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A UNIFYING NOTE ON FATOU'S LEMMA IN SEVERAL DIMENSIONS*†

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A general version of Fatou's lemma in several dimensions is presented. It subsumes the Fatou lemmas given by Schmeidler (1970), Hildenbrand (1974), Cesari–Suryanarayana (1978) and Artstein (1979). Also, it is equivalent to an abstract variational existence result that extends and generalizes results by Aumann–Perles (1965), Berliocchi–Lasry (1973), Artstein (1974) and Balder (1979) in several respects.

1. Main results. Let $(T, \mathscr{T}, \mu)$ be a finite measure space and m a prescribed dimension. Let $\mathscr{L}_1^m \equiv \mathscr{L}_1^m(T, \mathscr{T}, \mu)$ be the space of all integrable functions from T into $\mathbb{R}^m$. For any $y \in \mathbb{R}^m$ we shall define y^+, y^- in $\mathbb{R}^m$ by $(y^+)^i \equiv \max(y^i, 0)$, $i = 1, \ldots, m$ and $y^- \equiv (-y)^+$. In this note we introduce a pair of equivalent existence results, one of which is the following version of Fatou's lemma in several dimensions.

FATOU LEMMA. *Suppose* $\{f_k\} \subset \mathscr{L}_1^m$ *is such that*
(1) $\{f_k^-\}$ *is uniformly integrable,*
(2) $\lim_k \int f_k \, d\mu$ *exists (in* $\mathbb{R}^m$*).*
Then there exists $f_* \in \mathscr{L}_1^m$ *with*
(3) $f_*(t)$ *is a limit point of* $\{f_k(t)\}$ *a.e. in* T,
(4) $\int f_* \, d\mu \leqslant \lim_k \int f_k \, d\mu$.

This lemma subsumes similar results by Schmeidler (1970), Hildenbrand (1974), Cesari–Suryanarayana (1978) and Artstein (1979). To begin with, it clearly generalizes Schmeidler's original result; this is obtained by setting $f_k^- \equiv 0$ for all k. Further, the result in Cesari–Suryanarayana (1978, 2.2) follows from it, since by (3) certainly $f_*(t)$ belongs to the closure of $\{f_k(t)\}$ for a.e. t in T. (The stronger result in (3) is essential for a lot of applications!) Moreover, Cesari and Suryanarayana require $(T, \mathscr{T}, \mu)$ to be nonatomic. (Since they do not provide a proof of their version of Fatou's lemma, it is not possible to determine whether this restriction could be lifted.) Because the original version of Fatou's lemma in Hildenbrand (1974, p. 69) is a special case of the later result by Artstein (1979), it suffices to show that the latter result also follows from our version of Fatou's lemma (Hildenbrand (1974) requires additionally that $\{f_k(t)\}$ be pointwise bounded).

COROLLARY 1. *Suppose* $\{f_k\} \subset \mathscr{L}_1^m$ *is such that*
$\{f_k\}$ *is uniformly integrable,*
$\lim_k \int f_k \, d\mu$ *exists.*

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Then there exists $f_* \in \mathscr{L}_1^m$ *with*

$f_*(t)$ *is a limit point of* $\{f_k(t)\}$ *a.e. in* T,

$\int f_* \, d\mu = \lim_k \int f_k \, d\mu$.

PROOF. Apply the Fatou lemma to $\{(f_k, -f_k)\}$. This gives existence of $(f_*, -f^*) \in \mathscr{L}_1^{2m}$ with $(f_*(t), -f^*(t))$ a limit point of $\{(f_k(t), -f_k(t))\}$ a.e. in T and $\int f_* \leqslant \lim_k \int f_k \leqslant \int f^*$. The former property gives that $f_*(t) = f^*(t)$ a.e. in T. Q.E.D.

Before stating other consequences of the Fatou lemma, we shall need some definitions and notation. Suppose S is a metrizable Lusin space (alias standard Borel space) (Dellacherie–Meyer (1975, III.15)).[1] The Borel σ-algebra on S is denoted by $\mathscr{B}(S)$.

The set of all Borel measurable functions from T into S is denoted by $\mathscr{M}(T; S)$. A function $g: T \times S \to (-\infty, +\infty]$ is said to be a normal integrand on $T \times S$ if g is $\mathscr{T} \times \mathscr{B}(S)$-measurable and $g(t, \cdot)$ is lower semicontinuous on S for every t in T. The set of all (nonnegative) normal integrands on $T \times S$ is denoted by $\mathscr{G}(T; S)$ $[\mathscr{G}^+(T; S)]$. The set of all $g \in \mathscr{G}^+(T; S)$ such that $g(t, \cdot)$ is inf-compact[2] on S for every t in T is denoted by $\mathscr{H}(T; S)$. For any $g \in \mathscr{G}(T; S)$, $u \in \mathscr{M}(T; S)$ we shall write

$$I_g(u) \equiv \int g(t, u(t))\mu(dt) \equiv \int g^+(t, u(t))\mu(dt) - \int g^-(t, u(t))\mu(dt),$$

where $g^+ \equiv \max(g, 0)$, $g^- \equiv \max(-g, 0)$, with the understanding that $(+\infty) - (+\infty) \equiv +\infty$ by convention. Also, for any $g \in \mathscr{G}(T; S)$, $h \in \mathscr{H}(T; S)$ the symbolism $g^- \ll h$ will indicate the following growth property of h with respect to $g^- \equiv \max(-g, 0)$: for every $\epsilon > 0$ there exists $f_\epsilon \in \mathscr{L}_1$ such that on $T \times S$

$$g^-(t, x) \leqslant \epsilon h(t, x) + f_\epsilon(t).$$

Now let X_1, X_2 be metrizable Lusin spaces and n a prescribed number. We have the following abstract variational existence result as a consequence of the Fatou lemma. Later, we shall prove a very weak version of this result from which the Fatou lemma follows. Hence, the two results are in fact equivalent.

