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FATOU'S LEMMA FOR MULTIFUNCTIONS WITH UNBOUNDED VALUES

ERIK J. BALDER AND CHRISTIAN HESS

For multifunctions having unbounded sets as values we give Fatou-type inclusions for the Kuratowski limes superior, in finite as well as infinite dimensions. These are derived from similar, known Fatou-type inequalities for single-valued multifunctions (i.e., ordinary functions), that is, from Balder's unifying Fatou lemma in case the image set is finite-dimensional, and from an update of related results by Balder in the infinite dimensional case. For this extension from the single-valued to the multivalued situation a lemma due to Hess, used to prove earlier Fatou-type inclusions, is of critical importance, Also, an asymptotic correction term, introduced here, plays an important role. The two main results thus obtained subsume and extend an entire class of comparable Fatou lemmas.

1. Introduction. Fatou's lemma in several dimensions, formulated for ordinary functions (i.e., single-valued multifunctions) is known to be of some use in the theory of competitive equilibria (Hildenbrand 1974) and in proving existence results for variational problems in economics and optimal control theory (Balder 1984 a, c). The first such lemmas were given by Aumann (1965) and Schmeidler (1970), and the result was successively extended by Hildenbrand-Mertens (1971), Cesari-Suryanarayana (1978), and Artstein (1979) in different directions. All these results were unified by Balder (1984a), who based his proof on the tightness approach for Young measures. A formulation in terms of polar cones was given later in Olech (1987) and Balder (1991) (see also Balder 1988); it has also been adopted here (cf. Page 1991 for a recent development in another direction). Infinite-dimensional results of a simpler, approximate nature were obtained by Khan-Majumdar (1984), Yannelis (1988, 1991), Balder (1988), and Castaing (1987).

Fatou-type results for multifunctions were motivated by developments in mathematical economics. At first, results of this kind were derived very directly from the single-valued Fatou lemmas mentioned above (Aumann 1965, Schmeidler 1970, Hildenbrand 1974, Yannelis 1988). Later, also a different approach, based on the use of support functions, was pursued in infinite dimensions by Pucci-Vitillaro (1984), Hiai (1985), (Aumann, 1965, Schmeidler 1970, Hildenbrand 1974, Yannelis 1988) and Hess (1986, 1991); this approach leans heavily on the approximate nature of Fatou-type results in infinite dimensions, and would therefore seem to be hardly of any use in finite dimensions, i.e., in the most challenging case. The results in Pucci and Vitillaro (1984, Theorem 3.5) and Hiai (1985, Theorem 2.8(2)), concern multifunctions with bounded values. In Hess (1991, Theorem 4.3) (see also Hess (1986, Theorem 3.14) Hess gave the first Fatou-type result for multifunctions with unbounded values. Such an extension of the scope of Fatou-type lemmas is highly desirable, since it applies to epigraphic multifunctions, canonically associated to normal integrands. The proof of Hess consists of an adaptation of Hiai's proof by means of a truncation argument.

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In this paper we confirm the soundness of the more traditional approach to Fatou lemmas for multifunctions, by showing that single-valued results can be used in very general multivalued situations, both in finite and infinite dimensions. The introduction of an asymptotic correction term in the critical inclusion affords formulations of a great precision. Our main Fatou lemma in finite dimensions, Theorem 3.2, is entirely new. Also, Theorem 3.3, our main approximate Fatou lemma in infinite dimensions, substantially generalizes the results of Hiai (1985), Hess (1991) and several others. Moreover the precision of Theorem 3.2 is so great that the unifying Fatou lemma of Balder (1984a), upon which the proof in the finite-dimensional case rests, also follows again as a specialization to single-valued multifunctions (Corollary 4.3). In infinite dimensions the same holds true for the single-valued Fatou lemmas in Balder (1988, Theorems 2.1, 2.4), which are subsumed and sharpened in Corollary 4.4. An auxiliary result from convex analysis, due to Hess (1991, Lemma 4.1), plays an important role in deriving the multivalued Fatou lemmas from the single-valued ones, in both finite and infinite dimensions.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a separable Banach space (in the finite-dimensional case we simply set $X =: \mathbf{R}^d$, and take for $\|\cdot\|$ the Euclidean norm). As usual, the *radius* of a bounded set A (its maximum distance to the origin) is denoted by $\|A\| := \sup_{x \in A}\|x\|$. The dual of X will be denoted by X^*; the dual norm on X^* is denoted by $\|\cdot\|^*$ and the corresponding closed unit ball of X^* by B^*. Topological notions on X will be referred to by the symbol s or w, according to whether we consider them for the *strong* (i.e., norm-) topology, or the *weak* topology (i.e., $\sigma(X, X^*)$). Of course, in the finite-dimensional situation these topologies coincide with the usual Euclidean topology. Moreover, it is standard to indicate *sequential* versions of notions for the w-topology by adding 'seq' to the notation. For instance, s-cl A commonly indicates the norm-closure of a set A, and w-seq-cl A the weak sequential closure of A. A subset A of X is said to be *locally w-compact* if every point of A has an open neighborhood which is relatively w-compact. Also A is said to be w-ball compact if it has a w-compact intersection with every closed ball.

