Applications of duality – highlights 8

November 2010

Description of **zero-sum** game.

Two players, I ("row player") and II ("column player").

Player I's actions: numbered $1, \ldots, m$.

Player II's actions: numbered $1, \ldots, n$.

Payoff matrix: $m \times n$-matrix $P = (p_{ij})_{i,j}$.

Interpretation: p_{ij} Euros is what I must pay to II if

I chooses action I and II chooses action j.

Def: A **pure equilibrium pair** is pair of actions $(\bar{i}, \bar{j})$ such that

$$\forall_{1 \leq i \leq m} \;\; p_{\bar{i}\bar{j}} \leq p_{i\bar{j}} \text{ and } \forall_{1 \leq j \leq n} \;\; p_{\bar{i}\bar{j}} \geq p_{\bar{i}j}.$$

i.e., such that

$$\max_j p_{\bar{i}j} = p_{\bar{i}\bar{j}} = \min_i p_{i\bar{j}}.$$

Obs: equilibrium discourages unilateral deviations.

However, pure equilibrium rarely exists.

Remedy: ”gamble if you must” (von Neumann and gamblers (± 1700))

Def: A **mixed action** for player I is a probability vector from

$$S_I := \{x \in \mathbb{R}^m_+ : \sum_{i=1}^{m} x_i = 1\}$$

and a **mixed action** for player II is a probability vector from

$$S_{II} := \{u \in \mathbb{R}^n_+ : \sum_{j=1}^{n} u_j = 1\}$$

Observation 1: degenerate mixed actions (= unit vectors) give original actions.

Net result of choices x by I and u by II:

payoff outcome p_{ij} gets probability $x_i \times u_j$.

Observation 2: presumes independence.

Hence *expected* payoff for player I is

$$E(x, u) := \sum_{i=1}^{m} \sum_{j=1}^{n} x_i u_j p_{ij} = x^t P u$$

and for player II it is $-E(x, u)$.

Def. A **mixed equilibrium pair** is pair $(\bar{x}, \bar{u})$ in $S_I \times S_{II}$ such that

$\forall_{x \in S_I} E(\bar{x}, \bar{u}) \leq E(x, \bar{u})$ and $\forall_{u \in S_{II}} E(\bar{x}, \bar{u}) \geq E(\bar{x}, u)$,

so again unilateral deviations are disadvantageous.

Above inequalities can be rewritten as

$$\forall_{x \in S_I, u \in S_{II}} E(\bar{x}, u) \leq E(\bar{x}, \bar{u}) \leq E(x, \bar{u})$$

and also as

$$\max_{u \in S_{II}} \bar{x}^t P u = \bar{x}^t P \bar{u} = \min_{x \in S_I} x^t P \bar{u}.$$

Observe, similar to LP, that

$$\max_{u \in S_{II}} \bar{x}^t P u = \max_j (P^t \bar{x})_j = \max_j \bar{x}^t P^j,$$

where $P^j :=$ j-th column of P.

Likewise

$$\min_{x \in S_I} x^t P \bar{u} = \min_i (P \bar{u})_i.$$

Define player I's optimization problem as

$$(P_I) \quad \inf_{x \in S_I} \max_j x^t P^j.$$

Minimizes I's maximum expected amount to be paid.

Define player II's optimization problem as

$$(P_{II}) \quad \sup_{u \in S_{II}} \min_i (Pu)_i.$$

Maximizes II's minimum expected amount to be received.

Obs: $\inf(P_I)$ and $\sup(P_{II})$ are *attained* (Weierstrass).

Trick: for every $x \in S_I$

$$\max_j x^t P^j = \inf\{r \in \mathbb{R} : r \geq x^t P^j \forall_{1 \leq j \leq n}\}.$$

So can rewrite (P_I) equivalently as convex program:

$$(P) \inf_{x \geq 0, r \in \mathbb{R}} \{r : x^t P^j - r \leq 0, j = 1, \ldots, n, 1 - \sum_{i=1}^{m} x_i = 0\}.$$

Observe: Slater's CQ holds for (P).

For Lagrangian dual define

$$\theta(u, v) := \inf_{x \geq 0, r \in \mathbb{R}} r + \sum_j u_j (x^t P^j - r) + v(1 - \sum_i x_i).$$

Then calculation gives

$$\theta(u, v) = \begin{cases} v & \text{if } \sum_j u_j = 1 \text{ and } v \leq \min_i (Pu)_i, \\ -\infty & \text{otherwise.} \end{cases}$$

Lagrangian dual (D) of (P) is

$$\sup_{u \geq 0, v} \theta(u, v) = \sup_{u \in S_I, v \leq \min_i (Pu)_i} v = \sup_{u \in S_I} \min_i (Pu)_i,$$

so equivalent to player II's problem (P_{II}).

Conclusion: $\bar{v} := \min(P_I) = \max(P_{II})$

Consequences:
(i) a mixed equilibrium pair exists.

(ii) a pair $(\bar{x}, \bar{u}) \in S_I \times S_{II}$ is mixed equilibrium pair if and only if

$$\bar{x} \text{ optimal for } (P_I) \text{ and } \bar{u} \text{ optimal for } (P_{II})$$

(iii) (by CS): if $(\bar{x}, \bar{u})$ is mixed equilibrium pair, then

$\forall_i \ \bar{x}_i((P\bar{u})_i - \bar{v}) = 0$ (equalizing property for I),

i.e., $\bar{x}_i > 0 \Rightarrow (P\bar{u})_i = \bar{v}$, and

$\forall_j \ \bar{u}_j((P^t\bar{x})_j - \bar{v}) = 0$ (equalizing property for II),

i.e., $\bar{u}_j > 0 \Rightarrow (P^t\bar{x})_j = \bar{v}$.

Example. Let

$$P = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

Observe (P_I) can be reduced to interval optimization:

$$(\mathbb{I}) \quad \inf_{0 \leq x_2 \leq 1} \max_{1 \leq j \leq 4} [p_{1j}(1 - x_2) + p_{2j}x_2],$$

which gives $\bar{x}_2 = 2/5$ and then $\bar{x}_1 = 3/5$.

Hence $\bar{v} = \inf(P_I) = \inf(\mathbb{I}) = 3$ follows.

Use contrapositive equalizing property for I:

$\bar{x}^t P = (14/5, 11/5, 3, 3) \Rightarrow \bar{u}_1 = \bar{u}_2 = 0.$

Next, use equalizing property for II:
$\bar{x}_1, \bar{x}_2 > 0 \Rightarrow (P\bar{u})_1 = (P\bar{u})_2 = \bar{v} = 3$

Hence, $\bar{u}_3 + 5\bar{u}_4 = 3$ and $6\bar{u}_3 = 3$, so $\bar{u} = (0, 0, 1/2, 1/2)^t$.