PROPOSITION 1. *Suppose* $\{(x_k, \mu_k)\} \subset \mathscr{M}(T; X_1 \times X_2)$ *satisfies*

(5) $\{x_k\}$ *converges in measure to* $x_0 \in \mathscr{M}(T; X_1)$,

(6) $\sup_k I_h(u_k) < +\infty$ *for some* $h \in \mathscr{H}(T; X_2)$.

Suppose also that $\{g_1, \ldots, g_n\} \subset \mathscr{G}(T; X_1 \times X_2)$ *is such that*

(7) $\{g_i^-(\cdot, x_k(\cdot), u_k(\cdot))\}$ *is uniformly integrable*, $i = 1, \ldots, n$.

Then there exist a subsequence $\{u_{k_j}\}$ *of* $\{u_k\}$ *and* $u_* \in \mathscr{M}(T; X_2)$ *with*

(8) $u_*(t)$ *is a limit point of* $\{u_{k_j}(t)\}$ *a.e. in* T,

(9) $I_h(u_*) \leqslant \sup_k I_h(u_k)$,

(10) $I_{g_i}(x_0, u_*) \leqslant \liminf_{k_j} I_{g_i}(x_{k_j}, u_{k_j})$, $i = 1, \ldots, n$.

PROOF. Let β denote the supremum in (6). By (7) $\{I_{g_i}(x_k, u_k)\}$ is bounded from below by a constant for every i, $1 \leqslant i \leqslant n$. Hence, there is a subsequence $\{(x_{k_j}, u_{k_j})\}$ of $\{(x_k, u_k)\}$ such that for every i, $1 \leqslant i \leqslant n$, $\{I_{g_i}(x_{k_j}, u_{k_j})\}$ converges to some $\beta_i \in (-\infty, +\infty]$ and $\{I_h(x_{k_j}, u_{k_j})\}$ to some $\beta_{n+1} \in [0, \beta]$. Rather than extracting a subsequence once more, we may suppose without loss of generality that $x_{k_j}(t) \to x_0(t)$ a.e. in T, by (5) (Hildenbrand (1974, p. 47)). Let E denote the (possibly empty) set of those i, $1 \leqslant i \leqslant n$, for which $\beta_i < +\infty$. Define f_{k_j} to consist of $e + 1$ component functions $g_i(\cdot, x_{k_j}(\cdot), u_{k_j}(\cdot))$, $i \in E$, and $h(\cdot, u_{k_j}(\cdot))$. Here e stands for the number of indices in E. In view of (7), $g_i(\cdot, x_{k_j}(\cdot), u_{k_j}(\cdot))$ is integrable whenever $I_{g_i}(x_{k_j}, u_{k_j}) < +\infty$, so our construction of E gives $f_{k_j} \in \mathscr{L}_1^{e+1}$ for sufficiently large k_j. Now condition (1) of the

[1] For instance, every Polish (separable metric complete) space is metrizable Lusin.

[2] $\{x \in S : g(t, x) \leqslant \beta\}$ is compact for every $\beta \in \mathbb{R}$.

Fatou lemma holds by (7), and (2) holds in view of the choice of $\{(x_{k_j}, u_{k_j})\}$. By the Fatou lemma there exists $f_* \in \mathscr{L}_1^{e+1}$ such that $f_*(t)$ is a limit point of $\{f_{k_j}(t)\}$ a.e. in T and $\int f_*^i \leqslant \beta_i$ for every $i \in E \cup \{e+1\}$. It follows that for a.e. t in T there exists a subsequence $\{k_1\}$ of $\{k_j\}$—quite possibly depending upon t—such that $g_i(t, x_{k_1}(t), u_{k_1}(t)) \to f_*^i(t)$ for every $i \in E$ and with $h(t, u_{k_1}(t)) \to f_*^{e+1}(t) < +\infty$. By inf-compactness of $h(t, \cdot)$, $\{u_{k_1}(t)\}$ contains a subsequence converging to some $u_t \in X_2$. Note that u_t belongs to $\Omega(t) \equiv \bigcap_n \mathrm{cl} \bigcup_{k_j \geqslant n} \{u_{k_j}(t)\}$.[3] From the facts that $x_{k_j}(t) \to x_0(t)$ a.e. in T and that $g_i(t, \cdot, \cdot)$ is lower semicontinuous, we conclude that for a.e. t in T there exists $u_t \in \Omega(t)$ such that $g_i(t, x_0(t), u_t) \leqslant f_*^i(t)$, $i \in E$, and $h(t, u_t) \leqslant f_*^{e+1}(t)$. By Himmelberg (1975, Theorem 6.1) the graph of the multifunction Ω is easily seen to be $\mathscr{T} \times \mathscr{B}(X_2)$-measurable. Hence, the set of all $(t, x) \in T \times X_2$ such that $x \in \Omega(t)$, $g_i(t, x_0(t), x) \leqslant f_*^i(t)$, $i \in E$, and $h(t, x) \leqslant f_*^{e+1}(t)$ is also $\mathscr{T} \times \mathscr{B}(X_2)$-measurable. By Aumann's measurable selection theorem (Himmelberg (1975, Theorem 5.2)) there exists $u_* \in \mathscr{M}(T; X_2)$ such that a.e. in T $u_*(t) \in \Omega(t)$, $g_i(t, x_0(t), u_*(t)) \leqslant f_*^i(t)$ for each $i \in E$ and $h(t, u_*(t)) \leqslant f_*^{e+1}(t)$. This gives $I_h(u_*) \leqslant \int f^{e+1} \leqslant \beta$ and $I_{g_i}(x_0, u_*) \leqslant \beta_i$ for all i, $1 \leqslant i \leqslant n$, since the inequality holds trivially when $i \notin E$. Q.E.D.