For any sequence (A_k) of subsets of X[1] the *sequential Kuratowski w-limes superior* $w\text{-seq-Ls}_k\, A_k$ is defined as the set (possibly empty) of all $x \in X$ for which there exists a subsequence (A_{k_j}) of (A_k) such that $x = w\text{-lim}_j\, x_{k_j}$ for certain $x_{k_j} \in A_{k_j}$. In case the A_k happen to be singletons $\{a_k\}$, we write $w\text{-seq-Ls}_k a_k$ instead of the more formal expression $w\text{-seq-Ls}_k\{a_k\}$. We shall also encounter the *nonsequential* Kuratowski limes superior $w\text{-Ls}_k A_k$ of (A_k), which is defined as the intersection of all closures $w\text{-cl}\bigcup\{A_k\colon k \geqslant p\}$, $p \in \mathbf{N}$. Of course, the inclusion

$$w\text{-seq-Ls}_k A_k \subset w\text{-Ls}_k A_k$$

holds always. Clearly, when the topology on X is metrizable (as in the finite-dimensional situation) it becomes an identity, but below we shall give a more interesting situation where the identity is also valid.

Convex analysis, and especially asymptotic cones, will play a role in this paper; everything needed in this respect can be found in Castaing and Valadier (1977, I). We summarize some key notions and their notation: For any nonempty subset A of X the *support function* of A is the functional $x^* \to s(x^*|A)$ from X^* into $(-\infty, +\infty]$, with

$$s(x^*|A) := \sup_{x \in A} \langle x, x^* \rangle.$$

[1]By such notation the domain of the index k is tacitly understood to be the set $\mathbf{N}$ of natural numbers.

Recall that the *effective domain* dom $s(\cdot|A)$ of this support function is the convex cone of all $x^* \in X^*$ for which $s(x^*|A) < +\infty$. The *polar cone* of a cone $C \subset X^*$ is defined as the set of all $x \in X$ such that $\langle x, x^* \rangle \leqslant 0$ for all $x^* \in C$; this polar cone is denoted by C^*. The *asymptotic cone* As A of an arbitrary set $A \subset X$ is defined as the polar cone of dom $s(\cdot|A)$; thus, by definition, it coincides with As(cl co A), the asymptotic cone of the closed convex hull of A. When A is closed and convex, the asymptotic cone As A can be characterized as the largest cone D in X for which $x + D \subset A$ for every $x \in A$.

Further, we follow Hess (1990) in introducing the following notation. By $\mathscr{R}$ we denote the collection of all w-closed and w-ball-compact subsets of X. Note already that when X is finite-dimensional (or even reflexive), $\mathscr{R}$ consists precisely of all w-closed subsets of X. Also, by $\mathscr{L}$ we denote the collection of all sets whose closed convex hull is locally w-compact and does not contain any line. As observed in Hess (1990), any w-closed set in $\mathscr{L}$ must belong to $\mathscr{R}$. Note also that when X is finite-dimensional, any closed set is automatically locally compact; hence, in that case $\mathscr{L}$ consists of all sets whose closed convex hull does not contain any lines. As promised above, we can now state an interesting situation where the identity

$$w\text{-seq-Ls}_k A_k = w\text{-Ls}_k A_k$$

is valid: it occurs when there exists a set $C \in \mathscr{L}$ such that $\bigcup_k A_k \subset C$. This follows by a simple application of Hess (1990, Proposition 3.7).

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Recall that such as measure space can always be decomposed into a *purely atomic* part Ω^{pa} and a *nonatomic* part Ω^{na} (Hildenbrand 1974). Recall also that the *outer integral* over Ω of a nonmeasurable, extended real-valued function $\psi: \Omega \to [-\infty, +\infty]$ is defined by

$$\int_\Omega^* \psi \, d\mu := \inf\left\{\int_\Omega \phi \, d\mu : \phi \in \mathscr{L}^1_{\mathbf{R}}, \phi \geqslant \psi \text{ a.e.}\right\}.^2$$

See Balder (1984b), for instance. The set of all Bochner-integrable X-valued functions on $(\Omega, \mathscr{F}, \mu)$ is denoted by $\mathscr{L}^1_X$. By separability of X,we have $f \in \mathscr{L}^1_X$ if and only if $f: \Omega \to X$ is measurable, with respect to $\mathscr{F}$ and $\mathscr{B}(X)$, and $\int_\Omega \|f\| \, d\mu < +\infty$.[3] Recall that a sequence $(\phi_k)_k$ in $\mathscr{L}^1_{\mathbf{R}}$ is *uniformly integrable* if

$$\lim_{\alpha \to \infty} \sup_k \int_{\{|\phi_k| \geqslant \alpha\}} |\phi_k| d\mu = 0.$$

In this paper a sequence $(\psi_k)_k$ of (possibly nonmeasurable) functions $\psi_k: \Omega \to \mathbf{R}$ is said to be *uniformly integrably bounded* if there exists a uniformly integrable sequence $(\phi_k)_k$ such that $|\psi_k(\omega)| \leqslant \phi_k(\omega)$ a.e. for all k. For any multifunction $G: \Omega \to 2^X$ the (possibly empty) set $\mathscr{L}^1_G$ of all *integrable selectors* of G is defined by

$$\mathscr{L}^1_G := \{u \in \mathscr{L}^1_X : u(\omega) \in G(\omega) \text{a.e.}\}.$$

Also, the *integral* of G is defined by

$$\int_\Omega G \, d\omega := \left\{\int_\Omega u \, d\mu : u \in \mathscr{L}^1_G\right\};$$

[2]The infimum over the empty set is $+\infty$ by definition.

