On subdifferential calculus – highlights 3

Let $f, g_1, \cdots, g_m : \mathbb{R}^n \to (-\infty, +\infty]$ be convex. Let $S \subset \mathbb{R}^n$ be convex. Let

$$Z := \{x \in S : g_1(x) \leq 0, \cdots, g_m(x) \leq 0\}.$$

Consider the convex optimization problem:

$$(P) \quad \inf_{x \in Z} f(x).$$

Fix $\bar{x} \in Z$. Let $I(\bar{x}) := \{i : g_i(\bar{x}) = 0\}$.

List of abbreviations used below:

REG = *regularity* means

$$\bar{x} \in \text{int dom } f \cap \cap_{i \in I(\bar{x})} \text{int dom } g_i.$$

CS = *complementary slackness* means

$$\bar{u}_i g_i(\bar{x}) = 0 \text{ for } i = 1, \cdots, m$$

NLI = *normal Lagrange inclusion* means

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{u}_i \partial g_i(\bar{x}) + \bar{\eta}$$

LI = *Lagrange inclusion* means

$$0 \in \bar{u}_0 \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{u}_i \partial g_i(\bar{x}) + \bar{\eta}.$$

(hence **NLI** means: **LI** with $\bar{u}_0 = 1$).
OAP = *obtuse angle property* means:

$$\bar{\eta}^t(x - \bar{x}) \leq 0 \text{ for all } x \in S (\Leftrightarrow \bar{\eta} \in N_S(\bar{x})).$$

Theorem 3.1: (i) Sufficient for optimality:

$$\exists_{\bar{u} \in \mathbb{R}^m_+, \bar{\eta} \in \mathbb{R}^n} \textbf{CS, NLI, OAP} \Rightarrow \bar{x} \text{ optimal.}$$

(ii) Necessary for optimality: if **REG** then

$\bar{x}$ optimal $\Rightarrow \exists_{\bar{u}_0 \in \{0,1\}, \bar{u} \in \mathbb{R}^m_+, \bar{\eta} \in \mathbb{R}^n, (\bar{u}_0, \bar{u}) \neq (0,0)}$**CS, LI, OAP**.

Remark 3.2: NLI $\Rightarrow$ **MP**. Conversely, **MP** $\Rightarrow$ **NLI** if $\bar{x} \in \text{int dom } f \cap \cap_{i \in I(\bar{x})} \text{int dom } g_i \cap S$. Here **MP** = *minimum principle*:

$$\bar{x} \in \text{argmin}_{x \in S}[f(x) + \sum_{i \in I(\bar{x})} \bar{u}_i g_i(x)].$$

Remark 3.3 (Slater's constraint qualification): If $\exists_{\tilde{x} \in S} \forall_i g_i(\tilde{x}) < 0$, then in Theorem 3.1$(ii)$ **LI** is always **NLI**, i.e., $\bar{u}_0 = 1$.

Proof of Theorem 3.1. Write $I := I(\bar{x})$. (i) By Remark 3.2 and **CS**:

$$\forall_{x \in S} f(x) + \sum_{i \in I} \bar{u}_i g_i(x) \geq f(\bar{x}).$$

Hence, for any $x \in Z$ we have

$$f(x) \geq f(x) + \sum_{i \in I} \bar{u}_i g_i(x) \geq f(\bar{x}).$$

So $\bar{x}$ is optimal.

(ii) Consider the auxiliary optimization problem

$$(P') \quad \inf_{x \in S} \phi(x),$$

where $\phi(x) := \max[f(x) - f(\bar{x}), \max_{1 \leq i \leq m} g_i(x)]$. Then $\bar{x}$ is also optimal for (P') By small KKT Thm. 2.10 and Remark 2.11 $\exists_\eta$ with **OAP** and with $-\eta \in \partial\phi(\bar{x})$. By D-M Theorem 3.1 this gives

$$-\eta \in \partial\phi(\bar{x}) = \mathrm{co}(\partial f(\bar{x}) \cup \cup_{i \in I} \partial g_i(\bar{x})).$$