COROLLARY 2. *Suppose* $h \in \mathscr{H}(T; X_2)$ *and* $\{g_0, g_1, \ldots, g_n\} \subset \mathscr{G}(T; X_2)$ *satisfy*
(11) $g_i^- \ll h$, $i = 0, 1, \ldots, n$.
For given constants $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$, *let* $\mathscr{M}$ *be the set of all* $u \in \mathscr{M}(T; X_2)$ *which satisfy the following constraints*
(12) $I_{g_i}(u) \leqslant \alpha_i$, $i = 1, \ldots, n$ *and* $I_h(u) \leqslant \alpha_{n+1}$.
Suppose that $\mathscr{M}$ *is nonempty; then there exists* $u_* \in \mathscr{M}$ *such that*
(13) $I_{g_0}(u_*) = \inf_{u \in \mathscr{M}} I_{g_0}(u)$.

PROOF. Since $\mathscr{M}$ is nonempty, there exists a minimizing sequence $\{u_k\}$ in $\mathscr{M}$ (i.e., $I_{g_0}(u_k) \to \inf_{\mathscr{M}} I_{g_0}$). Now (11)–(12) imply the validity of (6)–(7). It follows from applying Proposition 1 that there exist a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and $u_* \in \mathscr{M}(T; X_2)$ such that (8), (9) and (10) hold for $i = 0, 1, \ldots, n$. By our choice of $\{u_k\}$, u_* then satisfies (12) and (13). Q.E.D.

COROLLARY 3. *Suppose* T *is the unit interval* $[0, 1]$, *equipped with Lebesgue* σ*-algebra and measure. Suppose* $h \in \mathscr{H}(T; X_2)$ *and* $\{g_0, g_1, \ldots, g_n\} \subset \mathscr{G}(T; X_2)$ *satisfy*
(14) $g_i^- \ll h$, $i = 0, 1, \ldots, n$.
For given constants $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ *let* $\mathscr{M}'$ *be the set of all* $(t_0, t_1, u) \in [0, 1] \times [0, 1] \times \mathscr{M}(T; X_2)$ *which satisfy*
(15) $\int_{t_0}^{t_1} g_i(t, u(t))\,dt \leqslant \alpha_i$, $i = 1, \ldots, n$, *and* $I_h(u) \leqslant \alpha_{n+1}$.
Suppose that $\mathscr{M}'$ *is nonempty; then there exists* $(t_{0*}, t_{1*}, u_*) \in \mathscr{M}'$ *such that*
(16) $\int_{t_{0*}}^{t_{1*}} g_0(t, u_*(t))\,dt = \inf\{\int_{t_0}^{t_1} g_0(t, u(t))\,dt : (t_0, t_1, u) \in \mathscr{M}'\}$.

PROOF. $\mathscr{M}'$ being nonempty, there exists a minimizing sequence $\{(t_{0,k}, t_{1,k}, u_k)\}$ in $\mathscr{M}'$. By compactness of $[0, 1] \times [0, 1]$ we may suppose without loss of generality that $t_{0,k} \to t_{0*}$, $t_{1,k} \to t_{1*}$ for some t_{0*}, t_{1*} in $[0, 1]$. Take $X_1 \equiv \{0, 1\}$ and define x_k to be the characteristic function $1_{(t_{0,k}, t_{1,k})}$ of the interval $(t_{0,k}, t_{1,k})$. Define $g_i' \in \mathscr{G}(T; X_1 \times X_2)$ by $g_i'(t, x_1, x_2) \equiv x_1 \cdot g(t, x_2)$. Condition (5) of Proposition 1 is fulfilled by the choice of $\{(t_{0,k}, t_{1,k})\}$. For $\{g_0', \ldots, g_n'\}$ (6)–(7) hold by virtue of (14)–(15). By Proposition 1 there exist a subsequence $\{(t_{0,k_j}, t_{1,k_j}, u_{k_j})\}$ and $u_* \in \mathscr{M}(T; X_2)$ such that (9) holds for $i = 0, 1, \ldots, n$ [and with $x_0 = 1(t_{0*}, t_{1*})$]. This shows that (t_{0*}, t_{1*}, u_*) has the required properties. Q.E.D.

Like its counterpart, Proposition 1 is a novel result. We should point out that it strongly resembles a classical lower semicontinuity result for integral functionals; cf.

[3] Here cl stands for closure.

Cesari (1974b), Ioffe (1977), Balder (1981b, 1982, 1983a) and their references. There seem to be no references in the literature on Fatou's lemma that indicate the reciprocity between this lemma and abstract variational existence results.

