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An Existence Result for Optimal Economic Growth Problems

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An existence result for optimal control problems of Lagrange type with unbounded time domain is derived very directly from a corresponding result for problems with bounded time domain. This subsumes the main existence result of R. F. Baum [*J. Optim. Theory Appl.* **19** (1976), 89–116] and has the existence results for optimal economic growth problems of S.-I. Takekuma [*J. Math. Econom.* 7 (1980), 193–208] and M. J. P. Magill [*Econometrica* **49** (1981), 679–711; *J. Math. Anal. Appl.* **82** (1981), 66–74] as simple corollaries. In addition, a new notion of uniform integrability is used, which coincides with the classical notion if the time domain is bounded.

1. INTRODUCTION

In the theory of optimal control and its parent discipline, the calculus of variations, two major approaches to existence problems can clearly be discerned.

One approach, which originated with the early work of L. Tonelli [5], can be said to be carried by a certain tightness (or compactness) property of the derivative functions and a lower semicontinuity property of a fundamental integral functional associated with the control problem [6–11]. (The tightness notion mentioned above is a generalization of the tightness notion in topological measure theory [12].) From the description it is evident that this approach is entirely in the spirit of the classical Weierstrass existence theorem, well known from elementary calculus.

The other approach is based upon the seminormality concept of L. Tonelli and E. J. McShane [13, 14]. Their concept was reformulated by L. Cesari into the notion of property (Q) [15]. (This property is actually equivalent to a certain upper semicontinuity property of a Hamiltonian associated with the control problem [16, 17].)

Although these approaches do show a fair amount of theoretical overlap, the first approach is often much easier to apply than is the second one. The

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price to be paid for this is that the first approach demands much more from the derivative functions of the control problem.

In this paper the first approach will be followed to obtain a general existence result for optimal control problems whose time domain is unbounded. Such control problems frequently arise in systems theory, but have played a very minor role in the area of optimal control theory proper. Thus, the literature devoted to existence results for such problems is of a quite small size (cf. [1, 18]). In contrast, optimal control problems with unbounded time domain have been common to the theory of optimal economic growth since its inception [19, 20]. Nevertheless, it is surprising to note, following [3], that only recently some attention has been given to the existence problem in that area (cf. [2–4]). Our existence result will be derived in a very direct manner from well-known fundamental results concerning the first approach for problems with bounded time domain. It will be shown to imply the main existence result of R. F. Baum [1, Theorem 6.1]. Also, the main existence results of S.-I. Takekuma [2, Theorem 4.1] and M. J. P. Magill [3, Theorem 7.6; 4, Proposition 3.1] will follow by considering quite special cases. Finally, also the main sensitivity result of [2, Theorem 4.2] will be generalized in Appendix B.

2. Basic Results

Consider the variational problem

$$\inf\{J(x): x \in C\},$$

where C is a given nonempty subset of curves in $AC^m_{\mathrm{loc}}([0, \infty))$; cf. Appendix A. Here

$$J(x) \equiv \int_0^\infty L(t, x(t), \dot{x}(t))\, dt, \qquad x \in AC^m_{\mathrm{loc}}([0, \infty)), \tag{2.1}$$

defines an integral functional whose integrand $L: [0, \infty) \times \mathbb{R}^{2m} \to (-\infty, +\infty]$, the so-called *Lagrangian*, is supposed to be $\mathscr{L} \times \mathscr{B}^{2m}$-measurable.[1]

The integral in (2.1) always has a meaning, since we adopt the convention $(+\infty) - (+\infty) = +\infty$ in

$$\int f \equiv \int f^+ - \int f^-,$$

[1] That is, measurable with respect to the product of the Lebesgue σ-algebra on $[0, \infty)$ and the Borel σ-algebra on $\mathbb{R}^{2m}$.

for any $\mathscr{L}$-measurable function f: $[0, \infty) \to [-\infty, +\infty]$. Here $f^+ \equiv \max(f, 0)$ and $f^- \equiv \max(-f, 0)$.

Entirely in agreement with the first approach discussed in the previous paragraph, the existence of an optimal curve in C is assured if for some topology on $AC^m_{\mathrm{loc}}([0, \infty))$

(C) C is sequentially compact,

(LSC) J is sequentially lower semicontinuous on C.

A first problem to be tackled consists of specifying a topology which has the potential to achieve both (C) and (LSC). [Note that (C) is driving the coarseness of the topology up, while (LSC) is driving it down.] For various reasons—these have to do with the usual properties of the Lagrangian as well as analytical tractability—one takes the usual weak topology on $AC^m_{\mathrm{loc}}([0, \infty))$ for this purpose. Specification (C) above is then largely covered by the following simple extension of the well-known characterization of relative weak compactness in $AC^m([0, T])$. (The latter characterization is mainly a reformulation of the Dunford–Pettis criterion [21, II.25].)

THEOREM 2.1. *A subset C of $AC^m_{\mathrm{loc}}([0, \infty))$ is relatively weakly sequentially compact if and only if*

$$\{x(0) : x \in C\} \text{ is bounded} \tag{2.2}$$

and for each $T > 0$

$$\{\dot{x} | [0, T] : x \in C\} \text{ is uniformly integrable over } [0, T]. \tag{2.3}$$