[3]We consider the prequotient setting; of course, all results can easily be restated for the quotient setup.

this definition makes sense for an *arbitrary* multifunction G, for we allow $\mathscr{L}_G^1$—whence $\int_\Omega G\,d\mu$ itself— to be empty.

3. Main results. As in the previous section, let X be a separable Banach space and $(\Omega, \mathscr{F}, \mu)$ a finite measure space. Let (F_k) be a given sequence of multifunctions from Ω into 2^X. Observe that no measurability whatsoever is required for these multifunctions. We suppose the following basic hypotheses (H_0)-(H_6) to hold: there exist a sequence (G_k) of multifunctions from Ω into 2^X, a sequence (r_k) in $\mathscr{L}_{\mathbf{R}}^1$, and a subset L of X such that

(H_0) $F_k(\omega) \subset G_k(\omega) + r_k(\omega)L$ *a.e. for all* k,
(H_1) $G_k(\omega)$ *is w-compact a.e. for all* k,
(H_2) $\sup_k \int_\Omega^* \|G_k(\omega)\| \mu(d\omega) < +\infty$,
(H_3) $(r_k)_k$ *is uniformly integrable*,
(H_4) L *belongs to* $\mathscr{L}$.

Moreover, we suppose that there exists a multifunction R from Ω into 2^X such that the following hypothesis hold:

(H_5) $\bigcup_k F_k(\omega) \subset R(\omega)$ *a.e.*
(H_6) $R(\omega)$ *belongs to* $\mathscr{R}$ *a.e.*

REMARK 3.1. Hypotheses (H_5)–(H_6) hold *automatically* when the Banach space X is reflexive (this includes the finite-dimensional case), for then $X \in \mathscr{R}$, as we noticed earlier. Another obvious case when (H_5)–(H_6) hold, is when $\bigcup_k F_k(\omega)$ is a relatively w-compact set for a.e. ω in Ω.

We are now ready to state our principal results. Let C be the convex cone consisting of all $x^* \in X^*$ for which

$$\left(\max\left[0, s(-x^*|G_k(\cdot))\right]\right)_k \text{ is uniformly integrably bounded}$$

(see the previous section for the definition involved and note how this sidesteps the measurability issue for the multifunctions G_k). In finite dimensions our main result is the following Fatou-like lemma:

THEOREM 3.2 (FINITE DIMENSIONS). *Suppose that X is finite-dimensional. Under the hypotheses (H_0)–(H_4)*

$$\mathrm{Ls}_k \int_\Omega F_k\,d\mu \subset \int_\Omega F\,d\mu + \mathrm{As}(L - C^*),$$

where $F\colon \Omega \to 2^X$ is defined by

$$F(\omega) := \mathrm{Ls}_k F_k(\omega).$$

It is well-known that Fatou-type lemmas do not extend directly to infinite dimensions, because of the breakdown of Lyapunov's theorem (Rustichini 1989). In this situation much simpler, *approximate* versions form the natural counterparts of the results obtained in finite dimensions (see Rustichini and Yannelis 1991, Main Theorem, for a different way to resolve the problem). These are obtained by taking the closure on the right-hand side of the inclusion statement. Even more precision is achieved by us here by omitting the usual closure on the left-hand side (see Remark 3.5), and by splitting the measure space into its purely atomic and nonatomic parts:

THEOREM 3.3 (INFINITE DIMENSIONS). *Under the hypotheses* (H_0)–(H_6),

$$w\text{-seq-Ls}_k \int_\Omega F_k \, d\mu \subset \int_{\Omega^{pa}} F \, d\mu + s\text{-cl} \int_{\Omega^{na}} F \, d\mu + \mathrm{As}(L - C^*),$$

where the nonsequential limes superior multifunction $F\colon \Omega \to 2^X$ *is defined by*

$$F(\omega) := w\text{-Ls}_k F_k(\omega).$$

REMARK 3.4. If (H_6) in Theorem 3.3 is strengthened into

(H_6') $R(\omega)$ is w-closed and belongs to $\mathscr{L}$ a.e.

(recall that any w-closed set in $\mathscr{L}$ belongs to $\mathscr{R}$), then F in Theorem 3.3 can be replaced by the sequential limes superior multifunction F_{seq}, given by

$$F_{\text{seq}}(\omega) := w\text{-seq-Ls}_k F_k(\omega),$$

as a consequence of the identity $F(\omega) = F_{\text{seq}}(\omega)$, then valid a.e. (see §2).

REMARK 3.5. Note that always

$$w\text{-seq-Ls}_k \int_\Omega F_k \, d\mu = w\text{-seq-Ls}_k s\text{-cl} \int_\Omega F_k \, d\mu,$$

because s-closures and sequential s-closures coincide.

It is instructive to consider the example given by Hess (1991, Remark 4.4) because it illustrates the need for what we shall call the *asymptotic correction term* As $(L - C^*)$ in the right-hand side of the critical inclusion.