Corollary 2 extends the existence results obtained thus far for a well-known variational problem concerning the optimal selection of continuous-time allocation plans, to the case where the underlying measure space may have atoms. Influenced by Yaari (1964), the first such result was given by Aumann–Perles (1965) (for T, X_2 Euclidean and μ Lebesgue). Subsequently, this result was generalized by Berliocchi–Lasry (1973) (T, X_2 locally compact Polish and μ nonatomic), Artstein (1974) (T abstract, X_2 Polish and μ nonatomic) and Balder (1979) (T abstract, X_2 metrizable Lusin and μ nonatomic). Corollary 3 deals with a variable consumption period. In its present abstract form it seems to be a new result. Less abstract versions of it would turn out to be well-known existence results for optimal control with variable time "without convexity"; cf. Cesari (1974a) and forthcoming work by Balder (1983b). (Incidentally, it is interesting to note that a version of Fatou's lemma in several dimensions was also used to deal with existence results for optimal control "with convexity"; cf. Cesari-Suryanarayana (1978), Angell (1981).)

2. Proof of the Fatou lemma. We shall derive the Fatou lemma from a very weak version of Proposition 1, to be proven here by using what are essentially the main results of relaxed control theory combined with Lyapunov's theorem. This would seem to suggest a new approach to the Fatou lemma. Of course, since Proposition 1 was already shown to follow from the Fatou lemma, this also establishes the equivalence of this lemma and Proposition 1.

In the weak version of Proposition 1 presented below, Proposition C, we shall only need to take $X_1 \equiv \mathbb{N} \cup \{\infty\}$, $X_2 \equiv \mathbb{R}^m$. Observe that these are both locally compact Polish spaces. Before proving Proposition C we shall introduce some facts, notation and terminology about relaxed control functions. Practically all of this can be found in Berliocchi–Lasry (1973) and Warga (1972); in a more abstract setting it can be found in Balder (1979, 1981a–b).

Let S be a locally compact Polish space. Such a space is countable at infinity, so its Alexandrov (one point) compactification $\hat{S}$ is metrizable (Hildenbrand (1974, p. 15)). Let ρ stand for a fixed compatible metric on $\hat{S}$. Define $\mathscr{C}(\hat{S})$ to be the set of all continuous functions on $\hat{S}$ and $\mathscr{C}_e(\hat{S})$ to be the set of all elementary functions c in $\mathscr{C}(\hat{S})$ that are of the form $c = \gamma\rho(\cdot, x) + \gamma'$, $x \in \hat{S}$, $\gamma, \gamma' \in \mathbb{R}$.

LEMMA A. *For every normal integrand $g \in \mathscr{G}^+(T; S)$ $[g \in \mathscr{G}^+(T; \hat{S})]$ there exist a null set N and sequences $\{T_p\}$ in $\mathscr{T}$, $\{c_p\}$ in $\mathscr{C}_e(\hat{S})$ such that on $(T \backslash N) \times S$ $[(T \backslash N) \times \hat{S}]$*

$$g(t, x) = \sup_p 1_{T_p}(t) c_p(x).$$

PROOF (cf. Balder (1981a, proof of Theorem 1)). Let $\{x_i\}$ be a countable dense subset of S and let $\{r_j\}$ be an enumeration of the rationals. For $i, j, k \in \mathbb{N}$ we define $c_{ijk} \in \mathscr{C}_e(\hat{S})$ by $c_{ijk} \equiv r_j - k\rho(x_i, \cdot)$ and set $B_{ijk} \equiv \{t \in T : c_{ijk}(x) \leqslant g(t, x) \text{ for all } x \in S\}$. Then B_{ijk} is the projection of the set of all $(t, x) \in T \times S$ such that $c_{ijk}(x) > g(t, x)$ onto T. By a well-known projection theorem (Castaing–Valadier (1977, III.23)), the set B_{ijk} belongs to the completion of the σ-algebra $\mathscr{T}$ with respect to μ. Hence, there exists for every i, j, k a set T_{ijk} in $\mathscr{T}$ such that $T_{ijk} \subset B_{ijk}$ and $B_{ijk} \backslash T_{ijk}$ is contained in a null set N_{ijk}. Using the lower semicontinuity and nonnegativity of the function $g(t, \cdot)$, it is not hard to see that

$$\sup_{i,j,k} 1_{B_{ijk}}(t) c_{ijk}(x) = g(t, x) \quad \text{on} \quad T \times S.$$

Taking N to be the union of all N_{ijk}, the result for the first case now follows. The second case is proven in an entirely similar way. Q.E.D.

Let $M(\hat{S})$ $[M_1^+(\hat{S})]$ be the set of all signed bounded measures (probability measures) on $(\hat{S}, \mathscr{B}(\hat{S}))$. We shall equip these with the (relative) vague topology $\sigma(M(\hat{S}), \mathscr{C}(\hat{S}))$ (Dellacherie–Meyer (1975, III.54)). Let $\hat{L}_1 \equiv L_1(T, \mathscr{T}, \mu; \mathscr{C}(\hat{S}))$ be the space of (equivalence classes of) integrable functions from T into $\mathscr{C}(\hat{S})$. The topological dual of $\hat{L}_1$ can be identified with the space $\hat{L}_\infty \equiv L_\infty(T, \mathscr{T}, \mu; M(\hat{S}))$ of (equivalence classes of) essentially bounded Borel measurable functions from T into $M(\hat{S})$ (Ionescu-Tulcea (1969, VII.7); cf. Meyer (1966, p. 301) for a short proof). More precisely, $\hat{L}_\infty$ consists of (equivalence classes of) functions $\hat{\delta}: T \to M(\hat{S}))$ that are vaguely Borel and have

