General duality – highlights 6

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Recall from Thm. 3.1(iii):

$$\begin{aligned} h \textbf{ continuous} \text{ at } 0 \ &\Rightarrow \ \partial h(0) \neq \emptyset \Rightarrow \\ \Rightarrow \ \text{set of dual solutions} \ &= \ -\partial h(0) \neq \emptyset \end{aligned}$$

Seek to relax such continuity for special perturbation functions h, namely *polyhedral* h.

Def. $S \subset \mathbb{R}^n$ is **polyhedral** if

$$S = \cap_{j=1}^{J} \{x \in \mathbb{R} : y_j^t x \leq \alpha_j\}$$

for some $\{y_1, \ldots, y_J\} \subset \mathbb{R}$ and $\{\alpha_1, \ldots, \alpha_J\} \subset \mathbb{R}$. This is *special* type of convex set.

Def. $f : \mathbb{R}^n \to [-\infty, +\infty]$ is **polyhedral** if epi $f \subset \mathbb{R}^n \times \mathbb{R}$ is polyhedral. This is *special* type of convex function.

Prop. 1.2. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$, $f \not\equiv +\infty$, be polyhedral. Then f is of the following form: there exist a polyhedral set $P \subset \mathbb{R}^n$ and affine functions $a_1, \ldots, a_N : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \chi_P(x) + \max_{1 \le i \le N} a_i(x).$$

Moreover,

$$\partial f(x_0) \neq \emptyset \text{ for every } x_0 \in \text{dom } f.$$

Obs: The latter does **not** require the usual condition

$$f \text{ continuous at } x_0 \;\Leftrightarrow x_0 \in \textbf{int} \text{ dom } f \quad (*)$$

Example. a. Recall: for $n = 1$ the convex non-polyhedral function

$$f(x) := \begin{cases} 1 - \sqrt{1 - x^2} & \text{if } |x| \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

had $\partial f(1) = \emptyset$ "because" $(*)$ is not valid for $x_0 = 1$.

b. However, the polyhedral function

$$f(x) := \begin{cases} |x| & \text{if } |x| \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

has $\partial f(1) \neq \emptyset$, although $(*)$ is not valid for $x_0 = 1$.

Fenchel's duality theorem can now be modified: if (P_F) is polyhedral, then the sufficient condition for strong stability becomes much weaker than before:

Let A be $m \times n$-matrix. Consider

$$(P_F) \quad \inf_{x \in \mathbb{R}^n} [f(x) + g(Ax)],$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ and $g : \mathbb{R}^m \to (-\infty, +\infty]$ are **polyhedral** functions. Suppose $\inf(P_F) \in \mathbb{R}$.

Define associated *Fenchel dual problem*:

$$(D_F) \quad \sup_{q \in \mathbb{R}^m} [-f^*(A^t q) - g^*(-q)].$$

Fenchel's duality thm – polyhedral version

(i) For all $x \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$

$$-f^*(A^t q) - g^*(-q) \leq f(x) + g(Ax).$$

(ii) If $0 \in \cancel{\mathrm{int}}(\text{ dom } g - A(\text{dom } f))$, then

$$\inf(P_F) = \max(D_F).$$

Moreover, then $\bar{x} \in \mathbb{R}^n$ is optimal for (P_F) and $\bar{q} \in \mathbb{R}^k$ is optimal for (D_F) if and only if

$$A^t \bar{q} \in \partial f(\bar{x}) \text{ and } -\bar{q} \in \partial g(A\bar{x}).$$

Application: semidefinite programming duality

$\mathbb{S}^n$: set of all symmetric $n \times n$-matrices

$\mathbb{S}^n$ is Euclidean vector space with inner product

$$\langle X, Y \rangle := \operatorname{tr}(XY) = \sum_{i,j} X_{i,j} Y_{i,j}.$$

Two special convex cones in $\mathbb{S}^n$:

$\mathbb{S}^n_+$: set of all **positive semidefinite** matrices

$\mathbb{S}^n_{++}$: set of all **positive definite** matrices

Recall: $A \in \mathbb{S}^n_+ \Leftrightarrow \forall_x x^t A x \geq 0$

Recall: $A \in \mathbb{S}^n_{++} \Leftrightarrow \forall_{x \neq 0} x^t A x > 0$

Fact: $\mathbb{S}^n_+$ is closed and $\mathbb{S}^n_{++} = \operatorname{int} \mathbb{S}^n_+$.

Let $A_i \in \mathbb{S}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$.

The **primal semidefinite program** is

$$(P_{SDP}) \inf_{X \in \mathbb{S}^n_+} \{\mathrm{tr}(CX) : \mathrm{tr}(A_i X) = b_i, i = 1, \ \ldots, m\}.$$

Can be written as

$$(P_{SDP}) \inf_{X \in \mathbb{S}^n_+} \{\langle C, X \rangle : \langle A_i, X \rangle = b_i, i = 1, \ \ldots, m\},$$

Compare to standard linear program

$$\inf_{x \in \mathbb{R}^n_+} \{\langle c, x \rangle_{\mathbb{R}^n} : \langle a_i, x \rangle_{\mathbb{R}^n} = b_i, \ i = 1, \ \ldots, m\},$$

Define **dual problem** of (P_{SDP}) by

$$(D_{SDP}) \quad \sup_{q \in \mathbb{R}^m} \{b^t q : C - \sum_{i=1}^m q_i A_i \in \mathbb{S}^n_+\}.$$