EXAMPLE 3.6. Let $(\Omega, \mathscr{F}, \mu)$ be the Lebesgue unit interval, $X = \mathbf{R}$, $L = \mathbf{R}_+$, and

$$F_k(\omega) = \begin{cases} L & \text{if } \omega \in [0, 1/k], \\ \{0\} & \text{if } \omega \in (1/k, 1] \end{cases}.$$

Of course, for the pointwise Kuratowski limes superior this gives $F(\omega) \equiv \{0\}$. On the other hand, $\int_\Omega F_k \, d\mu = L$ for all k. Therefore, $L = \mathrm{Ls}_k \int_\Omega F_k \, d\mu \not\subset \int_\Omega F \, d\mu = \{0\}$. Nevertheless, $G_k \equiv \{0\}$ gives $C = X^* = \mathbf{R}$ and $C^* = \{0\}$. Therefore, adding the asymptotic correction term makes the critical inclusion into $L \subset \{0\} + \mathrm{As}(L - \{0\}) = L$, which is in agreement with Theorem 3.2.

The following counterexample shows that our hypothesis (H_4) for the closed convex hull of L not to have any lines, is vital for our results to be valid.

EXAMPLE 3.7. Let $(\Omega, \mathscr{F}, \mu)$ be the Lebesgue unit interval, $X = L = \mathbf{R}$, and

$$F_k(\omega) = \begin{cases} \{k\} & \text{if } \omega \in [0, 1/2], \\ \{-k\} & \text{if } \omega \in (1/2, 1] \end{cases}.$$

Then $\int_\Omega F_k \, d\mu = \{0\}$ for all k, so $\mathrm{Ls}_k \int_\Omega F_k \, d\mu = \{0\}$. On the other hand, it is evident that F, the pointwise Kuratowski limes superior, is empty-valued on Ω. So $\int_\Omega F \, d\mu$ is empty, as is the entire right-hand side of the critical inclusion by implication. Apart from (H_4), the other hypotheses hold obviously (take $G_k \equiv \{0\}$, $r_k \equiv 1$).

4. Consequences of the main results. In this section we shall derive some well-known Fatou-type lemmas from Theorems 3.2–3.3. Basically, these follow by introducing *additional* conditions, which affect the asymptotic correction term of the critical inclusion statement.

REMARK 4.1. If in Theorems 3.2–3.3, $-C^* \subset \text{As } L$, then obviously $\text{As}(L - C^*) = \text{As } L$. This applies in particular when we follow Hiai (1985), Hess (1991) in supposing that

$$(\|G_k(\cdot)\|)_k \text{ is uniformly integrably bounded,}^4$$

for then $C = X^*$, so $C^* = \{0\}$.

REMARK 4.2. In Theorems 3.2–3.3, $L \subset -C^*$, then clearly $\text{As}(L - C^*) = -C^*$. This applies in particular if we suppose (as in Hiai 1985),

$$F_k \text{ has } w\text{-compact values for all } k,$$

for then one can choose $G_k := F_k$, $L := \{0\}$; this time C can be equivalently expressed as the cone of all $x^* \in X^*$ such that

$$(\max[0, s(-x^* | F_k(\cdot))])_k \text{ is uniformly integrably bounded.}$$

As a particular instance of this, consider the case when the F_k's are single-valued:

$$F_k(\omega) \text{ is a singleton } \{f_k(\omega)\} \text{ for all } \omega \text{ and } k;$$

here $(f_k)_k$ is a sequence of functions in $\mathscr{L}^1_X$.[5] In this single-valued case the cone C consists of all $x^* \in X^*$ such that

$$(\max[0, -< f_k(\cdot), x^* >])_k \text{ is uniformly integrable.}$$

Remarks 3.1, 4.1 apply to the Fatou-type results of Hiai (1985) and Hess (1991) (the first part of Remark 4.2 applies to the result of Hiai as well). The result of Hiai in infinite dimensions (Hiai 1985, Theorem 2.8(2)) follows by either of the two remarks above, by noting that he has $\text{As}(L - C^*) = \{0\}$. Hence his critical inclusion becomes

$$w\text{-seq-Ls}_k s\text{-cl} \int_\Omega F_k \, d\mu \subset s\text{-cl} \int_\Omega F \, d\mu,$$

using Remark 3.5. The result of Hess (1991, Theorem 4.3) is also stated only in approximate, infinite-dimensional form; it follows directly from combining Remarks 3.5, 4.1 and the following additional condition (Hess, 1991, Condition (c2′), p. 640):

$$\text{As } L \subset s\text{-cl} \int_\Omega \text{cl co } F \, d\mu.$$

This gives his result

$$w\text{-seq-Ls}_k s\text{-cl} \int_\Omega F_k \, d\mu \subset s\text{-cl} \int_\Omega \text{cl co } F \, d\mu,$$

[4]This concept was defined in §2.