$$\operatorname*{ess\,sup}_t |\hat{\delta}(t)|_v < +\infty,$$

where $|\cdot|_v$ denotes the total variation norm. Let $\hat{R}$ be the set of (equivalence classes of) Borel measurable functions $\hat{\delta}: T \to M(\hat{S})$ such that $\hat{\delta}(t) \in M_1^+(\hat{S})$ a.e. in T; then evidently $\hat{R} \subset \hat{L}_\infty$. We shall equip $\hat{L}_\infty[\hat{R}]$ with the (relative) topology $\sigma(\hat{L}_\infty, \hat{L}_1)$; note that this makes $\hat{L}_\infty$ into a Hausdorff locally convex space. By abuse of notation we write for any $\hat{\delta} \in \hat{R}$, $g \in \mathscr{G}(T; \hat{S})$

$$I_g(\hat{\delta}) \equiv \int g(t, \hat{\delta}(t))\mu(dt) \equiv \int g^+(t, \hat{\delta}(t))\mu(dt) - \int g^-(t, \hat{\delta}(t))\mu(dt),$$

with the provision $(+\infty) - (+\infty) \equiv +\infty$. Here $g^+(t, \hat{\delta}(t)) \equiv \int_{\hat{S}} g^+(t, x)\hat{\delta}(t)(dx)$, etc. It follows easily from Lemma A that, modulo abuse in our notation, this integral is well defined. Our next result forms the centerpiece of relaxed control theory; cf. Warga (1972, IV), Castaing–Valadier (1977, V.2).

THEOREM B. (i) *$\hat{R}$ is compact and sequentially compact.*
(ii) *For every $\hat{g} \in \mathscr{G}^+(T; \hat{S})$ the function $I_{\hat{g}}: \hat{R} \to [0, +\infty]$ is lower semicontinuous.*

PROOF. (i) Compactness of the closed convex set $\hat{R}$ in the unit ball of $\hat{L}_\infty$ follows by the Alaoglu–Bourbaki theorem (Holmes (1975, 12.D)). Sequential compactness of $\hat{R}$ is seen as follows; cf. Nölle–Plachky (1967), Kirschner (1976). Given a sequence $\{\hat{\delta}_k\}$ in $\hat{R}$, let $\mathscr{T}'$ be the sub-σ algebra of $\mathscr{T}$ generated by $\{\hat{\delta}_k\}$. Since $M_1^+(\hat{S})$ is metrizable and separable, $\mathscr{T}'$ is countably generated. Hence, $\hat{L}_1' \equiv L_1(T, \mathscr{T}', \mu; \mathscr{C}(\hat{S}))$ is separable, so by Holmes (1975, 12.F) the set $\hat{R}'$ of all (equivalence classes of) $\mathscr{T}'$-measurable functions $\hat{\delta}: T \to M_1^+(\hat{S})$ is metrizable for the topology $\sigma(\hat{L}_\infty', \hat{L}_1')$, where $\hat{L}_\infty' \equiv L_\infty(T, \mathscr{T}', \mu; M(\hat{S}))$. Also, $\hat{R}'$ is compact for $\sigma(\hat{L}_\infty', \hat{L}_1')$. Hence, a subsequence of $\{\hat{\delta}_k\}$ converges to some $\hat{\delta}_0 \in \hat{R}'$ in the topology $\sigma(\hat{L}_\infty', \hat{L}_1')$. It follows easily from the conditional expectation result of Castaing–Valadier (1977, VIII.32) that this subsequence converges now also to $\hat{\delta}_0$ in $\sigma(\hat{L}_\infty, \hat{L}_1)$.

(ii) Given $g \in \mathscr{G}^+(T; \hat{S})$, we can apply Lemma A. In the notation of that lemma, define

$$\hat{g}_q \equiv \min\left\{q, \max\left[\sup_{p \leq q} 1_{T_p} c_p, 0\right]\right\}.$$

Then $\hat{g}_q \uparrow g$ on $(T \backslash N) \times \hat{S}$. Note that each $\hat{g}_q$ is a version of an equivalence class in $\hat{L}_1$. Hence, I_g is the supremum of a collection of continuous functions on $\hat{R}$, as follows by applying the monotone convergence theorem. Q.E.D.

We are now in a position to prove Proposition C, a weak version of Proposition 1. Let $\hat{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$ denote the usual Alexandrov compactification of $\mathbb{N}$ ("addition of point at infinity").

PROPOSITION C. *Suppose $\{u_k\} \subset \mathscr{M}(T; \mathbb{R}^m)$ satisfies*
(17) $\sup_k I_h(u_k) < +\infty$ *for some $h \in \mathscr{H}(T; \mathbb{R}^m)$.*

Suppose also that $g_0 \in \mathscr{G}^+(T; \mathbb{N} \times \mathbb{R}^m)$ *and* $\{g_1, \ldots, g_n\} \subset \mathscr{G}(T; \mathbb{R}^m)$ *are such that*

(18) $g_i^- \ll h$, $i = 1, \ldots, n$.