Proof. Necessity of (2.2)–(2.3) follows directly from the Dunford–Pettis criterion. To prove sufficiency, let $\{x_k\}_1^\infty$ be arbitrary in C. In view of (2.3), we shall invoke the Dunford–Pettis criterion for $T = n$, $n \in \mathbb{N}$. For $n = 1$ there exist a subsequence $\{x_k^1\}_1^\infty$ and $g^1 \in L_1^m([0, 1])$ such that g^1 is the weak limit in $L_1^m([0, 1])$ of $\{\dot{x}_k^1 | [0, 1]\}_1^\infty$. For $n = 2$ there exist a further subsequence $\{x_k^2\}_1^\infty$ and $g^2 \in L_1^m([0, 2])$, weak limit of $\{\dot{x}_k^2 | [0, 2]\}_1^\infty$. It is easy to see that $g^2(t) = g^1(t)$ for almost every t in $[0, 1]$. Continuing in this way, we arrive at the diagonal sequence $\{x_k^k\}_1^\infty$. From this sequence we can extract a final subsequence $\{x_k'\}_1^\infty$ which is such that the values $\{x_k'(0)\}_1^\infty$ converge to some point $\bar{x} \in \mathbb{R}^m$. [Here we use (2.2).] We now define $g_0 \in L^m_{1,\mathrm{loc}}([0, \infty))$ by setting $g_0(t) \equiv g^n(t)$ for $n - 1 \leqslant t \leqslant n$, $n \in \mathbb{N}$. Also, we define $x_0 \in AC^m_{\mathrm{loc}}([0, \infty))$ by

$$x_0(t) \equiv \bar{x} + \int_0^t g_0(t)\, dt, \qquad t > 0.$$

We then conclude from the above construction that $\{x'_k\}_1^\infty$ converges weakly to x_0.

COROLLARY 2.2. *A subset C of $AC^m_{\mathrm{loc}}([0,\infty))$ is relatively weakly sequentially compact if* (2.2) *holds and if there exists $g \in L^+_{1,\mathrm{loc}}([0,\infty))$ such that*

$$\textit{for each } x \in C \textit{ for almost every } t \textit{ in } [0,\infty)\, |\dot{x}(t)| \leqslant g(t).$$

We shall now occupy ourselves with specification (LSC). The following definition presents a useful extension of the uniform integrability notion to infinite measure spaces.

DEFINITION 2.3. A subset G of $L_1([0,\infty))$ is *strongly uniformly integrable* if for every $\varepsilon > 0$ there exists $h \in L_1^+([0,\infty))$ such that

$$\sup_{g \in G} \int_{\{|g| \geqslant h\}} |g| \leqslant \varepsilon.$$

When formulated for finite measure spaces, strong uniform integrability coincides with uniform integrability. For infinite measure spaces the situation is different, as the following results show.

PROPOSITION 2.4. *Suppose G is a strongly uniformly integrable subset of $L_1([0,\infty))$. Then G is uniformly integrable.*

Proof. Given $\varepsilon > 0$, let $h \in L_1^+([0,\infty))$ be as asserted in Definition 2.3. Then for each $c > 0$, $g \in G$,

$$\int_{\{|g| \geqslant c\}} |g| \leqslant \int_{\{|g| \geqslant h\}} |g| + \int_{\{h \geqslant c\}} h \leqslant \varepsilon + \int_{\{h \geqslant c\}} h,$$

and the latter integral converges to zero as c goes to infinity, uniformly in $g \in G$.

EXAMPLE 2.5. Let G be the set of functions $g_{\alpha,\beta}$, $0 \leqslant \alpha \leqslant 1$, $\beta > 0$, with $g_{\alpha,\beta}(t) \equiv \alpha$ if $-\beta \leqslant t \leqslant \beta$, $\equiv 0$ otherwise. Then G is uniformly integrable, but not strongly uniformly integrable.

Our next result concerns (LSC). It is a simple extension of a classical result on the lower semicontinuity of integral functionals; e.g., cf. [9, Theorem 1] or [10, Theorem 5, Case 1]. We use notation introduced in (2.1).

THEOREM 2.6. *Suppose that $L\colon [0,\infty) \times \mathbb{R}^{2m} \to (-\infty,\infty]$ is $\mathscr{L} \times \mathscr{B}^{2m}$-measurable with*

$$L(t,\cdot,\cdot) \text{ is lower semicontinuous for each } t \geqslant 0. \tag{2.4}$$

Suppose also that $\{x_k\}_1^\infty$ *is a sequence in* $AC_{\mathrm{loc}}^m([0,\infty))$ *which converges weakly to* $x_0 \in AC_{\mathrm{loc}}^m([0,\infty))$ *with*

$$L(t, x_0(t), \cdot) \text{ is convex for each } t \geqslant 0, \tag{2.5}$$

$$\{L^-(\cdot, x_k(\cdot), \dot{x}_k(\cdot))\}_1^\infty \text{ is strong uniformly integrable.} \tag{2.6}$$

Then

$$\liminf_{k\to\infty} J(x_k) \geqslant J(x_0).$$

Proof. Let us write $L_k \equiv L(\cdot, x_k(\cdot), \dot{x}_k(\cdot))$, etc. As a first step, suppose that L is nonnegative. Then for each $T > 0$,

$$\liminf_{k\to\infty} J(x_k) \geqslant \liminf_{k\to\infty} \int_0^T L_k \geqslant \int_0^T L_0,$$

by the references mentioned above. Hence, the desired inequality follows. As our second step, suppose that L is bounded from below by a function $g \in L_1([0,\infty))$. Since we may apply the first step to $L' \equiv L - g$, the result follows. As our final step, consider the general case. Let $\varepsilon > 0$ be arbitrary. By (2.6) there exists $g \in L_1^+([0,\infty))$ such that for all $k \in \mathbb{N}$

$$\int L_k \geqslant \int_{\{L_k \geqslant -g\}} L_k - \varepsilon = \int L_k'' + \int_{\{L_k < -g\}} g - \varepsilon \geqslant \int L_k'' - \varepsilon,$$

where $L'' \equiv \max(L, -g)$ inherits properties like (2.4)–(2.5) from L. Since the previous step applies to the integrand L'' and ε was taken to be arbitrary, it is easy to conclude that the desired inequality has been reached.

Remark 2.7. The level of generality in Theorem 2.6 can effortlessly be raised to that reached in [10]. In particular, Theorem 2.6 can be extended to the case where one investigates sequential lower semicontinuity of the integral functional J' defined by

$$J'(x, \xi) = \int_0^\infty L(t, x(t), \xi(t))\, dt, \tag{2.7}$$

where $x\colon [0,\infty) \to \mathbb{R}^m$ is measurable and $\xi \in L_{1,\mathrm{loc}}^m([0,\infty))$. Here the mode of convergence is that of pointwise (or pointwise almost everywhere) convergence in the first argument of J' and that of weak convergence in $L_{1,\mathrm{loc}}^m([0,\infty))$ in the second argument.