Thus, there are $u_i \geq 0$ with $\sum_{i \in \{0\} \cup I} u_i = 1$ and $\xi_0 \in \partial f(\bar{x})$, $\xi_i \in \partial g_i(\bar{x})$, $i \in I$, with

$$-\eta = \sum_{i \in \{0\} \cup I} u_i \xi_i.$$

Case 1: $u_0 = 0$. Then set $\bar{u}_i := u_i$ for $i \in \{0\} \cup I$ and $\bar{u}_i := 0$ otherwise. Note: $\bar{u} \neq 0$ by $\sum_{i \in I} u_i = 1$. Also set $\bar{\eta} := \eta$.
Case 2: $u_0 \neq 0$. Then $u_0 > 0$, so set $\bar{u}_i := u_i/u_0$ for $i \in \{0\} \cup I$ and $\bar{u}_i := 0$ otherwise. Also set $\bar{\eta} := \eta/u_0$. QED

Consider additionally: a $p \times n$-matrix A and $b \in \mathbb{R}^p$. Define $L := \{x : Ax = b\}$. Consider the *convex* *programming problem*

$(P) \inf_{x\in S}\{f(x) : g_1(x) \le 0, \cdots, g_m(x) \le 0, Ax-b=0\}.$

In previous notation:

$$(P) \quad \inf_{x\in Z\cap L} f(x).$$

Fix $\bar{x} \in Z \cap L$. Let $I(\bar{x}) := \{i : g_i(\bar{x}) = 0\}$.

"Convex" KKT theorem – inequalities and equalities

Corollary 3.5 (i) Sufficient for optimality:

$\exists_{\bar{u}\in\mathbb{R}^m_+,\bar{v}\in\mathbb{R}^p,\bar{\eta}\in\mathbb{R}^n}$**CS, NLI', OAP** $\Rightarrow \bar{x}$ optimal.

(ii) Necessary for optimality: if **REG'** then

$\bar{x}$ optimal $\Rightarrow \exists_{\bar{u}_0\in\{0,1\},\bar{u}\in\mathbb{R}^m_+,\bar{\eta}\in\mathbb{R}^n,(\bar{u}_0,\bar{u})\neq(0,0)}$**CS, LI', OAP**.

Here **REG'** means

$\bar{x} \in \text{int dom } f \cap \cap_{i\in I(\bar{x})}\text{int dom } g_i$ and int $S \cap L \neq \emptyset$.

NLI' = *normal Lagrange inclusion* means

$$0 \in \partial f(\bar{x}) + \sum_{i\in I(\bar{x})} \bar{u}_i \partial g_i(\bar{x}) + A^t\bar{v} + \bar{\eta}$$

LI' = *Lagrange inclusion* means

$$0 \in \bar{u}_0\partial f(\bar{x}) + \sum_{i\in I(\bar{x})} \bar{u}_i \partial g_i(\bar{x}) + +A^t\bar{v} + \bar{\eta}.$$

Proof. Claim: $\partial\chi_L(\bar{x}) = \text{im } A^t$. Now

$$\eta \in \partial\chi_L(\bar{x}) \Leftrightarrow \forall_{x, A(x-\bar{x})=0} \eta^t(x - \bar{x}) \stackrel{!}{=} 0.$$

So $\eta \in \partial\chi_L(\bar{x}) \Leftrightarrow \eta \in ((\text{im } A^t)^\perp)^\perp$. Hence

$$\eta \in \partial\chi_L(\bar{x}) \Leftrightarrow \eta \in \text{im } A^t.$$

Observe

$$(P) \quad \inf_{x \in S'} \{f(x) : g_1(x) \le 0, \cdots, g_m(x) \le 0\},$$

where $S' := S \cap L$. Thus, parts (i) and (ii) follow by Theorem 3.1, but now $\bar{\eta}$ as in Theorem 3.1 is replaced by $\eta' \in \partial\chi_{S'}$. From M-R Theorem:

$$\partial\chi_{S'}(\bar{x}) = \partial\chi_S(\bar{x}) + \partial\chi_L(\bar{x}),$$

because of int $S \cap L \neq \emptyset$. Therefore, $\eta' = \bar{\eta} + \eta$, with $\bar{\eta} \in \partial\chi_S(\bar{x})$ (i.e., **OAP**) and $\eta \in \partial\chi_L(\bar{x})$. By above

$$\exists_{\bar{v} \in \mathbb{R}^m} \eta = A^t \bar{v}.$$

QED