[5]Of course, in this case, $\int_\Omega F_k \, d\mu$ is only nonempty if iff$_k \in \mathscr{L}^1_X$.

but it is quite evident that also our inclusions used along the way to this specialization have greater precision. Note also that both Hiai (1985) and Hess (1991) consider only measurable multifunctions, whereas we allow arbitrary ones (with measurable *selectors*, to be sure). Further, Hiai supposes $\int_\Omega F\,d\mu$ to be nonempty; this condition is shown to be redundant in our results (which imply simply that s-cl$\int_\Omega F\,d\mu$—i.e., the right-hand side of Hiai's critical inclusion above—is nonempty whenever the left-hand side is so). A similar comment applies to the result by Hess (1991); we elaborate a little on this, because the statement of nonemptiness of $\int_\Omega F\,d\mu$ forms an integral part of Hess 1991, Theorem 4.3(i)). In Hess (1991) the sequence $(\text{dist}(F_k(\cdot), 0))_k$ is supposed to be uniformly integrable (Hess 1991, Condition (L1)): together with the measurability of the F_k's, as supposed in Hess (1991), this implies in particular that w-seq-$\text{Ls}_k \int_\Omega F_k$ is nonempty. We observe that uniform boundedness of the integrals in $(\int_\Omega \text{dist}(F_k(\cdot), 0)\,d\mu)_k$ would already be enough for such nonemptiness; see Theorem 5.5, 5.7 in Hess (1990). Hence, under the additional conditions of Hess (1991), Remark 4.1 also ensures (independently) the nonemptiness of $\int_\Omega F\,d\mu$.

In Pucci and Vitillaro (1984, Theorem 3.5) the authors state a Fatou-type result for w-compact-valued multifunctions where the "dominating" multifunction R of (H_5)–(H_6) has w-compact values. This result follows directly from our Theorem 3.3, using Remarks 3.1, 4.2. Notice that their notion of limes superior is rather less general than ours (by Remark (2) in Pucci and Vitillaro 1984, p. 89 their limes superior of a sequence of subsets is equal to the closed convex hull of our sequential w-limes superior). Likewise, by the same remarks, the Fatou lemma of Yannelis (1988, 1991) for multifunctions, subject to the same kind of domination, follows directly from Theorem 3.3.

Amrani's Fatou lemma for Pettis integrals of multifunctions (Amrani 1991, Theorem 3.2) is not directly comparable to our results, because it contains additional features to ensure Pettis integrability. Another Fatou-type lemma which does not relate directly to the results presented here, is the one by Castaing-Clauzure ((1991, Theorem 4.2). However, a substantial generalization of their result can be obtained by observing (Balder and Hess) that Balder (1989b, Theorem 2.2) continues to hold when its multifunction g takes values in $\mathscr{R}$. The validity of this improvement is immediately evident from the proof of Balder (1989b, Theorem 2.2) (it suffices to repeat the inf-compactness argument given in the proof of Corollary 5.3 below). *Inter alia*, it thus follows that in (Castaing, and Clauzure (1991, Theorem 4.2) condition (i) and the Radon-Nikodym property for X can be completely omitted.

Remark 4.2 applies in particular when the multifunctions (F_k) are single-valued. In finite dimensions Theorem 3.2 thus extends the unifying Fatou lemma of Balder (1984a, b, 1991). By implication, this is also true for the Fatou lemmas which the unifying lemma subsumes: i.e., those of Artstein (1979), Cesari-Suryanarayana (1978), Hildenbrand-Mertens (1971) and Schmeidler (1970). Since Balder's result not only follows from Theorem 3.2, but will also be used in its proof, we shall state it in full, giving an independent reference:

COROLLARY 4.3 (BALDER 1994, PROPOSITION 3.7). *Suppose that X is finite-dimensional. Suppose that $(f_k)_k$ is a sequence in $\mathscr{L}^1_X$ such that*

$$\sup_k \int_\Omega \|f_k(\omega)\| \mu(d\omega) < +\infty.$$

Then

$$\text{Ls}_k \int_\Omega f_k\,d\mu \subset \int_\Omega F_0\,d\mu - C_0^*,$$

where $F_0(\omega) := \mathrm{Ls}_k f_k(\omega)$, *and where* C_0 *is the cone of all* $x^* \in X^*$ *for which*

$$(\max[0, -\langle x^*, f_k \rangle])_k \text{ is uniformly integrable.}$$

The result also follows immediately from the proof of the main result in Balder (1984a). The following result not only follows from Theorem 3.3 (again by Remark 4.2), but will also be used to prove it. An independent proof will be furnished in the next section.

COROLLARY 4.4. *Suppose that* $(f_k)_k$ *is a sequence in* $\mathscr{L}_X^1$ *such that*

$$\sup_k \int_\Omega \|f_k(\omega)\| \, \mu(d\omega) < +\infty,$$

with

$$f_k(\omega) \in R(\omega) \quad a.e. \text{ for all } k,$$

where R *is a multifunction from* Ω *into* X *having values in* $\mathscr{R}$. *Then*

$$w\text{-seq-Ls}_k \int_\Omega f_k \, d\mu \subset \int_{\Omega^{pa}} F_0 \, d\mu + s\text{-cl} \int_{\Omega^{na}} F_0 \, d\mu - C_0^*,$$

where $F_0(\omega) := w\text{-}\mathrm{Ls}_k f_k(\omega)$, *and where* C_0 *is the cone of all* $x^* \in X^*$ *for which*

$$(\max[0, -\langle x^*, f_k \rangle])_k \text{ is uniformly integrable.}$$

REMARK 4.5. In Corollary 4.4, if the sets $R(\omega)$ are not only w-closed but belong to $\mathscr{L}$ a.e., then the multifunction F_0 appearing in Corollary 4.4 has a measurable modification. This follows by Hess (1990, Theorem 4.4), because then

$$w\text{-}\mathrm{Ls}_k f_k(\omega) = w\text{-seq-Ls}_k f_k(\omega) \text{a.e.}$$

by what we observed before.

