Homework CAO, Part 1 – October, 2012

Due: Turn in your solutions by 1 December, 11 p.m. Make precisely 8 out of 12 problems.

Problem 1. Do "the remainder of the proof is left as an exercise" on p. 12 of [OSC], just before Theorem 2.17.

Problem 2. Let $f : S \to (-\infty, +\infty]$ be a function on the convex set $S \subset \mathbb{R}^n$

a. Prove that f is quasiconvex on S if and only if for every pair $x, y \in S$ and every $\lambda \in [0, 1]$ $f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y))$ (recall the definition of quasiconvexity from Exercise 2.2).

b. Let $f : [a, b] \to \mathbb{R}$ be a continuous and quasiconvex function on the interval $[a, b]$. Prove that at least one of the following is true: (i) f is (monotonically) nondecreasing , (ii) f is nonincreasing or (iii) there exists a real number c, with $a < c < b$, such that f is nonincreasing on $[a, c]$ and nondecreasing on $[c, b]$.

Problem 3. Exercise B.2 in [OSC] (i.e., of "On Subdifferential Calculus").

Problem 4. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an arbitrary function. Use the Fenchel-Moreau theorem to prove the following identity: for every $x_0 \in \mathbb{R}^n$

$$f^{**}(x_0) = \sup_{q, q \leq f} \{q(x_0) : q : \mathbb{R}^n \to (-\infty, +\infty] \text{ is l.s.c. and convex}\},$$

i.e., $f^{*}*$ is the largest lower semicontinuous and convex function that is less than or at most equal to f.

Problem 5. Let $f(x_1, x_2) := \exp(x_1 + x_2^2)$.
a. Demonstrate: f is convex on $\mathbb{R}^2$.
b. Calculate explicitly the function f^*.
c. Calculate explicitly the function f^{**} and check that it is equal to f.

Problem 6. Exercise 2.12 of [OSC].

Problem 7. Exercise 2.15 of [OSC].

Problem 8. Exercise 2.17 of [OSC]

Problem 9. Exercise 2.18 of [OSC]

Problem 10. Exercise 2.20 of [OSC]

Problem 11. Exercise 3.1 of [OSC]

Problem 12. Let $p : \mathbb{R}^n \to (-\infty, +\infty]$ be positively homogeneous (i.e., $p(\alpha x) = \alpha p(x)$ holds for all $\alpha \geq 0$ and all $x \in \mathbb{R}^n$), **lower semicontinuous** and subadditive (i.e., $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathbb{R}^n$). Prove: there exists a closed convex set $C \subset \mathbb{R}^n$ such that $p(x) = \sup_{\xi \in C} \xi^t x$ for each $x \in \mathbb{R}^n$. *Hint:* The desired C must satisfy $p = \chi_C^*$.