Then there exist a subsequence $\{u_{k_j}\}$ *of* $\{u_k\}$ *and* $u_* \in \mathscr{M}(T; \mathbb{R}^m)$ *with*

(19) $I_h(u_*) \leq \sup_k I_h(u_k)$,

(20) $I_{g_0}(\infty, u_*) \leq \liminf_{k_j} I_{g_0}(k_j, u_{k_j})$,

(21) $I_{g_i}(u_*) \leq \liminf_{k_j} I_{g_i}(u_{k_j})$, $i = 1, \ldots, n$.

PROOF. By (17)–(18) the sequences $\{I_{g_0}(k, u_k)\}$ and $\{I_{g_i}(u_k)\}$, $i = 1, \ldots, n$, are bounded below, so there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{I_{g_0}(k_j, u_{k_j})\}$ converges to some $\beta_0 \in [0, \infty]$ and $\{I_{g_i}(u_{k_j})\}$ converges to some $\beta_i \in (-\infty, +\infty]$.

Secondly, we shall show now that to a subsequence of $\{u_{k_j}\}$, there corresponds a generalized limit $\delta_1 \in \hat{R}$ such that $\delta_1(t)$ is a point measure a.e. in T_1. Here T_1 denotes the purely atomic part of $(T, \mathscr{T}, \mu)$, (a version of) the essential supremum of all atoms. Let $T_0 \equiv T \backslash T_1$ be the nonatomic part. For $S \equiv \mathbb{R}^m$ we shall apply Theorem B to the case where $(T, \mathscr{T}, \mu)$ is replaced by $(T_0, \mathscr{T}_0, \mu_0)$. Here $\mathscr{T}_0 \equiv \mathscr{T} \mid T_0$, $\mu_0 \equiv \mu \mid \mathscr{T}_0$. Also, we write correspondingly $\hat{L}^0_\infty$, $\hat{L}^0_1$, $\hat{R}^0$, etc. The Alexandrov compactification of $S \equiv \mathbb{R}^m$ is denoted by $\hat{\mathbb{R}}^m \equiv \mathbb{R}^m \cup \{\omega\}$. Since $\mathbb{R}^m$ is open in $\hat{\mathbb{R}}^m$, each function $u_k \mid T_0$ also belongs to $\mathscr{M}(T_0; \hat{\mathbb{R}}^m)$. Hence, the mapping $t \mapsto$(point mass at $u_k(t)$) defines (a version of) an element ϵ_{u_k} in $\hat{R}^0$. By Theorem B(i) $\{u_{k_j}\}$ has a subsequence $\{u_{k_l}\}$ such that $\{\epsilon_{u_{k_l}}\} \subset \hat{R}^0$ converges to some $\delta_0 \in \hat{R}^0$ in the topology $\sigma(\hat{L}^0_\infty, \hat{L}^0_1)$. We can split T_1 into at most countably many atoms A_j (modulo null sets this decomposition is unique). Every function u_{k_l} is equal to a constant $u^j_{k_l} \in \mathbb{R}^m$ almost everywhere on an atom A_j.

By Lemma A, applied to h, there corresponds to each atom A_j an inf-compact function $h_j: \mathbb{R}^m \to [0, +\infty]$ such that $h(t, \cdot) = h_j$ for a.e. t in A_j. (Note that the restrictions of the functions 1_{T_p} to A_j are a.e. constant zero or constant one.) By (17) we have for each A_j that $\sup_{k_l} h_j(u^j_{k_l}) \leq \beta / \mu(A_j)$, where $\beta \equiv \sup_k I_h(u_k)$. By inf-compactness of h_j, each sequence $\{u^j_{k_l}\}$ has a subsequence converging to some $u^j_* \in \mathbb{R}^m$. By an obvious diagonal extraction argument it follows that a subsequence $\{u_{k_r}\}$ of $\{u_{k_l}\}$ exists with the following properties: for every j, $\{u^j_{k_r}\}$ converges to $u^j_* \in \mathbb{R}^m$ and $\{\epsilon_{u_{k_r}}\} \subset \hat{R}^0$ converges to $\delta_0 \in \hat{R}^0$ in $\sigma(\hat{L}^0_\infty, \hat{L}^0_1)$. Define $\delta_1 \in \hat{R}$ by $\delta_1(t) \equiv \delta_0(t)$ if $t \in T_0$, $\delta_1(t) \equiv$ (point mass at u^j_*) if $t \in A_j$. By the simple nature of the weak star topology of $L_\infty(T_1, \mathscr{T} \mid T_1, \mu \mid T_1; M(\hat{\mathbb{R}}^m))$, it is easy to check that $\{\epsilon_{u_{k_r}}\} \subset \hat{R}$ (so with domain extended to T) converges to δ_1 in $\sigma(\hat{L}_\infty, \hat{L}_1)$.