Because of Remarks 3.1 and 4.2, Corollary 4.4 generalizes the two single-valued Fatou-type in Theorems 2.1 and 2.4 of Balder (1988). Consequently, the single-valued Fatou-type lemma of Khan-Majumdar (1984, Theorem 2) follows as well (their result has $(f_k)_k$ even taking values in a single w-compact subset of X). In contrast to Corollary 4.3, for which we cited two references independent of this paper, Corollary 4.4 is new, since it is somewhat sharper than Balder (1988, Theorems 2.1, 2.4), which both follow from it. For this reason a proof of Corollary 4.4–of course independent of Theorems 3.2, 3.3—is provided at the end of the paper.

5. Proofs of the main results. In this section we shall first prove Theorem 3.2 from Corollary 4.3 and Theorem 3.3 from Corollary 4.4. Next to some convex analysis, recalled already in §2, we shall need the following lemma from Hess (1991).

LEMMA 5.1 (HESS 1991, LEMMAS 2.1, 4.1). *Let* L *belong to* $\mathscr{L}$. *Then* dom $s(\cdot|L) =$ dom $s(\cdot|\mathrm{cl\,co}\,L)$ *has a nonempty interior for the Mackey topology, and for any* x_0^* *in this interior there exists a constant* γ, *only depending upon* L, *such that for every* w-*compact convex* $K \subset X$, *every* $r \geqslant 0$ *and every point* $x \in K + rL$,

$$\|x\| \leqslant \|K\| + \gamma[s(x_0^*|K) + r - \langle x, x_0^* \rangle].$$

PROOF. By Castaing and Valadier (1977, I.15) the interior of dom $s(\cdot|L)$ is nonempty. By the same result and by the given w-compactness and convexity of K, for any x_0^* in this interior.

$$W_\beta := \{x \in K + rL : \langle x, -x_0^* \rangle \leqslant \beta\}$$

is a w-compact subset of X for any $\beta \in \mathbf{R}$. Moreover, W_β is also nonempty, if we choose $\beta > -s(x_0^*|K + rL)$. As in Castaing and Valadier (1977, I.24), one obtains that for every $x^* \in X^*$,

$$s(x^*|W_\beta) = \inf_{\lambda \in R_+} \left[s(x^* + \lambda x_0^* | K + rL) + \lambda\beta \right].$$

In particular, this implies $s(x^*|W_\beta) \leqslant s(x^* + x_0^*|K + rL) + \beta$ for all $x^* \in X^*$, so obviously

$$s(x^*|W_\beta) \leqslant s(x^*|K) + s(x_0^*|K) + rs(x^* + x_0^*|L) + \beta.$$

By the choice of x_0^*, the function $s(\cdot + x_0^*|L) = s(\cdot + |\operatorname{cl}\operatorname{co} L)$ is Mackey-continuous at the origin; hence it is also continuous for the dual norm topology. So there exists $\alpha > 0$ such that $s(x^* + x_0^*|L) \leqslant 1$ for all $x^* \in \alpha B^*$. Of course, we also have $s(x^*|K) \leqslant \alpha\|K\|$ for all $x^* \in \alpha B^*$. Hence, it follows that

$$\alpha\|W_\beta\| = \sup_{x^* \in \alpha B^*} s(x^*|W_\beta) \leqslant \alpha\|K\| + s(x_0^*|K) + r + \beta;$$

observe that the way α was defined involved only the set L. For any $x \in K + rL$ we must have $\langle x, -x_0^* \rangle \leqslant \beta$ for some $\beta \in \mathbf{R}$ large enough. Then $x \in W_\beta$, and the inequality just reached gives the desired inequality in the statement of the lemma, with $\gamma := 1/\alpha$. □

PROOF OF THEOREM 3.2. Let a be an arbitrary element of $\mathrm{Ls}_k \int_\Omega F_k \, d\mu$. Then there exist a subsequence of (F_k) (for convenience we take the entire sequence, which can—at least notationally—always be achieved by renumbering), and corresponding integrable selectors $f_k \in \mathscr{L}^1_{F_k}$, such that $a = \lim_k \int_\Omega f_k \, d\mu$. Hence, $a \in \mathrm{Ls}_k \int_\Omega f_k \, d\mu$. Also, by (H_0)–(H_1), Lemma 5.1 gives the existence of $x_0^* \in X^*$ and a constant γ such that for all k,

$$\|f_k(\omega)\| \leqslant (1 + \gamma\|x_0^*\|^*)\|G_k(\omega)\| + \gamma[r_k(\omega) - \langle f_k(\omega), x_0^* \rangle] \quad \text{a.e.}$$

This implies that $\sup_k \int_\Omega \|f_k\| \, d\mu < +\infty$, by (H_2)–(H_3) and the convergence $\int_\Omega \langle f_k, x_0^* \rangle d\mu \to \langle a, x_0^* \rangle$. Thus, the one condition of Corollary 4.3 holds, and we obtain

$$a \in \int_\Omega F_0 \, d\mu - C_0^*,$$

where $F_0(\omega) := \mathrm{Ls}_k f_k(\omega)$, and where C_0 is the cone of all $x^* \in X^*$ for which

$$(\max[0, -\langle f_k(\cdot e), x^* \rangle])_k \text{ is uniformly integrable.}$$

