General duality II – highlights 7

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$\mathbb{S}^n$: set of all symmetric $n \times n$-matrices

$\mathbb{S}^n$ is Euclidean space, i.e. vector vector space with

$$\langle X, Y \rangle := \text{tr}(XY) = \sum_{i,j} X_{i,j} Y_{i,j}.$$

as inner product. Then for every $X \in \mathbb{S}^n$

$$X = \sum_{i,j,j \geq i} X_{i,j} E^{i,j} = \sum_{i,j,j \geq i} \langle X, E^{i,j} \rangle E^{i,j}.$$

Here for $k < l$:

$$(E^{k,l})_{i,j} := \begin{cases} \frac{1}{2}\sqrt{2} & \text{if } (i,j) \in \{(k,l),(l,k)\} \\ 0 & \text{otherwise} \end{cases}$$

and for $k = l$

$$(E^{k,k})_{i,j} := \begin{cases} 1 & \text{if } (i,j) = (k,k) \\ 0 & \text{otherwise} \end{cases}$$

Two special convex cones in $\mathbb{S}^n$:

$\mathbb{S}^n_+$: set of all **positive semidefinite** matrices

$\mathbb{S}^n_{++}$: set of all **positive definite** matrices

Fact: $\mathbb{S}^n_+$ is closed and $\mathbb{S}^n_{++} = \operatorname{int} \mathbb{S}^n_+$.

Let $A_i \in \mathbb{S}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$.

The **primal semidefinite program** is

$(P_{SDP}) \inf_{X \in \mathbb{S}^n_+} \{\operatorname{tr}(CX) : \operatorname{tr}(A_i X) = b_i, i = 1, \ldots, m\}$.

Can be written as

$$(P_{SDP}) \quad \inf_{X \in K} \{\langle C, X \rangle : \mathcal{A}(X) = b\}.$$

Here $K := \mathbb{S}^n_+$ (whence $K^* = -\mathbb{S}^n_+$) and

$$\mathcal{A}(X) := (\langle A_i, X \rangle)_{i=1}^m .$$

Compare to standard linear primal program

$$\inf_{x \in \mathbb{R}^n_+} \{< c, x >: Ax = b\},$$

with $< c, x >:= c^t x =$ inner product on $\mathbb{R}^n$.

Define **semidefinite dual problem** by

$$(D_{SDP}) \quad \sup_{q \in \mathbb{R}^m} \{b^t q : C - \sum_{i=1}^m q_i A_i \in \mathbb{S}^n_+\}.$$

Compare to standard linear dual program

$$\sup_{q \in \mathbb{R}^m} \{b^t q : c - A^t q \in \mathbb{R}^n_+\}.$$

SDP duality theorem

(i) If (P_{SDP}) has feasible solution in $\mathbb{S}^n_{++}$, then

$$\inf(P_{SDP}) = \max(D_{SDP}),$$

provided $\sup(D_{SDP}) \in \mathbb{R}$.

(ii) If $\exists_{\tilde{q}\in\mathbb{R}^m}\ C - \tilde{q}_i A_i \in \mathbb{S}^n_{++}$ then

$$\min(P_{SDP}) = \sup(D_{SDP}),$$

provided $\inf(P_{SDP}) \in \mathbb{R}$.

Follows by Fenchel duality thm. in $\mathbb{S}^n$, specialized to

$$f(X) := \langle C, X\rangle + \chi_K(X), g(y) := \chi_{\{b\}}(y).$$

Here $g^*(z) = b^t z$, whence $-g^*(-q) = b^t q$, and

$$f^*(Y) = \sup_{Y\in K} \langle Y - C, X\rangle = \begin{cases} 0 & \text{if } Y - C \in K^* \\ +\infty & \text{otherwise} \end{cases}$$

whence

$$f^*(\mathcal{A}^* q) = \begin{cases} 0 & \text{if } \mathcal{A}^* q - C \in K^* = -\mathbb{S}^n_+ \\ +\infty & \text{otherwise} \end{cases}$$

Here $\mathcal{A}^*$, the **adjoint** of $\mathcal{A}$, is defined by

$$\forall_{q\in\mathbb{R}^m, X\in\mathbb{S}^n} < q, \mathcal{A}(X) >= \langle \mathcal{A}^* q, X\rangle,$$

which gives

$$\mathcal{A}^* q = \sum_{i=1}^m q_i A_i.$$