Thirdly, we shall show that (19)–(21) hold with u_* replaced by δ_1. Define $\hat{h} \in \mathscr{G}^+(T; \hat{\mathbb{R}}^m)$ by $\hat{h}(t, x) \equiv h(t, x)$ if $x \in \mathbb{R}^m$ and $\hat{h}(t, \omega) \equiv +\infty$. (By definition of the topology on $\hat{\mathbb{R}}^m$, lower semicontinuity of $\hat{h}(t, \cdot)$ on $\hat{\mathbb{R}}^m$ is equivalent to inf-compactness of $h(t, \cdot)$ on $\mathbb{R}^m$.) By Theorem B(ii)

$$I_{\hat{h}}(\delta_1) \leq \liminf_{k_r} I_{\hat{h}}(\epsilon_{u_{k_r}}) = \liminf_{k_r} I_h(u_{k_r}) \leq \beta < +\infty.$$

Hence, for a.e. t in T the point ω "at infinity" cannot belong to the carrier of the probability measure $\delta_1(t)$. By the above this implies that $I_h(\delta_1) = I_{\hat{h}}(\delta_1) \leq \beta$, which proves (19) with u_* replaced by δ_1. Applying Lemma A with $S \equiv \hat{\mathbb{N}} \times \hat{\mathbb{R}}^m$, equipped with the sum $\rho_1 + \rho_2$ of compatible metrics ρ_1 on $\hat{\mathbb{N}}$ and ρ_2 in $\hat{\mathbb{R}}^m$, we find that there exist a null set N, sequences $\{T_p\} \subset \mathscr{T}$ and $\{c_p\} \subset \mathscr{C}_e(T; \hat{\mathbb{N}} \times \hat{\mathbb{R}}^m)$ such that

$$g_0(t, k, x) = \sup_p 1_{T_p}(t) c_p(k, x) \qquad \text{on} \quad (T \backslash N) \times \hat{\mathbb{N}} \times \mathbb{R}^m.$$

Define $\hat{g}_q \equiv \min\{q, \max[\sup_{p \leq q} 1_{T_p} c_p, 0]\}$; since elementary functionals are Lipschitz-continuous, it follows that for each $\hat{g}_q$ there exists $K_q > 0$ such that

$$|\hat{g}_q(t, k, x) - \hat{g}_q(t, k', x')| \leq K_q\left[\rho_1(k, k') + \rho_2(x, x')\right]$$

on $T \times \hat{\mathbb{N}} \times \hat{\mathbb{R}}^m$. Now by the monotone convergence theorem

$$I_{g_0}(k_r, u_{k_r}) = \lim_q \uparrow I_{\hat{g}_q}(k_r, u_{k_r}) \quad \text{and} \quad I_{g_0}(\infty, \delta_1) = \lim_q \uparrow I_{\hat{g}_q}(\infty, \delta_1),$$

where the latter identity follows from the fact that $\delta_1(t)$ is not supported by ω for a.e. t in T. In view of this, it is enough to prove that

$$\liminf_{k_r} I_{\hat{g}_q}(k_r, u_{k_r}) \geqslant I_{\hat{g}_q}(\infty, \delta_1)$$

for arbitrary $q \in \mathbb{N}$. By Lipschitz continuity of $\hat{g}_q(t, \cdot, \cdot)$ it follows that

$$I_{\hat{g}_q}(k_r, u_{k_r}) - I_{\hat{g}_q}(\infty, \delta_1) \geqslant -K_q \rho_1(k_r, \infty) + I_{\hat{g}_q}(\infty, u_{k_r}) - I_{\hat{g}_q}(\infty, \delta_1).$$

Since $\{\epsilon_{u_{k_r}}\}$ is known to converge to δ_1 and since $(t, x) \mapsto \hat{g}_q(t, \infty, x)$ clearly belongs to $\mathscr{G}^+(T; \mathbb{R}^m)$, it follows from Theorem B(ii) that (20) holds with u_* replaced by δ_1. For every $i = 1, \ldots, n$, $\epsilon > 0$ there exists by (18) a function $f_{i,\epsilon} \in \mathscr{L}_1$ such that $g_{i,\epsilon} \equiv g_i + \epsilon h + f_{i,\epsilon}$ is nonnegative and belongs to $\mathscr{G}^+(T; \mathbb{R}^m)$. By applying Lemma A in the usual way it follows that there exist a null set N_ϵ and $\hat{g}_{i,\epsilon} \in \mathscr{G}^+(T; \hat{\mathbb{R}}^m)$ such that $g_{i,\epsilon}(t, x) = \hat{g}_{i,\epsilon}(t, x)$ on $(T \backslash N_\epsilon) \times \mathbb{R}^m$. Define $\hat{R}(\hat{h})$ to be the set of all $\delta \in \hat{R}$ such that $I_{\hat{h}}(\delta) \leqslant \beta$. It is elementary to prove that for every $\delta \in \hat{R}(\hat{h})$

$$\sup_{\epsilon > 0} \left[I_{\hat{g}_{i,\epsilon}}(\delta) - \epsilon\beta - \int f_{i,\epsilon} \right] = I_{g_i}(\delta), \qquad i = 1, \ldots, n.$$

(Note that for every $\delta \in \hat{R}(\hat{h})$ the point ω is not carried by $\delta(t)$ for a.e. t in T.) By Theorem B(ii) this means that I_{g_i} is lower semicontinuous on $\hat{R}(\hat{h})$. Hence (21) holds with u_* replaced by δ_1.