Moreover, in this case we may interpret the integration in (2.7) as taking place with respect to an arbitrary σ-finite measure "dt" on $[0,\infty)$. Of course, local integrability must then be interpreted accordingly (integrability over all sets with finite measure).

3. Main Existence Result

The results of the previous section will be used to derive an existence result for the optimal control problem

$$\inf_{(x,u)\in\Omega} I(x,u).$$

Here Ω denotes the set of admissible pairs (x,u), $x\in AC^m_{\text{loc}}([0,\infty))$ and u: $[0,\infty)\to\mathbb{R}^r$ a $\mathscr{L}$-measurable function, such that for almost every t

$$x(t)\in A(t), \qquad u(t)\in U(t,x(t)),$$
$$\dot{x}(t)=f(t,x(t),u(t)).$$

Here A denotes a multifunction from $[0,\infty)$ into $\mathbb{R}^m$ with $\mathscr{L}\times\mathscr{B}^m$-measurable graph $\mathscr{A}$; U denotes a multifunction from $\mathscr{A}$ into $\mathbb{R}^r$ whose graph

$$M\equiv\{(t,x,u):t\in[0,\infty),x\in A(t),u\in U(t,x)\}$$

is $\mathscr{L}\times\mathscr{B}^{m+r}$-measurable. Also, f: $M\to\mathbb{R}^m$ is a $\mathscr{L}\times\mathscr{B}^{m+r}$-measurable function. Finally, I is defined by

$$I(x,u)\equiv\int_0^\infty f_0(t,x(t),u(t))\,dt, \qquad (x,u)\in\Omega,$$

where $f_0: M\to(-\infty,\infty]$ is a $\mathscr{L}\times\mathscr{B}^{m+r}$-measurable cost function. We shall frequently use the following notation: Define for $(t,x)\in\mathscr{A}$

$$Q(t,x)\equiv\{(z^0,z):z^0\geqslant f_0(t,x,u),$$
$$z=f(t,x,u),\ u\in U(t,x)\}.$$

Definition 3.1. The multifunction Q from A into $\mathbb{R}^{m+1}$ is said to have *property* (K) at $(t,x)\in\mathscr{A}$ if

$$Q(t,x)=\bigcap_{\delta>0}\operatorname{cl}\{\bigcup Q(t,x'):x'\in A(t),|x'-x|\leqslant\delta\}.$$

In the literature this property is also referred to as property (U). Note that property (K) at $(t,x)\in\mathscr{A}$ implies that the set $Q(t,x)$ is closed.

In what follows we shall also denote the section of the set M at any $t\geqslant 0$ by $M(t)$; in other words:

$$M(t)\equiv\{(x,u):x\in A(t),\ u\in U(t,x)\}.$$

First, we shall state a lower closure result; this makes the connection between our main existence result and Section 2 more transparent.

THEOREM 3.2. *Suppose that for each* $t \geqslant 0$

$$f(t, \cdot, \cdot) \text{ is continuous on } M(t), \tag{3.1}$$

$$f_0(t, \cdot, \cdot) \text{ is lower semicontinuous on } M(t), \tag{3.2}$$

$$A(t) \text{ is closed,} \tag{3.3}$$

$$M(t) \text{ is closed.} \tag{3.4}$$

Suppose also that $\{(x_k, u_k)\}_1^\infty$ *is a sequence in* Ω *with* $\{x_k\}_1^\infty$ *converging weakly to* $x_0 \in AC_{\mathrm{loc}}^m([0, \infty))$, *which is such that*

$$Q \text{ has property } (K) \text{ at } (t, x_0(t)) \text{ for each } t \geqslant 0, \tag{3.5}$$

$$Q(t, x_0(t)) \text{ is convex for each } t \geqslant 0, \tag{3.6}$$

$$\{f_0^-(\cdot, x_k(\cdot), u_k(\cdot))\}_1^\infty \text{ is strongly uniformly integrable,} \tag{3.7}$$

$$\liminf_{k \to \infty} I(x_k, u_k) < +\infty. \tag{3.8}$$

Then there exists a $\mathscr{L}$-measurable function $u^* : [0, \infty) \to \mathbb{R}^r$ *such that*

$$(x_0, u^*) \in \Omega,$$
$$I(x_0, u^*) \leqslant \liminf_{k \to \infty} I(x_k, u_k).$$

Proof. To begin with, we note that by (3.3) and the properties of weak convergence, $x_0(t) \in A(t)$ for each $t \geqslant 0$. We define for $(t, x) \in \mathscr{A}$, $(\xi, \lambda) \in \mathbb{R}^{m+1}$,

$$l(t, x, \xi, \lambda) \equiv \inf\{\max(f_0(t, x, u), \lambda) : u \in U(t, x), f(t, x, u) = \xi\}, \tag{3.9}$$

with the usual convention that the infimum over an empty set is taken to be $+\infty$. For $(t, x) \notin \mathscr{A}$, we set $l(t, x, \cdot, \cdot) \equiv +\infty$. Thus, l: $[0, \infty) \times \mathbb{R}^{2m+1} \to (-\infty, +\infty]$ is the modified Lagrangian for the optimal control problem [11]. Note that we can also write for $(t, x) \in \mathscr{A}$, $(\xi, \lambda) \in \mathbb{R}^{m+1}$,

$$l(t, x, \xi, \lambda) = \inf\{z_0 : (z_0, \xi) \in Q(t, x), z_0 \geqslant \lambda\}. \tag{3.10}$$

It is therefore easy to see that if $Q(t, x)$ is closed, $(t, x) \in \mathscr{A}$, then for every $(\xi, \lambda) \in \mathbb{R}^{m+1}$, $l(t, x, \xi, \lambda) < +\infty$ implies that the infimum in (3.10) is attained. Hence, by (3.9), for every $t \geqslant 0$ and $\lambda \in \mathbb{R}$ the inequality $l(t, x_0(t), \dot{x}_0(t), \lambda) < +\infty$ implies the existence of $u \in U(t, x_0(t))$ with

$$\begin{aligned} f(t, x_0(t), u) &= \dot{x}_0(t), \\ \max(f_0(t, x_0(t), u), \lambda) &= l(t, x_0(t), \dot{x}_0(t), \lambda). \end{aligned} \tag{3.11}$$