Clearly, a.e., $F_0(\omega) \subset F(\omega) := \mathrm{Ls}_k F_k(\omega)$. Therefore, the proof is finished by proving the inclusion $-C_0^* \subset \mathrm{As}(L - C^*)$. Using the definition of asymptotic cone, it is easy to see that the set on the right (which coincides by definition with As $[\mathrm{cl}(L - C^*)]$) is the polar of the cone dom $s(\cdot|L)\cap -C$. For arbitrary $x^* \in$ dom $s(\cdot|L)\cap -C$ it follows elementarily that

$$\langle f_k(\omega), -x^*\rangle \geqslant -\max[0, s(x^*|G_k(\omega))] - r_k(\omega)s(x^*|L) \quad \text{a.e. for all } k.$$

Here the right-hand side is uniformly integrably bounded, by the definition of C and (H_3). So x^* belongs to $-C_0$. Taking polars gives the desired inclusion □

PROOF OF THEOREM 3.3. This proof proceeds along exactly the same lines, this time by an application of Corollary 4.4, explaining the appearance of the s-closure term on the right of the critical inclusion. □

We finish with a proof of Corollary 4.4. As was mentioned before, this essentially comes about by strengthening and combining the proofs of Theorems 2.1, 2.4 in Balder (1988). In a variation on the traditional themes of Young measure theory, we shall rely on results of Prohorov-Komlós-type, which were established in Balder (1990) §5) and (1991) as particular cases of a general approach involving Komlós' theorem (see also (Balder (1989b) for different applications). An analogous, more conventional Prohorov-type result at the same level of generality was given in Balder 1989a). Let $\mathscr{Y}_X$ denote the set of all Young measures (alias transition probabilities) from $(\Omega, \mathscr{F})$ into $(X, \mathscr{B}(X))$ (note that the Borel σ-algebras of (X, w) and (X, s) coincide). Also, let $\Rightarrow$ stand for classical weak (alias narrow) convergence in the set of all probability measures on $(X, \mathscr{B}(X))$ Dellacherie and Meyer 1978), and let supp ν stand for the support of a probability measure ν on $(X, \mathscr{B}(X))$.

THEOREM 5.2 (PROHOROV-KOMLÓS THEOREM (BALDER 1991, A.5, A.7)). *Suppose that (δ_l) is a sequence in $\mathscr{Y}_X$ such that there exists a function $h\colon \Omega \times X \to [0, +\infty]$ with the following two properties*:

$$h(\omega,\cdot) \text{ is inf-compact on } (X,w) \text{ for a.e. } \omega,$$
$$\sup_l \int_\Omega^* \left[\int_X h(\omega, x)\delta_l(\omega)(dx)\right] \mu(d\omega) = \sigma < +\infty.$$

Then there exist a subsequence (δ_m) of (δ_l) and a Young measure δ_* in $\mathscr{Y}_X$ such that for every sub-subsequence (δ_{m_j}),

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{m_j}(\omega) \Rightarrow \delta_*(\omega) \quad \text{as } n \to \infty$$

for a.e. ω. Moreover,

$$\delta_*(\omega)(w\text{-Ls}_l \operatorname{supp} \delta_l(\omega)) = 1 \quad a.e.$$

and

$$\int_\Omega^* \left[\int_X h(\omega, x)\delta_*(\omega)\,(dx)\right] \mu(d\omega) \leqslant \sigma.$$

More generally, in Balder (1990, §5) and (1991) the role of (X, w) is played by a completely regular Suslin space.[6]

[6]Actually, a *regular* Suslin space is automatically completely regular; we are indebted to Professor E. Lanéry (CEREMADE) for this observation.

Observe that any ordinary $(\mathscr{F}, \mathscr{B}(X))$-measurable function $f: \Omega \to X$ has a natural counterpart in $\mathscr{Y}_X$, namely the Young measure obtained by setting

$$\epsilon_f(\omega) := \text{Dirac measure at } f(\omega).$$

The Young measure ϵ_f is also called the *relaxation of f*.

COROLLARY 5.3. *Suppose that (f_l) is a sequence in $\mathscr{L}_x^1$ such that*

$$\sigma := \sup_l \int_\Omega \|f_l(\omega)\| \, \mu(d\omega) < +\infty,$$

with

$$f_l(\omega) \in R(\omega) \quad a.e. \text{ for all } l,$$

where R is a multifunction from Ω into X having values in $\mathscr{R}$. Then there exist a subsequence (f_m) of (f_l) and a transition probability δ_* from $(\Omega, \mathscr{F})$ into $(X, \mathscr{B}(X))$ such that for every sub-subsequence (f_{m_j}),

$$\frac{1}{n}\sum_{j=1}^{n} \epsilon_{f m_j}(\omega) \Rightarrow \delta_*(\omega) \quad as\ n \to \infty$$

for a.e. ω. Moreover,

$$\delta_*(\omega)(w\text{-}\mathrm{Ls}_l f_l(\omega)) = 1 \quad a.e.$$

and

$$\int_\Omega \left[\int_X \|x\| \delta_*(\omega)\,(dx) \right] \mu(d\omega) \leqslant \sigma.$$