Lastly, the proof is finished by showing that there exists $u_* \in \mathscr{M}(T; \mathbb{R}^m)$ such that

$$I_{g_0}(\infty, u_*) \leqslant I_{g_0}(\infty, \delta_1) \quad \text{and} \quad I_{g_i}(u_*) \leqslant I_{g_i}(\delta_1), \qquad i = 1, \ldots, n.$$

No harm is done if we denote from now on $g_0(t, \infty, x)$ as $g_0(t, x)$. For $i = 0, 1, \ldots, n$ we write

$$I^0_{g_i}(\delta_1) \equiv \int_{T_0} g_i(t, \delta_1(t)) \mu(dt), \qquad \text{etc.}$$

Define $\hat{R}^0(\hat{h})$ to be the set of all $\delta \in \hat{R}^0$ such that $I^0_{\hat{h}}(\delta) \leqslant \beta$. By Theorem B(i) $\hat{R}^0(\hat{h})$ is compact in $\hat{R}^0$. It is clear from the previous step that $I^0_{g_i}$ is lower semicontinuous (and affine) on $\hat{R}^0(\hat{h})$ for $i = 0, 1, \ldots, n$. Hence, the set P of all $\delta \in \hat{R}^0(\hat{h})$ with $I^0_{g_i}(\delta) \leqslant I^0_{g_i}(\delta_1)$, $0 \leqslant i \leqslant n$, is nonempty and compact. Therefore, it contains an extreme point of δ_* by the Krein–Milman theorem. By a consequence of Carathéodory's theorem δ_* is a convex combination of at most $n + 2$ extreme points in $\hat{R}^0(\hat{h})$ (Berliocchi–Lasry (1973, Proposition II.2)). By the same result, every extreme point of $\hat{R}^0(\hat{h})$ is the convex combination of at most two extreme points in $\hat{R}^0$. By Himmelberg (1975, Theorems 5.2, 9.3) there corresponds to every extreme point δ in $\hat{R}^0$ a function $u \in \mathscr{M}(T_0; \mathbb{R}^m)$ such that $\delta(t) = \epsilon_u(t)$ a.e. in T_0. We conclude that there exist at most $2n + 4$ coefficients $\alpha_j \geqslant 0$ and associated $v_j \in \mathscr{M}(T_0; \hat{\mathbb{R}}^m)$, such that $\sum \alpha_j = 1$ and $\delta_* = \sum \alpha_j \epsilon_{v_j}$. Since $\delta_* \in \hat{R}^0(h)$, we also know that for a.e. t in T_0 the measure $\delta_*(t)$ is not carried by ω. Hence, all v_j can be supposed to belong to $\mathscr{M}(T_0; \mathbb{R}^m)$. Writing temporarily $g_{n+1} \equiv h$, we find that

$$\sum \alpha_j I^0_{g_i}(v_j) = I^0_{g_i}(\delta_*) \leqslant I^0_{g_i}(\delta_1), \qquad i = 0, 1, \ldots, m + 1.$$

By a well-known extension of Lyapunov's theorem there exists $v_* \in \mathscr{M}(T_0; \mathbb{R}^m)$ with $I^0_{g_i}(\delta_*) = I^0_{g_i}(v_*)$, $i = 0, 1, \ldots, m$ [Castaing–Valadier (1977, IV.17)]. Now define u_*

$\in \mathcal{M}(T;\mathbb{R}^m)$ by $u_*(t) \equiv v_*(t)$ on T_0, $u_*(t) \equiv u_*^j$ on A_j. Then combining the above steps gives that (19)–(21) hold. Q.E.D.

Proof of the Fatou Lemma. Let us apply Proposition C to the following case. Take $n = 3m$. Define g_0 as follows. For $p \neq +\infty$ define $g_0(t,p,x) \equiv 0$ if $x \in \mathrm{cl}\bigcup_{k \geq p}\{f_k(t)\}$ and $g_0(t,p,x) \equiv +\infty$ if not. For $p = +\infty$ define $g_0(t,\infty,x) \equiv 0$ if $x \in \bigcap_{p=1}^{\infty} \mathrm{cl}\bigcup_{k \geq p}\{f_k(t)\}$ and $g_0(t,\infty,x) \equiv +\infty$ if not.

Measurability of g_0 follows by applying Himmelberg (1975, Theorem 6.1) and it is not hard to verify that $g_0(t,\cdot,\cdot)$ is lower semicontinuous on $\hat{\mathbb{N}} \times \mathbb{R}^m$. Hence $g_0 \in \mathcal{G}^+(T;\hat{\mathbb{N}} \times \mathbb{R}^m)$. Define $g_1, \ldots, g_m$ by $g_i(t,x) \equiv (x^+)^i$, $g_{m+1}, \ldots, g_{2m}$ by $g_{m+i}(t,x) \equiv (x^-)^i$ and $g_{2m+1}, \ldots, g_{3m}$ by $g_{2m+i}(t,x) \equiv -(x^-)^i$. Clearly all g_i, $1 \leq i \leq 3m$, belong to $\mathcal{G}(T;\mathbb{R}^m)$. Now by de la Vallée-Poussin's theorem (Dellacherie–Meyer (1975, II.22)) it follows from (1) that there exists a lower semicontinuous $h' : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(i) $\sup_k \int h'(|f_k^-|)\,d\mu < +\infty$,

(ii) $h'(\beta)/\beta \to +\infty$ as $\beta \to +\infty$.

Define now $h(t,x) \equiv |x^+| + h'(|x^-|)$. Then (17) is valid by (2) and (i). Also, (ii) implies that (18) holds. By an application of Proposition C it follows that there exist a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ and $f_* \in \mathcal{M}(T;\mathbb{R}^m)$ such that (19)–(21) hold mutatis mutandis. Now (19) implies that f_* is integrable, (20) implies (3) and it follows from (21) that

$$\int f_*^- = \lim_{k_j} \int f_{k_j}^-, \qquad \int f_*^+ \leq \liminf_{k_j} \int f_{k_j}^+ .$$

A fortiori (4) follows, in view of (2). Q.E.D.

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