Moreover, it follows by (3.10) from (3.6) in elementary fashion that

$$l(t, x_0(t), \cdot\,, \cdot\,) \text{ is convex for each } t \geqslant 0. \tag{3.12}$$

We shall now demonstrate that for each $t \geqslant 0$, $(\xi, \lambda) \in \mathbb{R}^{m+1}$,

$$l(t, \cdot, \cdot, \cdot) \text{ is lower semicontinuous at } (x_0(t), \xi, \lambda). \tag{3.13}$$

Let $t \geqslant 0$ be arbitrary and let $x^k \to x_0(t)$, $\xi^k \to \xi^0$ and $\lambda^k \to \lambda^0$, also arbitrary. Denote $l^k \equiv l(t, x^k, \xi^k, \lambda^k)$ and $\gamma \equiv \liminf_{k \to \infty} l^k$. We have to demonstrate that $\gamma \geqslant l(t, x_0(t), \xi^0, \lambda^0)$. In case $\gamma = +\infty$, this is trivial. Hence, we can suppose that $l^k < +\infty$ for all k without loss of generality. Let $\varepsilon > 0$ be arbitrary. By (3.10) there exists for each k $z_0^k \geqslant \lambda^k$ such that $(z_0^k, \lambda^k) \in Q(t, x^k)$ and $z_0^k \leqslant l^k + \varepsilon$. Hence, $\lambda^0 \leqslant \liminf_{k \to \infty} z_0^k \leqslant \gamma + \varepsilon$. Define $\gamma' \equiv \liminf_{k \to \infty} z_0^k$. Without loss of generality we can assume that $z_0^k \to \gamma'$ (otherwise we could restrict the considerations to a suitable subsequence). Since $x^k \to x_0(t)$ and $(z_0^k, \xi^k) \to (\gamma', \xi^0)$ it follows quickly from (3.5) and Definition 3.1 that $(\gamma', \xi^0) \in Q(t, x_0(t))$. Above we showed $\gamma' \geqslant \lambda^0$; so we conclude from (3.10) that $l(t, x_0(t), \xi^0, \lambda^0) \leqslant \gamma' \leqslant \gamma + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $l(t, x_0(t), \xi^0, \lambda^0) \leqslant \gamma$. This proves our claim. After this, we shall apply Theorem 2.6 to the integral functional whose integrand $\bar{l}$ is defined by taking $\bar{l}(t, \cdot\,, \cdot\,, \cdot\,)$ to be the lower semicontinuous hull of the function $l(t, \cdot\,, \cdot\,, \cdot\,), t \geqslant 0$. By elementary properties of such hulls it follows from (3.10) and (3.13) that

$$\bar{l}(t, x_0(t), \cdot\,, \cdot\,) = l(t, x_0(t), \cdot\,, \cdot\,) \qquad \text{for each } t \geqslant 0, \tag{3.14}$$

$$\bar{l}(t, x, \xi, \lambda) \geqslant \lambda \qquad \text{for each } (t, x, \xi, \lambda) \in [0, \infty) \times \mathbb{R}^{2m+1}. \tag{3.15}$$

Our next claim is

$$\bar{l} \text{ is } \mathscr{L} \times \mathscr{B}^{2m+1}\text{-measurable}. \tag{3.16}$$

To see this, introduce the function $e\colon [0, \infty) \times \mathbb{R}^{2m+1+r} \to (-\infty, +\infty]$ by setting for $(t, x, u) \in M$, $(\xi, \lambda) \in \mathbb{R}^{m+1}$,

$$\begin{aligned} e(t, x, \xi, \lambda, u) &\equiv \max(f_0(t, x, u), \lambda) \qquad \text{if } \ f(t, x, u) = \xi, \\ &\equiv +\infty \text{ otherwise.} \end{aligned}$$

Elsewhere, we set $e \equiv +\infty$. By (3.1)–(3.3), $e(t, \cdot, \cdot, \cdot, \cdot)$ is lower semicontinuous for each $t \geqslant 0$. Also, our initial measurability assumptions imply that e is $\mathscr{L} \times \mathscr{B}^{2m+1+r}$-measurable. Hence, e is a normal integrand in the sense of Rockafellar [22, Theorem 2A]. (Note that $[0, \infty)$, equipped with the Lebesgue σ-algebra and the Lebesgue measure, is complete.) Since for each $(t, x, \xi, \lambda) \in [0, \infty) \times \mathbb{R}^{2m+1}$

$$\bar{l}(t, x, \xi, \lambda) = \inf_{u \in \mathbb{R}^r} e(t, x, \xi, \lambda, u),$$

it follows from [22, Proposition 2R] that $\bar{l}$ is a normal integrand in the sense of Rockafellar. This implies (3.16).

After these technical preparations we shall now give the main steps of the proof. Rather than considering a suitable subsequence, we shall suppose without loss of generality that $\{I(x_k, u_k)\}_1^\infty$ converges as a whole to the number $\liminf_{k \to \infty} I(x_k, u_k)$. We introduce

$$\lambda_k(t) \equiv -f_0^-(t, x_k(t), u_k(t)), \qquad t \geqslant 0,$$

$$\Lambda_k(t) \equiv \int_0^t \lambda_k(\tau)\, d\tau, \qquad t \geqslant 0.$$

It follows from assumption (3.7), by Theorem 2.1, that $\{\Lambda_k\}_1^\infty \subset AC^1([0, \infty)) \subset AC^1_{\mathrm{loc}}([0, \infty))$ contains a subsequence which weakly converges to a certain element Λ_0 in $AC^1_{\mathrm{loc}}([0, \infty))$. Without loss of generality we may assume that $\{\Lambda_k\}_1^\infty$ as a whole converges to Λ_0. We can now summarize as follows:

$\{(x_k, \Lambda_k)\}_1^\infty$ converges weakly to (x_0, Λ_0) in $AC^{m+1}_{\mathrm{loc}}([0, \infty))$,

$\bar{l}$ is $\mathscr{L} \times \mathscr{B}^{2m+1}$-measurable,

$\bar{l}(t, \cdot, \cdot, \cdot)$ is lower semicontinuous for each $t \geqslant 0$,

$\bar{l}(t, x_0(t), \cdot, \cdot)$ is convex for each $t \geqslant 0$,

$\bar{l}_k(t) \equiv \bar{l}(t, x_k(t), \dot{x}_k(t), \lambda_k(t)) \geqslant \lambda_k(t)$ for each $t \geqslant 0$,

as follows from the above and (3.12)–(3.16). Hence, in view of assumption (3.7), all conditions of Theorem 2.6 have been met. We now get

$$\int_0^\infty \bar{l}(t, x_0(t), \dot{x}_0(t), \dot{\Lambda}_0(t))\, dt$$

$$\leqslant \liminf_{k \to \infty} \int_0^\infty \bar{l}_k \leqslant \liminf_{k \to \infty} I(x_k, u_k), \tag{3.17}$$

where the latter inequality follows from (3.9) and the obvious relation $\bar{l} \leqslant l$. By (3.8), (3.14) and (3.17) we have, for almost every $t \geqslant 0$,

$$\begin{aligned} l_0(t) &\equiv l(t, x_0(t), \dot{x}_0(t), \dot{A}_0(t)) \\ &= \bar{l}(t, x_0(t), \dot{x}_0(t), \dot{A}_0(t)) < +\infty. \end{aligned} \tag{3.18}$$

By (3.11) it then follows that for almost every $t > 0$ there exists u_t in $U(t, x_0(t))$ such that

$$\dot{x}_0(t) = f(t, x_0(t), u_t) \quad \text{and} \quad f(t, x_0(t), u_t) = l_0(t). \tag{3.19}$$

In view of (3.17)–(3.19), the proof is now finished by applying a measurable implicit function theorem in a standard way [23, Theorem III.38].

Remark 3.3. The set $M(t)$ is closed for each $t \geqslant 0$ if and only if the multifunction U has property (K) at each $(t, x) \in \mathscr{A}$ (this property is defined in analogy to Definition 3.1). Thus, supposition (3.4) can be expressed in the terminology of (3.5).

Remark 3.4. We observe that the lower closure result in Theorem 3.2 is merely the manifestation of a lower semicontinuity result for the integral functional with integrand $\bar{l}$. This theme is well known in existence theory [8]; it was modernized in [11] by introduction of the modified Lagrangian.

Remark 3.5. An improvement over the lower closure results commonly found in the literature is that property (K)—as well as closedness of the values of Q—merely has to be satisfied on the arc $\{(t, x_0(t)) : t \geqslant 0\}$. Using a different approach, a similar localization of such conditions was reached in [17].

It is now straightforward to forge the above lower closure result into an existence result. For this, one merely has to place any minimizing sequence for the control problem in the role of the sequence figuring in the lower closure problem. Before stating our main existence result, let us agree to define the set Ω_α for any $\alpha \in \mathbb{R}$ by

$$\Omega_\alpha \equiv \{(x, u) \in \Omega : I(x, u) \leqslant \alpha\}.$$

THEOREM 3.6. *Suppose that* (3.1)–(3.4) *hold. Suppose also that there exists* $\alpha \in \mathbb{R}$ *such that*

$$\{x(0) : (x, u) \in \Omega_\alpha\} \text{ is bounded,} \tag{3.20}$$

$$\begin{aligned} &\{f(\cdot, x(\cdot), u(\cdot))| [0, T] : (x, u) \in \Omega_\alpha\} \\ &\qquad \text{is uniformly integrable for each } T \geqslant 0, \end{aligned} \tag{3.21}$$

$$Q \text{ has property } (K) \text{ at every } (t, x) \in \mathscr{A}, \tag{3.22}$$

$$Q(t, x) \text{ is convex for each } (t, x) \in \mathscr{A}, \tag{3.23}$$

$$\{f_0^-(\cdot, x(\cdot), u(\cdot)) : (x, u) \in \Omega_\alpha\}$$
$$\text{is strongly uniformly integrable,} \tag{3.24}$$

$$\Omega_\alpha \text{ is nonempty.} \tag{3.25}$$

Then there exists an admissible pair $(\bar{x}, \bar{u}) \in \Omega$ *such that*

$$I(\bar{x}, \bar{u}) = \inf_{(x,u) \in \Omega} I(x, u).$$

Proof. Let α be as supposed. Clearly

$$\beta \equiv \inf_{(x,u) \in \Omega} I(x, u) = \inf_{(x,u) \in \Omega_\alpha} I(x, u).$$

There exists a sequence $\{(x_k, u_k)\}_1^\infty \subset \Omega_\alpha$ such that $\beta = \lim_{k \to \infty} I(x_k, u_k)$ by (3.25). Because of (3.20)–(3.21), the sequence $\{x_k\}_1^\infty$ contains a subsequence which converges weakly to some $\bar{x}$ in $AC_{\text{loc}}^m([0, \infty))$; this follows from applying Theorem 2.1. Without losing generality we may suppose that $\{x_k\}_1^\infty$ converges to $\bar{x}$ as a whole. By (3.22)–(3.24) the conditions of Theorem 3.2 have been met. We thus find that there exists $\bar{u}: [0, \infty) \to \mathbb{R}^r$ such that

$$(\bar{x}, \bar{u}) \in \Omega \qquad \text{and} \qquad I(\bar{x}, \bar{u}) \leqslant \beta.$$

Remark 3.7. Actually, supposition (3.22) can be generalized into

$$Q \text{ has property } (K) \text{ at every point in } \{(t, x(t)) : t \geqslant 0, (x, u) \in \Omega_\alpha\}.$$

A similar comment holds for (3.23). That this is true is easily seen from Theorem 3.2 and the proof of Theorem 3.6.