PROOF. The result will be derived directly from applying Theorem 5.2 to the sequence (ϵ_{f_l}). Define $h: \Omega \times X \to [0, +\infty]$ as follows:

$$h(\omega, x) := \begin{cases} \|x\| & \text{if } x \in r(\omega), \\ +\infty & \text{otherwise}. \end{cases}$$

Then for any $\omega \in \Omega$ and $\beta \in \mathbf{R}_+$ the set

$$\{x \in X : h(\omega, x) \leqslant \beta\} = \{x \in R(\omega) : \|x\| \leqslant \beta\}$$

is w-compact, precisely by ball-compactness of $R(\omega)$. Also, by

$$\int_X h(\omega, x)\epsilon_{f_l}(\omega)(dx) = h(\omega, f_l(\omega)) = \|f_l(\omega)\| \quad \text{a.e.}$$

we see that the conditions of Theorem 5.2 are met. This gives immediately the result (note that $\operatorname{supp} \epsilon_{f_l}(\omega) = \{f_l(\omega)\}$). □

PROOF OF COROLLARY 4.4. Let $a \in w\text{-seq-Ls}_k \int_\Omega f_k \, d\mu$ be arbitrary. Without loss of generality we may assume that $a = w\text{-lim}_k \int_\Omega f_k \, d\mu$. Without loss of generality, Ω^{pa} may be supposed to be the countable union of disjoint atoms A_i. On each such atom A_i the f_k's are all a.e. constant (say $f_k \equiv c_{k,i}$) and $\mu(A_i)\|c_{k,i}\| \leqslant \sup_k \int_\Omega \|f_k\| \, d\mu$. Since there exists a null set N such that $f_k(\omega) = c_{k,i} \in R(\omega)$ for all ω in the set $A_i \setminus N$ (which cannot be empty), it follows from the w-ball-compactness property of the sets $R(\omega)$ that $(c_{k,i})_k$ contains a convergent subsequence for each i (observe that by the Eberlein-Šmulian theorem w-compact sets are also sequentially w-compact, and vice versa). By an obvious diagonal extraction argument, this gives the existence of a subsequence (f_l) of (f_k) such that $f_*(\omega) := \lim_l f_l(\omega)$ exists a.e. on Ω^{pa}. This gives

$$w\text{-Ls}_l f_l(\omega) = \{f_*(\omega)\} \quad \text{a.e. in } \Omega^{pa}.$$

Let (f_m) and δ_* be as in Corollary 5.3. By what we know about the support of δ_*, it follows that

$$\delta_*(\omega) = \epsilon_{f_*}(\omega) \quad \text{a.e. in } \Omega^{pa}.$$

Also, by the inequality involving σ in Corollary 5.3 and by well-known properties of Bochner integration, the two integrals

$$b := \int_{\Omega^{pa}} f_* \, d\mu, \qquad c := \int_{\Omega^{na}} \left[\int_X x \delta_*(\omega)(dx) \right] \mu(d\omega)$$

are well defined. Define $g_{x^*}(\omega, x) := \langle x, -x^* \rangle$, $x^* \in X^*$. For every $x^* \in C_0$ the conditions of the Fatou-type Lemma A.7 in Balder (1991) are met. This gives that for every $x^* \in C_0$,

$$\int_\Omega \left[\int_X g_{x^*}(\omega, x) \delta_*(\omega)(dx) \right] \mu(d\omega) \leqslant \liminf_m \int_\Omega g_{x^*}(\omega, f_m(\omega)) \, \mu(d\omega),$$

i.e., $\langle b + c, -x^* \rangle \leqslant \langle a, -x^* \rangle$. Hence $a - b - c$ belongs to C_0^* the negative polar of C_0. The definition of b shows immediately that b belongs to $\int_{\Omega^{pa}} F_0 \, d\mu$. Finally, the support property of δ_* in Corollary 5.3 gives

$$\int_X x \delta_*(\omega)(dx) \in \operatorname{cl} \operatorname{co} F_0(\omega) \quad \text{a.e. in } \Omega^{na},$$

by a well-known property of barycenters. In view of the definition of c, it now remains to prove

$$\int_{\Omega^{na}} \operatorname{cl} \operatorname{co} F_0 \, d\mu \subset s\text{-cl} \int_{\Omega^{na}} F_0 \, d\mu.$$

But this inclusion holds by results from Hess (1990) and Hiai (1985): by Fatou's lemma the function $\omega \mapsto \liminf_k \|f_k(\omega)\|$ is integrable. Hence, by the fact that R takes its values in $\mathscr{R}$, it follows from Theorem 5.5 of Hess (1990) that $\mathscr{L}^1_{F_0}$ is nonempty. So the inclusion follows from Hiai, and Umegaki (1977, Theorem 1.5 and the proof of Theorem 4.2). □

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E. J. Balder: Mathematical Institute, University of Utrecht, P. O. Box 80.010, 3508 TA Utrecht, The Netherlands; e-mail: balder@math.ruu.nl
C. Hess: Centre de Recherche de Mathématiques de la Décision (CEREMADE), Université Paris Dauphine, 75775 Paris Cedex 16, France