measure on $S_{-i}^{(n')}$, such as $\tilde{\alpha}_{-i}^{(n')}$, is automatically extended to a probability measure on the larger set S_{-i}, simply by setting it to zero on $S_{-i} \backslash S_{-i}^{(n')}$; see p. 49 in ECR. I can now mimick arguments given in ECR by replacing the $\pi^{(n)}$ by the current $\rho^{(n')}$. This gives me (i)–(iii) stated below. Namely, just as in the proof of Lemma 1 in ECR, the tightness of $\{\rho^{(n')}\}_{n'}$, which results from combining the compactness of Z with the weak convergence of $\{\tilde{\alpha}_{-i}^{(n')}\}_{n'}$ to $\tilde{\beta}_{-i}$ as in ECR, implies

(i) $\{\rho^{(n')}\}_{n'}$ *contains a subsequence* $\{\rho^{(n'')}\}_{n''}$ *which converges weakly to some limit probability measure* ρ^*

by applying both Prohorov's theorem and its converse (both are recalled in ECR). Here I accept that both $\{\rho^{(n'')}\}_{n''}$ and ρ^* may depend on the initial choice of the fixed $\bar{s}_i \in S_i$. By Propositions 2 and 3 of ECR it follows from (i) that the support $\text{supp}\, \rho^*$ is contained in $\text{Ls}_{n''}\text{supp}\, \rho^{(n'')} \subset \text{Ls}_{n'}\text{supp}\, \rho^{(n')}$, whence in $\text{Ls}_{n'}\{(s_{-i}, \bar{q}^{(n')}(s_{-i})) : s_{-i} \in S_{-i}^{(n')}\}$. By $\bar{s}_{i,n'} \to \bar{s}_i$ and the definition of the Kuratowski limes superior set $L := \text{Ls}_n\{(s, q^{(n)}(s)) : s \in S^{(n)}\}$ in Lemma 1 of ECR, this implies

(ii) $\text{supp}\, \rho^*$ *is contained in the section* $L_{\bar{s}_i} := \{(s_{-i}, z) \in S_{-i} \times Z : (\bar{s}_i, s_{-i}, z) \in L\}$.

Because $\tilde{\alpha}_{-i}^{(n'')}$ is the S_{-i}-marginal of $\rho^{(n'')}$ for every n'', I also conclude directly from (i) that

(iii) $\tilde{\beta}_{-i}$ *is the marginal of* ρ^* *on* S_{-i};

cf. step 1 in the proof of Lemma 2 in ECR. Now for $n' \to \infty$ the left side in (2) converges to the left side of (1) by Lemma 4.b in ECR. Also, if I replace n' on the right of (2) by n'', then the resulting expression converges to $\int_{S_{-i} \times Z} z_i \rho^*(d(s_{-i}, z))$ for $n'' \to \infty$, by (i) and the identity (3). So by the above I obtain

$$\int_S q_i^{**} d\tilde{\beta} \geq \int_{S_{-i} \times Z} z_i \rho^*(d(s_{-i}, z)) = \int_{L_{\bar{s}_i}} z_i \rho^*(d(s_{-i}, z)),$$

where the equality on the right holds by (ii). Finally, I observe that for all $(s_{-i}, z) \in L_{\bar{s}_i}$ the definition of $\hat{q}^i$ gives $z_i \geq \hat{q}_i^i(\bar{s}_i, s_{-i})$, so the rightmost expression above is larger than or equal to

$$\int_{L_{\bar{s}_i}} \hat{q}_i^i(\bar{s}_i, s_{-i})\rho^*(d(s_{-i}, z)) = \int_{S_{-i}\times Z} \hat{q}_i^i(\bar{s}_i, s_{-i})\rho^*(d(s_{-i}, z)) = \int_{S_{-i}} \hat{q}_i^i(\bar{s}_i, \cdot)d\tilde{\beta}_{-i},$$

where I use *(ii)* for the first identity and *(iii)* for the second one. This proves (1).