Remark 3.8. Suppose that the following growth condition holds: For each $\varepsilon > 0$ there exists $g \in L_{1,\text{loc}}^+([0, \infty))$ such that

$$|f(t, x, u)| \leqslant g(t) + \varepsilon f_0(t, x, u) \qquad \text{for all } (t, x, u) \in M. \tag{3.26}$$

Then supposition (3.21) is satisfied.

Remark 3.9. Suppose that the cost function is bounded from below in the following way: There exists $h \in L_1([0, \infty))$ such that

$$f_0(t, x, u) \geqslant h(t) \qquad \text{for all } (t, x, u) \in M. \tag{3.27}$$

Then supposition (3.24) is satisfied.

Remark 3.10. Suppose that $r = m$ and that f is of the linear form

$$f(t, x, u) = B(t)\, x + u, \qquad (t, x, u) \in M, \tag{3.28}$$

where B is a measurable $m \times m$ matrix-valued function on $[0, \infty)$. Then, given (3.2) and (3.4), supposition (3.22) holds automatically. If, moreover, for each $(t, x) \in \mathscr{A}$,

$$f_0(t, x, \cdot) \text{ is convex,} \tag{3.29}$$

$$U(t, x) \text{ is convex,} \tag{3.30}$$

then supposition (3.23) holds, too.

Remark 3.11. Theorem 3.6 remains valid if we replace (3.22) by the following alternative supposition: There exists a function c: $[0, \infty) \times \mathbb{R}^r \to [0, \infty]$, $L \times B^r$-measurable, such that

$$c(t, \cdot) \text{ is inf-compact for each } t \geqslant 0,$$

$$\sup_{(x,u) \in \Omega_\alpha} \int_0^\infty c(t, u(t))\, dt < +\infty. \tag{3.31}$$

The proof of the above statement is virtually contained in the lower closure result of [11, Theorem 10]. (Basically, what one has to do is to replace the modified Lagrangian in the proof of Theorem 3.6 by a sequence of *approximate Lagrangians*, introduced in [11].)

Remark 3.12. Theorem 3.6 can easily be converted so as to hold for the more abstract state equation

$$\xi(t) = f(t, x(t), u(t)), \tag{3.32}$$

provided the modes of convergence are taken as in Remark 2.7 and the relative compactness condition for the trajectories is adapted to the new situation. More specifically, this means that we merely have to replace (3.20) by

$\{x : (x, u) \in \Omega_\alpha\}$ is relatively sequentially compact for the topology of pointwise (almost everywhere) convergence. (3.33)

Of course, in the more abstract situation f can map into a Euclidean space with arbitrary finite dimension.

Remark 3.13. Suppose $\Omega' \subset \Omega$ is such that for every sequence $\{(x_k, u_k)\}_0^\infty$ in Ω' with $\{x_k\}_1^\infty$ converging weakly in $AC^m_{\mathrm{loc}}([0, \infty))$ to x_0, we have that there exists $\bar{u}$ such that $(x_0, \bar{u}) \in \Omega'$ and $I(x_0, \bar{u}) \leqslant I(x_0, u_0)$. Then Theorem 3.6 continues to hold if we replace Ω by Ω' in the statement of that result. (This follows trivially from the above proof.)

Let us now compare Theorem 3.6 with the existence results reached in [1–4]. Inter alia, it is supposed in [1] that the multifunction Q has property (Q) at every $(t, x) \in \mathscr{A}$, that (3.26)–(3.27) hold and that f_0 and f are continuous on M. By the fact that property (K) is implied by property (Q) and in view of Remarks 3.8–3.9 it is easy to check that Theorem 6.1 in [1] indeed follows from our result above. In [2] a very simple linear control system is considered, where (3.28)–(3.30) hold with B identically equal to the zero matrix and with even $f_0(t, \cdot, \cdot)$ and $M(t)$ convex for each $t \geqslant 0$. From a special assumption for the sets $M(t)$ [2, Assumption 1] it follows directly that (3.26) must hold [2, Lemma 3.1]. Also, (3.24) holds by [2, Assumption 2(ii)]. It is now easy to verify that Theorem 4.1 of [2] follows from our Theorem 3.6. In [3] the state equation consists of a simple relation in integral form between trajectory and control function [3, Formula (7.1)], viz., a slight abstraction of the classical integral expression for the solution of a linear differential equation. Among other things, the function f_0 is supposed to be such that $f_0(t, \cdot, \cdot)$ is convex in both arguments with nonempty interior of its effective domain for each $t \geqslant 0$. To derive the existence result of [3], one has to take into account our Remarks 3.12–3.13: Define Ω' to be the set of admissible pairs in Ω satisfying the integral relation mentioned above. Further, define the state equation to be used in applying Theorem 3.6 by

$$\xi(t) = u(t). \tag{3.34}$$

Since (3.29)–(3.30) certainly hold in [3], it follows from Remark 3.10 that (3.22)–(3.23) are fulfilled. Also, (3.27) holds in [3], so by Remark 3.9 supposition (3.24) is valid. In view of [3, Assumption 3, formula (5.5)] and (3.34), supposition (3.21) holds in both cases considered there (cf. Remark 2.7). Further, (3.33) is satisfied in [3] by an application of the Alaoglu–Bourbaki theorem, possible by [3, Assumptions 2, 5]. Finally, the integral relation [3, formula (7.1)] is such that if $\{(x_k, u_k)\}_0^\infty$ is in Ω with $\{x_k\}_1^\infty$ converging to x_0 pointwise and with $\{u_k\}_1^\infty$ converging weakly in $L^r_{1,\mathrm{loc}}([0, \infty))$ to u_0 [cf. Remark 2.7 and (3.34)], then the fact that (x_k, u_k) satisfies the integral relation for all $k \in \mathbb{N}$ implies that (x_0, u_0) satisfies this relation, too. It is now easy to see that Theorem 7.6 in [3] follows from Theorem 3.6, provided we extend the latter result in the spirit of Remarks 3.12–3.13. In [4] suppositions (3.28)–(3.30) hold with B identically equal to the zero matrix and with $M(t)$ convex for each $t \geqslant 0$. Also, there is $f^0 : [0, \infty) \times \mathbb{R}^m \to (-\infty, +\infty]$ such that $f_0(t, x, u) = f^0(t, u)$, $(t, x, u) \in M$, where it is required that the effective domain of $f^0(t, \cdot)$ has a nonempty interior for each $t \geqslant 0$. Now (3.1)–(3.4) and (3.20) hold in [4]. Also, (3.21) is fulfilled in view of [4, conditions (2.4), (2.5)]. By Remark 3.10 and what we said above, (3.22)–(3.23) are valid. Finally, (3.24) follows from [4, conditions (2.7), (2.8)]. It is now easy to validate our claim that Theorem 3.6 implies [4, Proposition 3.1]. Moreover, we observe that all conditions connected with bounding f^0 also from above can be omitted in [4].

We conclude these evaluations with a simple pathological example, showing that the property (Q) supposition of [1, Theorem 6.1] can sometimes fail badly in a situation where our Theorem 3.6 can be applied.

EXAMPLE 3.14. Take $m = r = 2$. Denote $e \equiv (1, 1)$, $0 \equiv (0, 0)$ and set $A(0) = \{0\}$, $U(0, 0) = \{0\}$. Take for $t > 0$

$$\begin{aligned}
A(t) &= \mathbb{R}^2, \\
U(t, 0) &= \{(u_1, u_2) : u_1 = 0, u_2 \in \mathbb{R}\}, \\
U(t, x) &= \{0\} \qquad \text{if } x \neq 0 \text{ and } x \neq n^{-1}e,\ n \in \mathbb{N}, \\
U(t, x) &= \{(u_1, u_2) : u_1 \geqslant 0,\ u_2 = |x|^{-1} u_1\} \\
&\qquad \text{if } x = (2n+1)^{-1} e,\ n \in \mathbb{N}, \\
U(t, x) &= \{(u_1, u_2) : u_1 \geqslant 0,\ u_2 = |x|^{-1} u_1\} \\
&\qquad \text{if } x = (2n)^{-1} e,\ n \in \mathbb{N}.
\end{aligned}$$

Further, set $f_0(t, x, u) = |u|^2$ and $f(t, x, u) = u$, $(t, x, u) \in M$. The control problem is thus

$$\inf \left\{ \int_0^\infty |\dot{x}(t)|^2\, dt : x \in AC^2_{\mathrm{loc}}([0, \infty)),\ x(0) = 0,\ \dot{x}(t) \in U(t, x(t)) \text{ a.e.} \right\}.$$

It is not hard to verify that all suppositions used in Theorem 3.6 are fulfilled; in particular, Q has property (K) at every $(t, x) \in \mathscr{A}$. However, at all points of the optimal arc $\{(t, 0) : t > 0\}$ property (Q) even fails to hold in the x-variable alone (cf. Remark 3.5).

4. CONCLUSION

In this paper we demonstrated that for a large class of existence results there is actually no difference between the unbounded and bounded time domain case, provided that the concepts used in the classical case are extended appropriately. For the derivative functions this means that uniform integrability is replaced by local uniform integrability, and for the negative parts of the cost functions that uniform integrability is replaced by strong uniform integrability (as introduced in this paper).

It would be an easy task to raise the level of generality and abstraction of everything discussed here to that of our paper [11], which deals with lower closure results, and to introduce the additional features studied there. In fact, as our Remarks 3.5 and 3.7 indicate, even more can be achieved.

Appendix A

The linear space $AC^m_{\mathrm{loc}}([0, \infty))$ of *locally absolutely continuous m-dimensional functions* on $[0, \infty)$ is defined to consist of all continuous functions $x: [0, \infty) \to \mathbb{R}^m$ such that for each $T > 0$ the restriction $x|[0, T]$ of x to the interval $[0, T]$ belongs to the set $AC^m([0, T])$ of absolutely continuous m-dimensional functions on $[0, T]$ [24, IV.2.22]. Using [24, IV.12.2] it is easy to see that each $x \in AC^m_{\mathrm{loc}}([0, \infty))$ has a derivative $\dot{x}(t)$ at almost every point t in $[0, \infty)$ and that for each $t > 0$

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\, d\tau.$$

Moreover, $\dot{x}$ belongs to the set $L^m_{1,\mathrm{loc}}([0, \infty))$ of locally integrable m-dimensional functions on $[0, \infty)$. Thus, we can define the *weak topology* on $AC^m_{\mathrm{loc}}([0, \infty))$ as the initial topology with respect to the functionals

$$x \mapsto a \cdot x(0) + \int_0^t b(\tau) \cdot \dot{x}(\tau)\, d\tau, \qquad a \in \mathbb{R}^m, t > 0,$$
$$b \in L^m_\infty([0, \infty)).$$

It is important to observe that the topology of uniform convergence on compact subsets of $[0, \infty)$, which $AC^m_{\mathrm{loc}}([0, \infty))$ inherits from the set of all continuous functions on $[0, \infty)$, is weaker than the weak topology!

Appendix B

Regarding our comparing Theorem 3.6 with the existence result in [2], we should point out that the assumption that $f_0(t, \cdot, \cdot)$ is convex for each $t \geqslant 0$, also serves another purpose. Namely, it plays a crucial role in the proof of Theorem 4.2 in [2], the main sensitivity result of that reference.

We shall also show in this appendix that the sensitivity result remains valid under our more general assumption (3.29). At the same time, our argument makes it clear that this sensitivity result is actually a combination of Theorem 3.2, Fatou's lemma and an easy convexity property of the projection of Ω onto the set $AC^m_{\mathrm{loc}}([0, \infty))$.

Let us define, for each $p \in \mathbb{R}^m$,

$$\Omega(p) \equiv \{x : (x, u) \in \Omega, x(0) = p\},$$

$$i(p) \equiv \inf_{(x,u) \in \Omega(p)} I(x, u),$$

$$\Omega_0(p) \equiv \{x : (x, u) \in \Omega(p), I(x, u) = i(p)\}.$$

Note that we regard Ω_0 as a multifunction from $\mathbb{R}^m$ into $AC^m_{\mathrm{loc}}([0, \infty))$, where the latter space is equipped with the weak topology. Also, i is considered as a function on $\mathbb{R}^m$; its effective domain dom i is defined by

$$\operatorname{dom} i \equiv \{p : i(p) < +\infty\}.$$

THEOREM. *Suppose f has the linear structure of* (3.28). *Suppose also that* (3.3)–(3.4), (3.29) *are valid and that for each $t \geqslant 0$*

$$f_0(t, \cdot, \cdot) \text{ is continuous on } M(t), \tag{B.1}$$

$$A(t) \text{ is convex}, \tag{B.2}$$

$$M(t) \text{ is convex}. \tag{B.3}$$

Further, suppose that there exists $h \in L_1([0, \infty))$ such that

$$f_0(t, x, u) \leqslant h(t) \qquad \text{for all } (t, x, u) \in M, \tag{B.4}$$

and that (3.21) *and* (3.24) *hold for $\alpha = \int_0^\infty h$. Then the multifunction Ω_0 is upper semicontinuous on the relative interior of* dom i.

Proof. To prove upper semicontinuity, it is enough to show the following: Given an arbitrary sequence $\{p_k\}_1^\infty$ in dom i, which converges to a point p_0 in the relative interior of dom i, and for associated $x_k \in \Omega_0(p_k)$, $k \in \mathbb{N}$, there exists $x_0 \in \Omega_0(p_0)$ which is the limit of a subsequence of $\{x_k\}_1^\infty$. Let p_k, x_k be as given above. It follows from boundedness of the values $x_k(0) = p_k$, $k \in \mathbb{N}$, and (3.21) that $\{x_k\}_1^\infty$ has a subsequence which converges to some $x_0 \in AC^m_{\mathrm{loc}}([0, \infty))$; to see this, apply Theorem 2.1. It is clear from the kind of topology involved that $x_0(0) = p_0$ (cf. Appendix A). Note that for each $k \in \mathbb{N}$,

$$I(x_k, \dot{x}_k - Bx_k) \leqslant \int_0^\infty h = \alpha, \tag{B.5}$$

as implied by (3.28), ((B.4) and the fact that $i(p_k) < +\infty$. Given arbitrary $x \in \Omega(p_0)$, we now show that

$$I(x, \dot{x} - Bx) \geqslant I(x_0, \dot{x}_0 - Bx_0), \tag{B.6}$$

$$x_0(t) \in A(t) \quad \text{and} \quad \dot{x}_0(t) - B(t)\,x_0(t) \in U(t, x_0(t))$$
$$\text{for almost every } t \geqslant 0. \tag{B.7}$$

Together, these would then imply that $x_0 \in \Omega_0(p_0)$. Without loss of generality we can suppose that $\{x_k\}_1^\infty$ converges to x_0 as a whole. We are going to apply Theorem 3.2. By (3.4), (3.29), (B.1) and (B.3) we know that (3.5)–(3.6) hold, in view of Remark 3.10. Also, (3.7)–(3.8) hold by virtue of (3.21) and (B.5). We then have from Theorem 3.2 that (B.7) is true and that

$$I(x_0, \dot{x}_0 - Bx_0) \leqslant \liminf_{k \to \infty} I(x_k, \dot{x}_k - Bx_k). \tag{B.8}$$

Since p_0 lies in the relative interior of dom i, there exists $\delta > 0$ such that the intersection of the affine hull of dom i with the closed ball with radius δ around p_0 is contained in dom i. Without loss of generality we may suppose that $p_k \neq p_0$ for all k. Hence, for each $k \in \mathbb{N}$ there exists $q_k \in$ dom i, $|q_k - p_0| = \delta$, and γ_k, $0 \leqslant \gamma_k \leqslant 1$, such that

$$p_k = \gamma_k q_k + (1 - \gamma_k)\,p_0.$$

It is evident that

$$\gamma_k \to 0, \tag{B.9}$$

since the q_k remain at a fixed distance from p_0. For each $k \in \mathbb{N}$ the fact that $i(q_k) < +\infty$ implies that $\Omega(q_k)$ is nonempty and contains some y_k, say. We now form

$$z_k \equiv \gamma_k y_k + (1 - \gamma_k)\,x, \qquad k \in \mathbb{N}. \tag{B.10}$$

It follows easily from (3.28) and (B.2)–(B.3) that the projection of Ω onto $AC^m_{\mathrm{loc}}([0, \infty))$ must be convex. Hence, for each $k \in \mathbb{N}$,

$$z_k \in \Omega(p_k). \tag{B.11}$$

It is obvious that

$$z_k(0) \to p_0 = x_0(0). \tag{B.12}$$

Differentiating in (B.10), it follows from (B.9) and (3.21) that

$$\int_0^T |\dot{z}_k(t) - \dot{x}(t)|\,dt \to 0 \qquad \text{for each } T > 0. \tag{B.13}$$

Together with (B.12) this gives

$$z_k(t) \to x(t) \qquad \text{for each } t \geqslant 0. \tag{B.14}$$

In view of (B.4) and (B.11), we have that for each $k \in \mathbb{N}$

$$f_0(t, z_k(t), \dot{z}_k(t) - B(t) z_k(t)) \leqslant h(t) \quad \text{for almost every } t \geqslant 0. \tag{B.15}$$

Taking subsequences sufficiently many times, it follows by an application of Fatou's lemma, possible in view of (B.1), (B.13)–(B.15), that

$$\limsup_{k \to \infty} I(z_k, \dot{z}_k - Bz_k) \leqslant I(x, \dot{x} - Bx).$$

By definition of x_k we have for each $k \in \mathbb{N}$

$$I(x_k, \dot{x}_k - Bx_k) \leqslant I(z_k, \dot{z}_k - Bz_k),$$

in view of (B.11). If we combine this with (B.8) and (B.16), we reach (B.6) and the end of the proof.

Remark. Under the more restrictive assumption made in [2], the proof is virtually finished after deriving (B.8). The reason for this is to be found in the fact that i is then a convex function, hence continuous on the relative interior of its effective domain.

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